TIF 150: Information theory for complex systems

Time: August 29, 2008, 14.00 – 18.00.
Allowed material: Calculator (type approved according to Chalmers rules).
Examiner and teacher: Kristian Lindgren (7723131)

All answers and derivations must be clear and well motivated.
Grade limits: 25p for 3, 34p for 4, and 42p for 5.
The results will be available on September 9.

1. **Balance measurements.**
   Assume you have 5 balls of which 3 are of equal weight but different from the remaining 2 that are also of equal weight. You have at your disposal three balance measurments to find out which are the light and which are the heavy ones. Find such a procedure by using an information-theoretic approach. Answer the following questions.

   (a) What is the initial uncertainty (entropy)? Compare this with how much information you could gain in the worst case from three ideal measurements.
   (b) Show that the two reasonable ways to arrange the first measurement results in the same entropy, i.e., the same expected gain of information from the measurement.
   (c) What is the worst case outcome of your first measurement? Quantify this by calculating the uncertainty that remains.
   (d) For the complete procedure, identifying all balls in three measurements, it would be enough if you show that the worst case always works. (Worst case means that you get the outcome of a balance measurement where there are the most possibilities left, i.e., the remaing entropy is the highest.)

   (9 p)

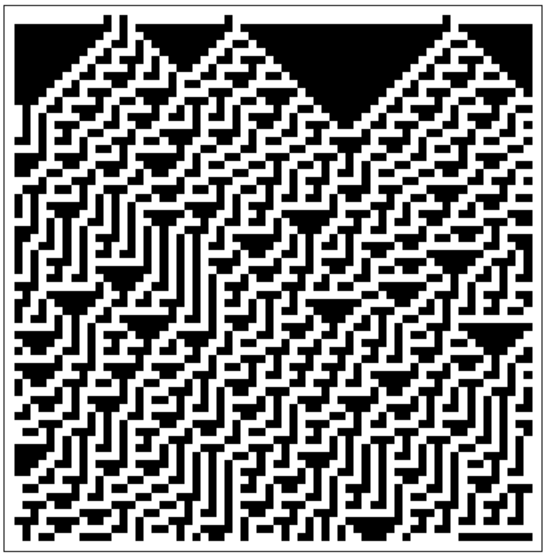
2. **Maximum entropy for a stochastic process.** Consider a class of stochastic processes that produce sequences of symbols A, B, and C, but where pairs of the symbol does not occur, e.g., … A B A C B A C B A B A B C …

   Find the stochastic process with these characteristics that has the highest entropy. What is the entropy of that process? What is the density information and the correlation information from lengths 2, 3 etc?

   (8 p)

3. CA information.

Consider a one-dimensional cellular automaton given by elementary rule 135, i.e., neighbourhoods 111, 000, 001, and 010 map to 1, but all others to 0. Let the initial state, at $t = 0$, be characterized by an uncorrelated sequence of 0's and 1's but with a density for 1's of 1/16 (Bernoulli distribution). The figure below illustrates the complex pattern that is generated by this rule from such an initial state.



(a) What is the initial entropy (at $t = 0$)? Derive the finite state automaton that characterizes the CA state after one time step ($t = 1$). What is the entropy at this time?

(b) Discuss briefly what happens with the entropy at time $t = 2$. (You need not derive the automaton.)

(c) If the initial state ($t = 0$) is derived from non-overlapping triplets of binary digits from π in the binary form of $\pi = 11.0010010000111111011010101000100001...$ so that a 1 is generated for the initial state only when the triplet is 111. Then we would get the same entropy for the initial state as in (a). But, assuming an infinite system, what would the algorithmic information per symbol in the initial state be? What would the algorithmic information per symbol be after one time step (at $t = 1$)?

(12 p)

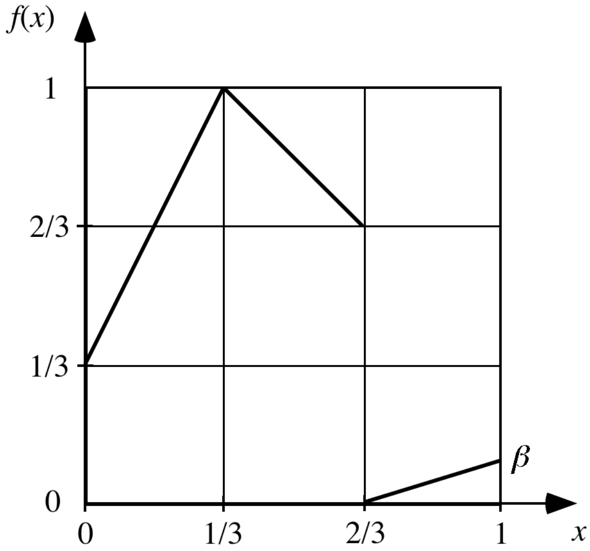
4. Decay of chemical information.

