TIF 150: Information theory for complex systems

Time: August 26, 2013, 12.00 – 16.00.
Place: Swedish Embassy, Teheran
Allowed material: Calculator
Teacher and examiner: Kristian Lindgren (+46-707574031)

All answers and derivations must be clear and well motivated.
Grade limits: 25p for 3, 34p for 4, and 42p for 5.
ECTS grades: 25p for E, 28p for D, 34p for C, 38p for B, 42p for A.
The results will be available on September 5.

1. Ball measurements.
Assume you have 3 pairs of balls, each of a different color: 6 in total with 2 blue, 2 red and 2 green. In each pair, one of the balls is heavy and the other is light (you don't know which is which). The 3 heavy balls weigh the same, and the three light balls weigh the same. You have at your disposal a balance scale.

(a) According to information theory, what is the Shannon entropy of the system? How much information would you get from an ideal measurement with the scale? Discuss briefly what conclusions can be drawn from these quantities.
(b) Find a procedure for finding out the weight of all the balls using as few measurements as possible.

(10 p)

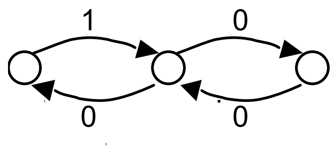
2. An equilibrium spin system. Consider a one-dimensional spin system where each site can be occupied by either $\uparrow$ or $\downarrow$:

…	$\uparrow$	$\downarrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\uparrow$	…	spin system
…	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$	…	external field

The system is under influence of an alternating (in space) external field so that the field at a site is the opposite of its neighbours field. If a spin is aligned with the local external field direction, this constitutes an energy contribution of $-J / 2 < 0$ to the total energy of the system, if the spin and the field is opposed, the energy contribution is $+J / 2 > 0$. Additionally, neighbouring spins also interact, with an energy contribution of $-J$ if the two spins are aligned, and $+J$ if they are opposed. If the average energy is u, what is the equilibrium distribution? Or, in other words, what are the probabilities over sequences of symbols that characterise the system? You need not to solve the equations but you should set up the equations that determine the solution. You may keep the inverse temperature instead of energy. Discuss how the system looks like in the limit of zero temperature.

(10 p)

3. **CA information.**
Consider a one-dimensional cellular automaton given by elementary rule R50, i.e., neighbourhoods 100, 101and 001 map to 1, but all others to 0. Let the initial state be characterized by the following finite state automaton (where it is assumed that if two arcs leave the same node they have equal probabilities)



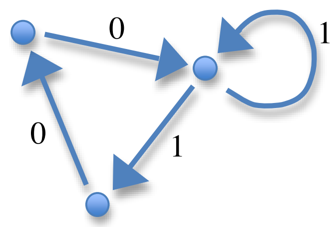
(a) What is the initial entropy (at $t = 0$)? Derive the finite state automaton that characterizes the CA state after one time step ($t = 1$). What is the entropy at this time?

(b) What is the entropy at $t = 2$?

(c) What does the entropy converge to in the t=infinity limit?

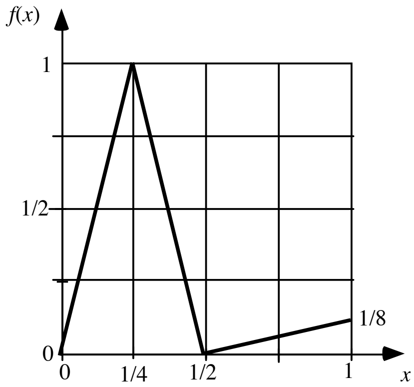
(11 p)

4. **Correlation complexity.** Below is a hidden Markov model. When two arcs leave a node it is assumed that they have the same probability. Determine the correlation complexity η.



(11 p)

5. Chaos and information. Let a piecewise linear map $f(x)$ be defined by the figure below, where the mapping is determined by $f(0) = f(1/2) = 0$, $f(1/4) = 1$, $f(1) = 1/8$.



