Information theory for complex systems (New course: 7.5 credits)

Time: March 13, 2008, 14.00 – 18.00.
Allowed material: Calculator (type approved according to Chalmers rules).
Examiner: Kristian Lindgren
Teacher: Olof Görnerup (772 3130)

All answers and derivations must be clear and well motivated.
Grade limits: 25p for 3, 34p for 4, and 42p for 5.
The results will be available on March 26.

1. **Balance measurements.**
   Assume you have 6 balls of which 3 are of equal weight and slightly heavier than the remaining 3 that are also of equal weight. You have at your disposal three balance measurments to find out which are the light and which are the heavy ones. Find such a procedure by using an information-theoretic approach. Answer the following questions (a-c).

   (a) What is the initial uncertainty (entropy)? Compare this with how much information you could gain in the worst case from three ideal measurements.
   (b) Show that the two reasonable ways to arrange the first measurement results in the same entropy, i.e., the same expected gain of information from the measurement.
   (c) What is the worst case outcome of your first measurement? Quantify this by calculating the uncertainty that remains.

   For the complete procedure, identifying all balls in three measurements, it would be enough if you show that the worst case always works.

   (8 p)

2. **An equilibrium spin system.** Consider a one-dimensional infinite sequence of states, A, B, and C, where a microstate can be illustrated as

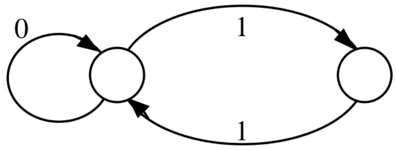
…	B	B	A	A	C	B	A	B	B	B	C	C	B	…

   There is a local contribution to energy, being $-J$ when two neighbouring states are the same, and $+J$ when they are different. If the average energy is u, what is the equilibrium distribution over microstates? Solve the problem by determining the probabilities involved in the equilibrium distribution? (You may give the answer as a function of temperature instead of energy.)

   (9p)

3. CA information.

Consider a one-dimensional cellular automaton given by elementary rule 136, i.e., neighbourhoods 111 and 011 map to 1, but all others to 0. Let the initial state, at $t = 0$, be characterized by the following finite state automaton



where the probabilities for choosing an arc is always the same (1/2) if there is a choice.

(a) Determine the finite state automaton that characterizes the state at time $t = 1$ and $t = 2$. What is the initial entropy and what is the entropy at $t = 1$? Does the entropy change from $t = 1$ to $t = 2$? (You need to provide an argument for your answer.)

(b) If the process that generates the initial state ($t = 0$), using the automaton in the figure, makes the choices in the left node depending on the consecutive binary numbers in the binary form of $\pi = 11.0010010000111111011010101000010001\ldots$ (taking the arc to the right only when there is a 1 in the sequence), the entropies would still be as derived in (a). But, what is the algorithmic information per symbol, for the three time steps $t = 0$, 1, and 2?

(12 p)

4. Decay of information and entropy production.

Use the reaction-diffusion dynamics for a closed chemical system,

$$\frac{d}{dt}c_i(\mathbf{x},t) = D_i \nabla^2 c_i(\mathbf{x},t) + F_i(\mathbf{c}(\mathbf{x},t))$$

to show that the destruction in total information

$$-\frac{dK}{dt} = -\frac{d}{dt}\int_0^L dx\, K[c_0;c(x)] = -\frac{d}{dt}\int_0^L dx \sum_{i=1}^{M} c_i(x) \ln\frac{c_i(x)}{p_{i0}}$$

can be written as a spatial intergral over local entropy production from diffusion σ_{diff} and reactions σ_{chem}, where

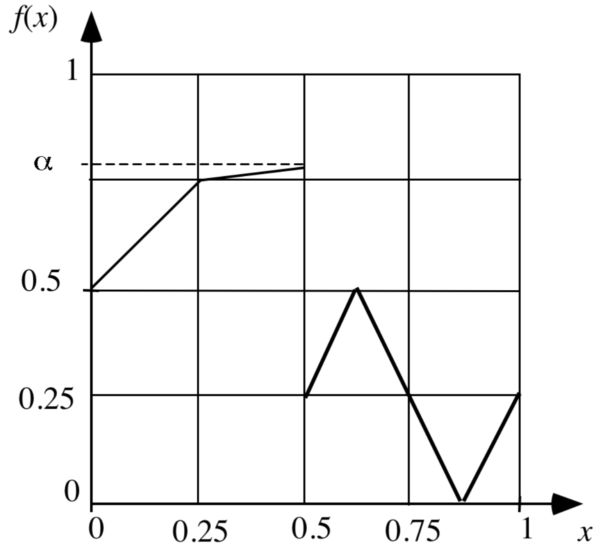
$$\sigma_{\text{diff}}(x,t) = \sum_i D_i \frac{1}{c_i(x,t)}\left(\frac{dc_i(x,t)}{dx}\right)^2, \text{ and}$$

$$\sigma_{\text{chem}}(x,t) = -\sum_i \left(\ln\frac{c_i(x,t)}{c_{i0}}\right) F_i(\mathbf{c}(x,t)) \ .$$

Assume periodic boundary conditions at x=0 and x=L, and that concentrations are normalised at every position, i.e., $\Sigma_i\, c_i(x, t) = 1$.