Show that, for a closed chemical system, the decay of chemical information is related to the flow towards smaller length scales j_r at the worst level of resolution ($r \to \infty$).

Assume, as we have done in the course, that we have reaction-diffusion dynamics with periodic boundary conditions at x=0 and x=L, and that concentrations are normalised at every position, i.e., $\Sigma_i\, c_i(x, t) = 1$.

(9 p)

5. Chaos and information. Let a piecewise linear mapping $f(x)$ be defined by the figure below, where $0 < \beta < 1/3$, and where the mapping is determined by $f(0) = 1/3$, $f(1/3) = 1$, $f(2/3) = 2/3$, $f(2/3^+) = 0$, and $f(1) = \beta$.



Consider the dynamical system

$$x_{t+1} = f(x_t) .$$

Starting with $\beta = 0$ and increasing that value, at what value of β does the system become chaotic? What is the dynamic behaviour of the system for β lower than this value?

Suppose that $\beta = 1/3$. Determine the invariant measure that characterizes the chaotic behaviour, and calculate the Lyapunov exponent λ. Find a partition that is generating, and calculate the measure entropy from the symbolic dynamics.

If you know that the system is in the region $x < 1/3$ at time t, how much information do you get if you observe the system in the same region again at time t+3?

(12 p)

Information theory for complex systems – useful equations

Basic quantities

$$I(p) = \log\frac{1}{p} \qquad S[P] = \sum_{i=1}^{n} p_i \log\frac{1}{p_i} \qquad K\left[P^{(0)};P\right] = \sum_{i=1}^{n} p_i \log\frac{p_i}{p_i^{(0)}}$$

Max entropy formalism (with $k + 1$ constraints) using the Lagrangian L

$$L\left(p_1,\ldots,p_n,\lambda_1,\ldots,\lambda_r,\mu\right) = S[P] + \sum_{k=1}^{r}\lambda_k\left(F_k - \sum_{i=1}^{n} p_i f_k(i)\right) + (\mu-1)\left(1-\sum_{i=1}^{n} p_i\right),$$

$$p_j = \exp\left(-\mu - \sum_{k=1}^{r}\lambda_k f_k(j)\right), \quad \mu(\lambda) = \ln\sum_{j=1}^{n}\exp\left(-\sum_{k}\lambda_k f_k(j)\right), \quad \frac{\partial\mu(\lambda)}{\partial\lambda_k} = -F_k$$

Symbol sequences

$$s = \lim_{n\to\infty}\sum_{x_1\ldots x_{n-1}} p(x_1\ldots x_{n-1})\sum_{x_n} p(x_n \mid x_1\ldots x_{n-1})\log\frac{1}{p(x_n \mid x_1\ldots x_{n-1})} =$$

$$= \lim_{n\to\infty}\left(S_n - S_{n-1}\right) = \Delta S_\infty = \lim_{n\to\infty}\frac{1}{n}S_n$$

$$k_1 = \log\nu - S_1, \qquad k_n = -S_n + 2S_{n-1} - S_{n-2} = -\Delta S_n + \Delta S_{n-1} = -\Delta^2 S_n \quad (n = 2,3,\ldots)$$

$$S_{\text{tot}} = \log\nu = \sum_{m=1}^{\infty} k_m + s$$

$$\eta = \sum_{m=1}^{\infty}(m-1)\,k_m = \lim_{m\to\infty}\lim_{n\to\infty}\sum_{\sigma_m} p(\sigma_m)\sum_{\tau_n} p(\tau_n \mid \sigma_m)\log\frac{p(\tau_n \mid \sigma_m)}{p(\tau_n)} = \lim_{m\to\infty}(S_m - m\,s)$$

Geometric information theory

$$p(r;x) = \frac{1}{\sqrt{2\pi}\,r}\int_{-\infty}^{\infty} dw\, e^{-w^2/2r^2}\, p(x-w) \quad , \qquad \left(-r\frac{\partial}{\partial r} + r^2\frac{d^2}{dx^2}\right)p(r;x) = 0$$

$$p_{\text{Gaussian}}(r;x) = \frac{1}{\sqrt{2\pi}\sqrt{b^2+r^2}}\exp\left(-\frac{x^2}{2(b^2+r^2)}\right)$$

$$K[p_0;p] = \int dx\, p(x)\ln\frac{p(x)}{p_0(x)} = \int_0^{\infty}\frac{dr}{r}\int dx\, k(r,x), \qquad k(r,x) = r^2 p(r;x)\left(\frac{d}{dx}\ln p(r;x)\right)^2$$

$$d(r) = D_{\text{E}} - r\frac{\partial}{\partial r}\int d\mathbf{x}\, p(r;\mathbf{x})\ln\frac{1}{p(r;\mathbf{x})}$$