Consider the dynamical system

$$x_{t+1} = f(x_t) \ .$$

(ii) Determine the invariant measure that characterizes the chaotic behaviour, and *from this* calculate the Lyapunov exponent λ. Find a partition that is generating, and calculate the measure entropy *from the symbolic dynamics*.

(8 p)

Information theory for complex systems – useful equations

Basic quantities

$$I(p) = \log\frac{1}{p} \qquad S[P] = \sum_{i=1}^{n} p_i \log\frac{1}{p_i} \qquad K\left[P^{(0)};P\right] = \sum_{i=1}^{n} p_i \log\frac{p_i}{p_i^{(0)}}$$

Max entropy formalism (with $k+1$ constraints) using the Lagrangian L

$$L\left(p_1,\ldots,p_n,\lambda_1,\ldots,\lambda_r,\mu\right) = S[P] + \sum_{k=1}^{r} \lambda_k \left(F_k - \sum_{i=1}^{n} p_i f_k(i)\right) + \left(\mu-1\right)\left(1-\sum_{i=1}^{n} p_i\right) ,$$

$$p_j = \exp\left(-\mu - \sum_{k=1}^{r} \lambda_k f_k(j)\right), \quad \mu(\lambda) = \ln\sum_{j=1}^{n} \exp(-\sum_{k} \lambda_k f_k(j)), \quad \frac{\partial\mu(\lambda)}{\partial\lambda_k} = -F_k$$

Symbol sequences

$$s = \lim_{n\to\infty} \sum_{x_1\ldots x_{n-1}} p(x_1\ldots x_{n-1}) \sum_{x_n} p(x_n \mid x_1\ldots x_{n-1}) \log\frac{1}{p(x_n \mid x_1\ldots x_{n-1})} =$$

$$= \lim_{n\to\infty}\left(S_n - S_{n-1}\right) = \Delta S_\infty = \lim_{n\to\infty}\frac{1}{n} S_n$$

$$k_1 = \log\nu - S_1, \qquad k_n = -S_n + 2S_{n-1} - S_{n-2} = -\Delta S_n + \Delta S_{n-1} = -\Delta^2 S_n \quad (n = 2,3,\ldots)$$

$$S_{\mathrm{tot}} = \log\nu = \sum_{m=1}^{\infty} k_m + s$$

$$\eta = \sum_{m=1}^{\infty}(m-1)\,k_m = \lim_{m\to\infty}\lim_{n\to\infty}\sum_{\sigma_m} p(\sigma_m)\sum_{\tau_n} p(\tau_n \mid \sigma_m)\log\frac{p(\tau_n \mid \sigma_m)}{p(\tau_n)} = \lim_{m\to\infty}(S_m - m\,s)$$

Geometric information theory

$$p(r;x) = \frac{1}{\sqrt{2\pi}\,r}\int_{-\infty}^{\infty} dw\, e^{-w^2/2r^2}\, p(x-w) \quad , \qquad \left(-r\frac{\partial}{\partial r} + r^2\frac{d^2}{dx^2}\right)p(r;x) = 0$$

$$p_{\mathrm{Gaussian}}(r;x) = \frac{1}{\sqrt{2\pi}\sqrt{b^2+r^2}}\exp\left(-\frac{x^2}{2(b^2+r^2)}\right)$$

$$K[p_0;p] = \int dx\, p(x)\ln\frac{p(x)}{p_0(x)} = \int_0^{\infty}\frac{dr}{r}\int dx\, k(r,x), \qquad k(r,x) = r^2 p(r;x)\left(\frac{d}{dx}\ln p(r;x)\right)^2$$

$$d(r) = D_{\mathrm{E}} - r\frac{\partial}{\partial r}\int d\mathbf{x}\, p(r;\mathbf{x})\ln\frac{1}{p(r;\mathbf{x})}$$