(9 p)

5. Chaos and information. Let a mapping $f(x)$ be defined by the figure below, with $f(0)=1/2, f(1/4)=3/4, f(1/2)=\alpha, f(5/8)=1/2, f(3/4)=f(1)=1/4, f(7/8)=0$, where $3/4 < \alpha < 1$.



Consider the dynamical system

$$x_{t+1} = f(x_t) \; .$$

Starting with $\alpha=3/4$ and increasing that value, at what value of α does the system become chaotic? What is the dynamic behaviour of the system for α close to 3/4?

Suppose that $\alpha = 7/8$. Determine the invariant measure that characterizes the chaotic behaviour, and calculate the Lyapunov exponent λ. Find a partition that is generating, and calculate the measure entropy from the symbolic dynamics.

If you know that the system is in the region $x < 1/8$ at time t, how much information do you get if you observe the system in the same region again at time $t+4$?

(12 p)

Information theory for complex systems – useful equations

Basic quantities

$$I(p)=\log\frac{1}{p} \qquad S[P]=\sum_{i=1}^{n}p_i\log\frac{1}{p_i} \qquad K\left[P^{(0)};P\right]=\sum_{i=1}^{n}p_i\log\frac{p_i}{p_i^{(0)}}$$

Max entropy formalism (with $k+1$ constraints) using the Lagrangian L

$$L\left(p_1,\ldots,p_n,\lambda_1,\ldots,\lambda_r,\mu\right)=S[P]+\sum_{k=1}^{r}\lambda_k\left(F_k-\sum_{i=1}^{n}p_i f_k(i)\right)+\left(\mu-1\right)\left(1-\sum_{i=1}^{n}p_i\right),$$

$$p_j=\exp\left(-\mu-\sum_{k=1}^{r}\lambda_k f_k(j)\right),\ \ \mu(\lambda)=\ln\sum_{j=1}^{n}\exp(-\sum_{k}\lambda_k f_k(j)),\ \ \frac{\partial\mu(\lambda)}{\partial\lambda_k}=-F_k$$

Symbol sequences

$$s=\lim_{n\to\infty}\sum_{x_1\ldots x_{n-1}}p(x_1\ldots x_{n-1})\sum_{x_n}p(x_n\mid x_1\ldots x_{n-1})\log\frac{1}{p(x_n\mid x_1\ldots x_{n-1})}=$$

$$=\lim_{n\to\infty}\left(S_n-S_{n-1}\right)=\Delta S_\infty=\lim_{n\to\infty}\frac{1}{n}S_n$$

$$k_1=\log\nu-S_1,\qquad k_n=-S_n+2S_{n-1}-S_{n-2}=-\Delta S_n+\Delta S_{n-1}=-\Delta^2 S_n\quad(n=2,3,\ldots)$$

$$S_{\text{tot}}=\log\nu=\sum_{m=1}^{\infty}k_m+s$$

$$\eta=\sum_{m=1}^{\infty}(m-1)k_m=\lim_{m\to\infty}\lim_{n\to\infty}\sum_{\sigma_m}p(\sigma_m)\sum_{\tau_n}p(\tau_n\mid\sigma_m)\log\frac{p(\tau_n\mid\sigma_m)}{p(\tau_n)}=\lim_{m\to\infty}(S_m-m\,s)$$

Geometric information theory

$$p(r;x)=\frac{1}{\sqrt{2\pi}\,r}\int_{-\infty}^{\infty}dw\,e^{-w^2/2r^2}\,p(x-w)\quad,\qquad\left(-r\frac{\partial}{\partial r}+r^2\frac{d^2}{dx^2}\right)p(r;x)=0$$

$$p_{\text{Gaussian}}(r;x)=\frac{1}{\sqrt{2\pi}\sqrt{b^2+r^2}}\exp\left(-\frac{x^2}{2(b^2+r^2)}\right)$$

$$K[p_0;p]=\int dx\,p(x)\ln\frac{p(x)}{p_0(x)}=\int_0^{\infty}\frac{dr}{r}\int dx\,k(r,x),\qquad k(r,x)=r^2p(r;x)\left(\frac{d}{dx}\ln p(r;x)\right)^2$$

$$d(r)=D_{\text{E}}-r\frac{\partial}{\partial r}\int d\mathbf{x}\,p(r;\mathbf{x})\ln\frac{1}{p(r;\mathbf{x})}$$

