Information theory for complex systems – FFR050

Time: March 11, 2004.
Allowed material: anything except other person.
Examiner/teacher: Kristian Lindgren, tel. 772 3131.

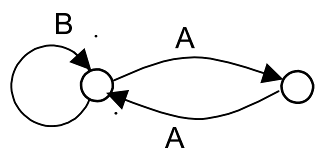
1. **Balance information.** Suppose that you have four balls that all look the same, two of equal heavier weight and two of equal lighter weight. Assuming this knowledge, what is the uncertainty of the system. How many measurements using a balance would theoretically be needed in order to sort out which are the heavier and which are the lighter ones? Is there a procedure that accomplishes this?

(5 p)

2. **ABC-CA.** Assume that a three-state cellular automaton, with states A, B, and C, develops according to a rule that depends on the cell and its right neighbour, so that

t	AA	AB	AC	BA	BB	BC	CA	CB	CC
$t+1$	C	A	C	A	C	C	B	A	B

Let the initial state be characterised by the following finite state automaton



where the probabilities for choosing an arc is always the same (1/2) if there is a choice. What is the initial entropy ($t = 0$), and what is the entropy at $t = 1$ and $t = 2$?

(12 p)

3. Spin system. Suppose that in a one-dimensional discrete spin system, described by a row of spins, up or down, the interaction is with second nearest neighbours only (i.e., position x interacts with positions $x + 2$ and x – 2). Parallel spins at this distance contribute with an energy –J and anti-parallel spins with an energy +J. What is the equilibrium state of this system? Use the maximum entropy formalism, and express the probabilities that characterise the equilibrium as functions of temperature (or of β).

(10 p)

4. Space-time information. The space-time pattern for a deterministic cellular automaton rule has entropy $s = 0$. This can be understood from the definition of entropy based on the 2-dimensional block entropy

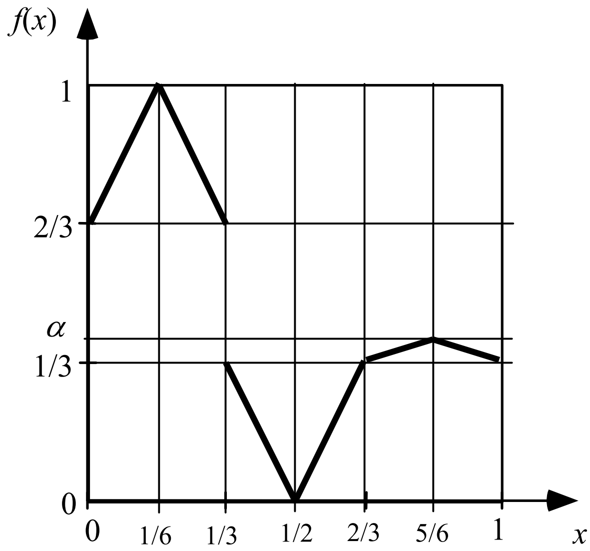
$$s = \lim_{m\to\infty} \frac{1}{m^2} S_{m\times m}$$

Then the edges of such a space-time block contains all information that a deterministic CA rule of range r=1 needs to generate the complete $m \times m$ space-time block. This implies that the block entropy grows only linearly with m and the entropy s is zero.

When noise is added to the rule the resulting entropy of the space-time pattern is larger than zero. Assume that the rule is an almost reversible rule, for example R60, and that noise is added so that the result of the rule is the opposite with a probability 1%. In the limit $t\to\infty$ the system reaches a stationary state with a maximal one-dimensional entropy (in the spatial direction, for each time step), regardless of the initial state. But, how large is the entropy for the space-time pattern when the stationary state is reached?

(10 p)

5. Chaos and information. Let a mapping $f(x)$ be defined by the figure below (so that $f(0) = f(1/3) = 2/3, f(1/6) = 1, f(1/3) = f(2/3) = f(1) = 1/3,\ f(1/2) = 0,\ f(5/6) = \alpha$).



Consider the dynamical system

$$x_{t+1} = f(x_t) .$$

a) At which value of α does the system become chaotic?

b) What is the behaviour for small α (below the critical value in (a))? Describe qualitatively only.

Assume from now on that $\alpha = 1/2$.

c) Find the invariant measure μ that characterises the chaotic behaviour, and determine the corresponding Lyapunov exponent λ by using that measure. Find a partition that has a symbolic dynamics with a measure entropy s_μ that equals the Lyapunov exponent (as one should expect).

c) Suppose that we at a certain time t observe the system in the region given by $x < 1/6$. If we find the system in this region again three time steps later (at $t + 3$), how much information do we gain by this observation?

(13 p)

Information theory for complex systems: Answers to exam March 11, 2004

1. Balance information:
   $S = \ln 6$
   2 measurements

2. ABC-CA
   $s(t) = 2/3 \ln 2$ for $t = 0, 1, 2$

3. Spin system

   $p(ijk) = [4 (1 + \exp(2\beta J)]^{-1}$ for i and k anti-parallel
   $p(ijk) = [4 (1 + \exp(-2\beta J)]^{-1}$ for i and k parallel
   regardless of j

   (cf. spin system example in the Lecture Notes)

4. Space-time entropy

   $s = -0.01 \ln 0.01 - 0.99 \ln 0.99$

   (the entropy contribution only comes from the noise)

5. Chaos and information

   a) $\alpha = 1/3 + 1/24 = 9/24$
   b) stable periodic, period 3
   c) Six equally sized intervals: A_1, A_2, B_1, B_2, C_1, C_2, but B_2 is never reached.
      $\mu(A_1) = 1/6$, $\mu(A_2) = 1/6$, $\mu(B_1) = 1/3$, $\mu(C_1) = 1/6$, $\mu(C_2) = 1/6$, gives
      $\lambda = 2/3 \ln 2$
      $A_1 \to C_1 \cup C_2$, $A_2 \to C_1 \cup C_2$, $C_1 \to B_1$, $C_2 \to B_1$, $B_1 \to A_1 \cup A_2$, gives
      $\mu = 2/3 \ln 2$
   d) $\ln 2$