Chemical systems and information flow

$$E = k_{\mathrm{B}}T_0\frac{N}{V}K, \qquad K = \int_V d\mathbf{x}\, K[c_0;c(\mathbf{x})] = \int_V d\mathbf{x}\sum_{i=1}^{M} c_i(\mathbf{x})\ln\frac{c_i(\mathbf{x})}{c_{i0}}, \quad \sum_{i=1}^{M} c_i(\mathbf{x},t) = 1$$

$$K = V\sum_{i=1}^{M}\bar{c}_i \ln\frac{\bar{c}_i}{c_{i0}} + \int_V d\mathbf{x}\sum_{i=1}^{M} c_i(\mathbf{x})\ln\frac{c_i(\mathbf{x})}{\bar{c}_i} = K_{\text{chem}} + K_{\text{spatial}}$$

$$K_{\text{spatial}} = \int_0^\infty \frac{\partial r}{r}\int d\mathbf{x}\, k(r,\mathbf{x})$$

$$k(r,\mathbf{x}) = r^2\sum_i \tilde{c}_i(r,\mathbf{x})\left(\nabla\ln\frac{\tilde{c}_i(r,\mathbf{x})}{c_{i0}}\right)^2 = r^2\sum_i\frac{(\nabla\tilde{c}_i(r,\mathbf{x}))^2}{\tilde{c}_i(r,\mathbf{x})} = \left(-r\frac{\partial}{\partial r} + r^2\nabla^2\right)\sum_i\tilde{c}_i(r,\mathbf{x})\ln\frac{\tilde{c}_i(r,\mathbf{x})}{c_{i0}}$$

$$\dot{c}_i(\mathbf{x},t) = \frac{d}{dt}c_i(\mathbf{x},t) = D_i\nabla^2 c_i(\mathbf{x},t) + F_i(\mathbf{c}(\mathbf{x},t)) + b_i(c_{i,\,\text{res}} - c_i(\mathbf{x},t))$$

$$\sigma(\mathbf{x},t) = \sum_i\left(D_i\frac{(\nabla c_i(\mathbf{x},t))^2}{c_i(\mathbf{x},t)} - F_i(\mathbf{c}(\mathbf{x},t))\ln\frac{c_i(\mathbf{x},t)}{c_{i0}}\right)$$

$$j_r(r,\mathbf{x},t) = \sum_i\left(D_i\frac{(\nabla\tilde{c}_i(r,\mathbf{x},t))^2}{\tilde{c}_i(r,\mathbf{x},t)} - \tilde{F}_i(\mathbf{c}(\mathbf{x},t))\ln\frac{\tilde{c}_i(r,\mathbf{x},t)}{c_{i0}}\right),\ \tilde{F}_i(\mathbf{c}(\mathbf{x},t)) = \exp(\tfrac{r^2}{2}\nabla^2)F_i(\mathbf{c}(\mathbf{x},t))$$

$$\mathbf{j}(r,\mathbf{x},t) = -r^2\nabla\sum_i\tilde{F}_i(\mathbf{c}(\mathbf{x},t))\ln\frac{\tilde{c}_i(r,\mathbf{x},t)}{c_{i0}},\quad J(r,x,t) = -\sum_i b_i(\tilde{c}_i + c_{i,\,\text{res}})[r\nabla\ln\tilde{c}_i]^2$$

$$\dot{k}(r,\mathbf{x},t) = r\frac{\partial}{\partial r}j_r(r,\mathbf{x},t) - \nabla\cdot\mathbf{j}(r,\mathbf{x},t) + J(r,x,t)$$

Chaos and information

$$\int dx\mu(x)\varphi(x) = \lim_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T-1}\varphi(f^k(x(0))),$$

$$\lambda = \lim_{t\to\infty}\frac{1}{t}\sum_{k=0}^{t-1}\ln\left|f'(x(k))\right| = \int dx\mu(x)\ln\left|f'(x)\right|$$

$$h(\mu,\mathbf{A}) = \lim_{n\to\infty}\frac{1}{n}H(B^{(n)}) = \lim_{n\to\infty}\left(H(B^{(n+1)}) - H(B^{(n)})\right),\quad s_\mu = \lim_{diam(\mathbf{A})\to 0}h(\mu,\mathbf{A}),\quad s_\mu = \lambda$$