Chemical systems and information flow

$$E=k_{\text{B}}T_0\frac{N}{V}K,\qquad K=\int_V d\mathbf{x}\,K[c_0;c(\mathbf{x})]=\int_V d\mathbf{x}\sum_{i=1}^{M}c_i(\mathbf{x})\ln\frac{c_i(\mathbf{x})}{c_{i0}},\quad\sum_{i=1}^{M}c_i(\mathbf{x},t)=1$$

$$K = V\sum_{i=1}^{M}\bar{c}_i \ln\frac{\bar{c}_i}{c_{i0}} + \int_V d\mathbf{x}\sum_{i=1}^{M} c_i(\mathbf{x})\ln\frac{c_i(\mathbf{x})}{\bar{c}_i} = K_{\text{chem}} + K_{\text{spatial}}$$

$$K_{\text{spatial}} = \int_0^{\infty}\frac{\partial r}{r}\int d\mathbf{x}\, k(r,\mathbf{x})$$

$$k(r,\mathbf{x}) = r^2\sum_i \tilde{c}_i(r,\mathbf{x})\left(\nabla\ln\frac{\tilde{c}_i(r,\mathbf{x})}{c_{i0}}\right)^2 = r^2\sum_i\frac{(\nabla\tilde{c}_i(r,\mathbf{x}))^2}{\tilde{c}_i(r,\mathbf{x})} = \left(-r\frac{\partial}{\partial r} + r^2\nabla^2\right)\sum_i \tilde{c}_i(r,\mathbf{x})\ln\frac{\tilde{c}_i(r,\mathbf{x})}{c_{i0}}$$

$$\dot{c}_i(\mathbf{x},t) = \frac{d}{dt}c_i(\mathbf{x},t) = D_i\nabla^2 c_i(\mathbf{x},t) + F_i(\mathbf{c}(\mathbf{x},t)) + b_i(c_{i,\,\text{res}} - c_i(\mathbf{x},t))$$

$$\sigma(\mathbf{x},t) = \sum_i\left(D_i\frac{(\nabla c_i(\mathbf{x},t))^2}{c_i(\mathbf{x},t)} - F_i(\mathbf{c}(\mathbf{x},t))\ln\frac{c_i(\mathbf{x},t)}{c_{i0}}\right)$$

$$j_r(r,\mathbf{x},t) = \sum_i\left(D_i\frac{(\nabla\tilde{c}_i(r,\mathbf{x},t))^2}{\tilde{c}_i(r,\mathbf{x},t)} - \tilde{F}_i(\mathbf{c}(\mathbf{x},t))\ln\frac{\tilde{c}_i(r,\mathbf{x},t)}{c_{i0}}\right),\ \tilde{F}_i(\mathbf{c}(\mathbf{x},t)) = \exp(\tfrac{r^2}{2}\nabla^2)F_i(\mathbf{c}(\mathbf{x},t))$$

$$\mathbf{j}(r,\mathbf{x},t) = -r^2\nabla\sum_i\tilde{F}_i(\mathbf{c}(\mathbf{x},t))\ln\frac{\tilde{c}_i(r,\mathbf{x},t)}{c_{i0}},\quad J(r,x,t) = -\sum_i b_i(\tilde{c}_i + c_{i,\,\text{res}})[r\nabla\ln\tilde{c}_i]^2$$

$$\dot{k}(r,\mathbf{x},t) = r\frac{\partial}{\partial r}j_r(r,\mathbf{x},t) - \nabla\cdot\mathbf{j}(r,\mathbf{x},t) + J(r,x,t)$$

Chaos and information

$$\int dx\mu(x)\varphi(x) = \lim_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T-1}\varphi(f^k(x(0))),$$

$$\lambda = \lim_{t\to\infty}\frac{1}{t}\sum_{k=0}^{t-1}\ln|f'(x(k))| = \int dx\mu(x)\ln|f'(x)|$$

$$h(\mu,\mathbf{A}) = \lim_{n\to\infty}\frac{1}{n}H(B^{(n)}) = \lim_{n\to\infty}\left(H(B^{(n+1)}) - H(B^{(n)})\right),\quad s_\mu = \lim_{diam(\mathbf{A})\to 0}h(\mu,\mathbf{A}),\quad s_\mu = \lambda$$

Algorithmic information

$$H_U(\alpha_m) = \min_{U(P,X)=\alpha_m} l(P) + l(X)$$

$$L_U(\omega_d) = \sum_{\gamma\in\omega_d}\min\left(H_U(\gamma), l(\gamma)\right),\quad L_U^{(d)}(\alpha_m) = \min_{\omega_d}\left(L_U(\omega_d)\right)$$

$$C_U^{(d)}(\alpha_m) = L_U^{(d-1)}(\alpha_m) - L_U^{(d)}(\alpha_m)$$

$$m = \sum_{k=2}^{m}C_U^{(d)}(\alpha_m) + L_U^{(m)}(\alpha_m)$$