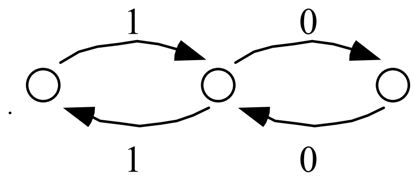
Information theory for complex systems

Time: Monday 5th March 2001, 8.45-12.45, in ML18.
Allowed material: anything except other person.
Examiner/teacher: Kristian Lindgren/Anders Eriksson ext 3451.

1. Suppose that elementary rule 129 governs the time evolution of a cellular automaton, and that the initial state (at $t = 0$) is characterised by the following finite automaton

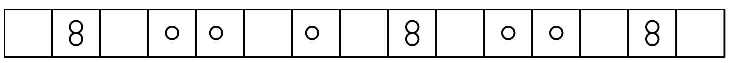


where the probabilities for choosing 0 and 1 from the middle node are equal (1/2). What is the entropy s after one time step ($t = 1$) and after two steps ($t = 2$)?

Suppose instead that the probabilities for choosing 0 and 1 in the middle node differ, say $p(0) < ½ < p(1)$. Describe in a qualitative way how this affects entropy and correlations at the first time step ($t = 1$).

(13 p)

2. Suppose that in a one-dimensional discrete system, described by a row of "land pieces" or cells, there live these "dot" agents. By inspecting what they are doing you quickly find the following characteristics. There is a density ρ of "dots", some of them living alone at a "piece of land", and some of them that have joined in a "marriage" so that two share the same "piece of land". You also find that whenever there is a married couple in one cell, the adjacent cells (to the left and to the right) are always free — the married couple really want to be on their own or others avoid them. You also find by studying several systems of this kind that the fraction of "dots" that are married is a certain constant α.



Now, before analysing more in detail the statistics of the "dot world", you want to design a probabilistic description of the system that is consistent with the observations above and that has a maximum in its (Shannon) entropy.

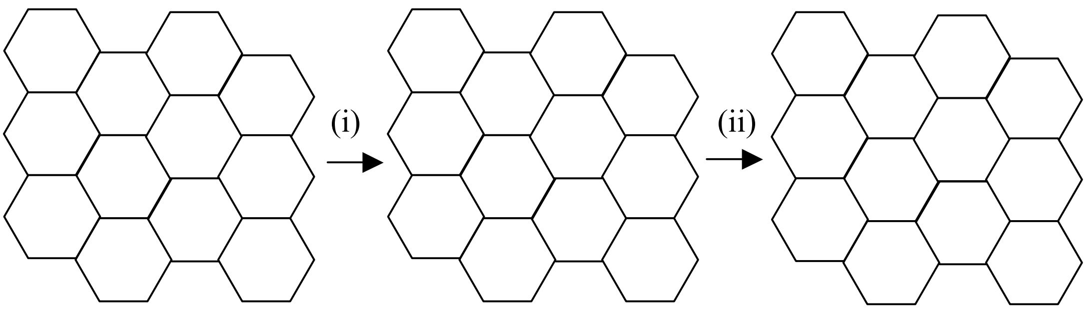
Use the maximum entropy formalism to find a probabilistic description[1] that obeys the constraints described above, including the parameters ρ and α. Assume that the system is of infinite length.

(12 p)

[1] This means that you should find probabilities for blocks of "states", as we do when we solve for the equilibrium state in a spin system.

3. A “lattice gas” is a two-dimensional cellular automaton that consists of a lattice of hexagonal cells. Each cell may contain up to six different particles each of them having unit speed with a certain direction. Particles may only take one of the six directions pointing to a neighbouring cell, and there may only be one particle in each direction. Particles keep their directions unless they collide.

   In each time step of the lattice gas the following happens (all in parallel). (i) Particles move to adjacent cells according to the direction they are pointing to. (ii) If several particles enter the same cell, there may be a “collision”. This actually only happens if exactly two particles enter a cell and if they meet “head on”. Such a “head on” collision results in the particles leaving the cell (in the next time step), but with each of them turning 60° to the right. In all other cases, when one or several particles enter a cell, they just pass through each other continuing in the same direction as before. This means that we have a fully deterministic microscopic dynamics. These two parts that constitute a time step is illustrated in the following figure. Note that in the bottom right there is a “head on” collision.

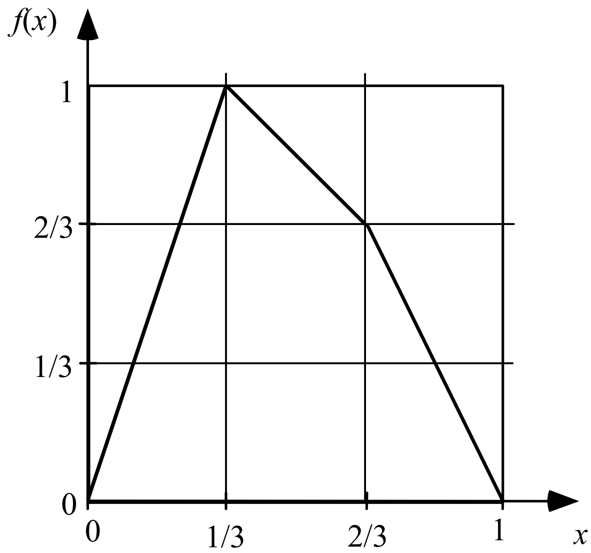
   

   First, discuss qualitatively what is happening with the (Shannon) entropy in a system like this. Consider, for example, an infinite system, with an initial state where we have clusters of particles (with speeds in random directions).

   Secondly, discuss qualitatively what happens with the entropy if we modify the “collision rule” so that a head on collision results in either (a) both particles turning 60° to the right, or (b) both particles turning 60° to the left; both outcomes with equal probability (50%). All other collisions are kept unchanged.

   (12 p)

4. Let a mapping $f(x)$ be defined by the figure below, where $f(1/3) = 1$, $f(2/3) = 2/3$, and $f(0) = f(1) = 0$.



Consider the dynamical system

$$x_{t+1} = f(x_t) .$$

Find the invariant measure μ that characterises the chaotic behaviour, and determine the corresponding Lyapunov exponent λ by using that measure. Show that there is a partition that has a symbolic dynamics with a measure entropy s_μ that equals the Lyapunov exponent (as one should expect).

Suppose that we at a certain time t observe the system in the region given by $x > 2/3$. If we find the system in this region again two time steps later, how much information do we gain by this observation?

(13 p)