Information theory for complex systems

Time: Wednesday March 13, 2002, 13.00-18.30, in FL61.
Allowed material: anything except other person.
Examiner/teacher: Kristian Lindgren (only available on 0707574031), Anders Eriksson (at his office).

1. **Dots lost.** In Swedish we have these strange characters Å, Ä, and Ö. Sometimes, when writing email, for example, the keyboard is not equipped with these characters, and we just drop the dots and circles by instead using A, A, and O, respectively. Still one can read and understand the message, at least most of the time.

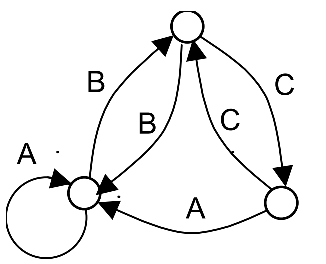
   Imagine that we would do this – dropping the dots and circles – with the whole language at once. What would the effect be on the entropy *s*, i.e., the entropy that also takes into account correlations? What is the interpretation of that (summarised in one to three sentences)? (You may think of this in English if you prefer, for example, by considering replacing all occurrences of "u" with "o". Good lock!)

   (7 p)

2. **ABC-CA.** Assume that a three-state cellular automaton, with states A, B, and C, develops according to a rule that depends on the cell and its right neighbour, so that

t	AA	AB	AC	BA	BB	BC	CA	CB	CC
$t+1$	B	C	A	C	A	B	B	A	C

   Let the initial state be characterised by the following finite state automaton



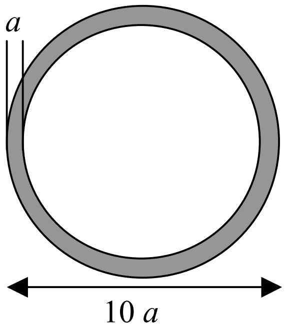
   where the probabilities for choosing an arc is always the same (1/2). What is the initial entropy ($t = 0$), and what is the entropy at $t = 1$ and $t = 2$?

   (12 p)

3. **Dots are back.** Consider an infinite two-dimensional lattice system (of square cells) in which dots are distributed. The cells can be either empty or inhabited (by one or two dots). Free living dots have an "energy level" of 0, while pairs (two dots in one cell) have the energy –J (where J is a positive constant). Assume that there is a certain density of dots ρ, and that there is a given average energy u (per cell). Use the maximum entropy formalism to characterise an "equilibrium" state of this system as a function of an "inverse temperature" (β). What is the entropy? What happens in the limit of a "zero temperature"?

   (9 p)

4. **Dimension of a ring.** Consider a pattern in a two-dimensional space that is described as a ring with a certain thickness a and a certain diameter $10a$, schematically depicted in the figure below.



The ring may be described by a uniform probability distribution $p(x, y)$ that is constant within the grey area and zero elsewhere. By introducing the resolution dependent probability density $p(r; x, y)$, one may study how the entropy $S(r)$ changes when the resolution is made worse (r increases).

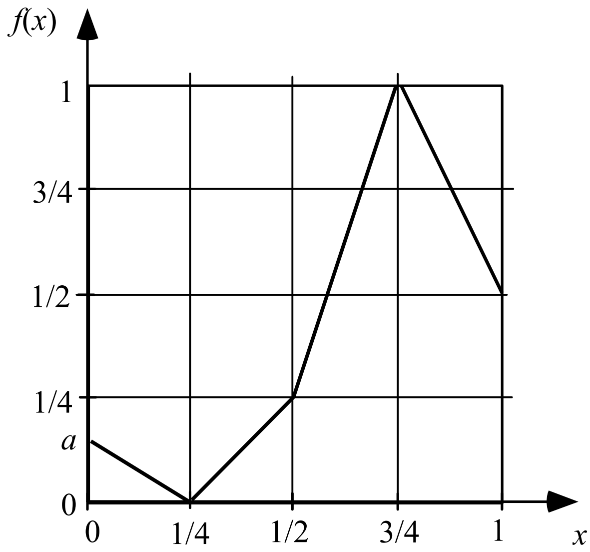
<u>Discuss</u> how the quantity

$$r\frac{\partial}{\partial r}\int dxdy\; p(r;x,y)\ln\frac{1}{p(r;x,y)}$$

depends on r. It should <u>not</u> be necessary with calculations (which are quite complicated), but you should be able to discuss this in a more schematic way, relating resolution level r to characteristic lengths in the system, for example a.

(9 p)

5. **Chaos and information.** Let a mapping $f(x)$ be defined by the figure below, where $f(1/4) = 0, f(1/2) = 1/4,\ f(3/4) = 1,\ f(1) = 1/2$. Let $a = f(0)$ be a parameter so that $0 < a < 1$.



Consider the dynamical system

$$x_{t+1} = f(x_t) .$$

a) What is the behaviour for $a = 1/8$? At which value of a becomes the system chaotic?

Assume from now on that <u>a = 3/4</u>.

b) Find the invariant measure μ that characterises the chaotic behaviour, and determine the corresponding Lyapunov exponent λ by using that measure. Find a partition that has a symbolic dynamics with a measure entropy s_μ that equals the Lyapunov exponent (as one should expect).

c) Suppose that we at a certain time t observe the system in the region given by $x > 3/4$. If we find the system in this region again three time steps later (at $t + 3$), how much information do we gain by this observation?

(13 p)