Chemical systems and information flow

$$E = k_{\text{B}}T_0\frac{N}{V}K, \qquad K = \int_V d\mathbf{x}\, K[c_0;c(\mathbf{x})] = \int_V d\mathbf{x}\sum_{i=1}^{M} c_i(\mathbf{x})\ln\frac{c_i(\mathbf{x})}{c_{i0}}, \quad \sum_{i=1}^{M} c_i(\mathbf{x},t) = 1$$

$$K = V\sum_{i=1}^{M} \bar{c}_i \ln\frac{\bar{c}_i}{c_{i0}} + \int_V d\mathbf{x} \sum_{i=1}^{M} c_i(\mathbf{x}) \ln\frac{c_i(\mathbf{x})}{\bar{c}_i} = K_{\text{chem}} + K_{\text{spatial}}$$

$$K_{\text{spatial}} = \int_0^\infty \frac{\partial r}{r} \int d\mathbf{x}\, k(r,\mathbf{x})$$

$$k(r,\mathbf{x}) = r^2 \sum_i \tilde{c}_i(r,\mathbf{x}) \left(\nabla \ln\frac{\tilde{c}_i(r,\mathbf{x})}{c_{i0}}\right)^2 = r^2 \sum_i \frac{(\nabla \tilde{c}_i(r,\mathbf{x}))^2}{\tilde{c}_i(r,\mathbf{x})} = \left(-r\frac{\partial}{\partial r} + r^2\nabla^2\right) \sum_i \tilde{c}_i(r,\mathbf{x}) \ln\frac{\tilde{c}_i(r,\mathbf{x})}{c_{i0}}$$

$$\dot{c}_i(\mathbf{x},t) = \frac{d}{dt} c_i(\mathbf{x},t) = D_i \nabla^2 c_i(\mathbf{x},t) + F_i(\mathbf{c}(\mathbf{x},t)) + b_i(c_{i,\,\text{res}} - c_i(\mathbf{x},t))$$

$$\sigma(\mathbf{x},t) = \sum_i \left(D_i \frac{(\nabla c_i(\mathbf{x},t))^2}{c_i(\mathbf{x},t)} - F_i(\mathbf{c}(\mathbf{x},t)) \ln\frac{c_i(\mathbf{x},t)}{c_{i0}} \right)$$

$$j_r(r,\mathbf{x},t) = \sum_i \left(D_i \frac{(\nabla \tilde{c}_i(r,\mathbf{x},t))^2}{\tilde{c}_i(r,\mathbf{x},t)} - \tilde{F}_i(\mathbf{c}(\mathbf{x},t)) \ln\frac{\tilde{c}_i(r,\mathbf{x},t)}{c_{i0}} \right),\ \tilde{F}_i(\mathbf{c}(\mathbf{x},t)) = \exp(\tfrac{r^2}{2}\nabla^2) F_i(\mathbf{c}(\mathbf{x},t))$$

$$\mathbf{j}(r,\mathbf{x},t) = -r^2 \nabla \sum_i \tilde{F}_i(\mathbf{c}(\mathbf{x},t)) \ln\frac{\tilde{c}_i(r,\mathbf{x},t)}{c_{i0}}, \quad J(r,x,t) = -\sum_i b_i(\tilde{c}_i + c_{i,\,\text{res}})[r\nabla \ln \tilde{c}_i]^2$$

$$\dot{k}(r,\mathbf{x},t) = r\frac{\partial}{\partial r} j_r(r,\mathbf{x},t) - \nabla \cdot \mathbf{j}(r,\mathbf{x},t) + J(r,x,t)$$

<u>Chaos and information</u>

$$\int dx \mu(x) \varphi(x) = \lim_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T-1} \varphi(f^k(x(0))),$$

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t-1} \ln|f'(x(k))| = \int dx \mu(x) \ln|f'(x)|$$

$$h(\mu,\mathbf{A}) = \lim_{n\to\infty} \frac{1}{n} H(B^{(n)}) = \lim_{n\to\infty} \left(H(B^{(n+1)}) - H(B^{(n)}) \right), \quad s_\mu = \lim_{diam(\mathbf{A})\to 0} h(\mu,\mathbf{A}), \quad s_\mu = \lambda$$

<u>Algorithmic information</u>

$$H_U(\alpha_m) = \min_{U(P,X)=\alpha_m} l(P) + l(X)$$

$$L_U(\omega_d) = \sum_{\gamma\in\omega_d} \min\left(H_U(\gamma), l(\gamma)\right), \quad L_U^{(d)}(\alpha_m) = \min_{\omega_d}\left(L_U(\omega_d)\right)$$

$$C_U^{(d)}(\alpha_m) = L_U^{(d-1)}(\alpha_m) - L_U^{(d)}(\alpha_m)$$

$$m = \sum_{k=2}^{m} C_U^{(d)}(\alpha_m) + L_U^{(m)}(\alpha_m)$$