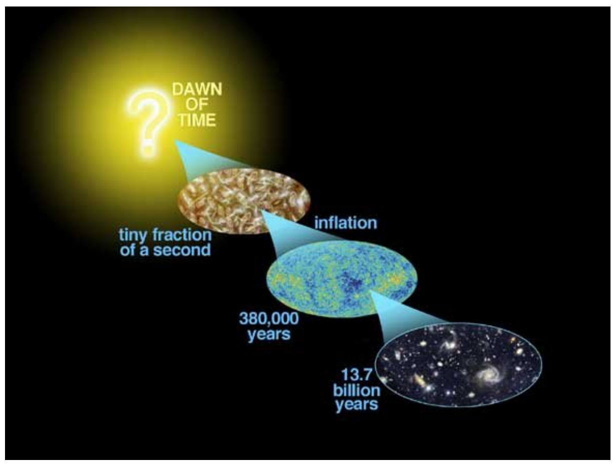
HOME EXAM

–GRAVITATION AND COSMOLOGY (FFM071)–
SPRING SEMESTER 2013

LECTURER: DANIEL PERSSON
ASSISTANT TEACHER: CHRISTIAN VON SCHULTZ



Instructions: Please use the conventions of Weinberg for all problems[1], and write out explicitly the formulas you are using. Unless otherwise specified, write all derivations explicitly.

There are 10 problems and the maximum score is 80p.

Deadline is Monday 25/3 at 10 am.

[1]For covariant derivatives you may use Weinberg's notation, e.g. $V_{\mu;\nu}$, or the more standard one $D_\nu V_\mu$.

Problem 1. Indicate whether the following statements are true or false. Only answer **true** or **false**; no calculations or motivations needed! *Every wrong answer (or no answer) gives a negative contribution to the total score.*

• The energy density of relativistic massive particles scales like the volume of the spatial universe.

• The particle horizon gives an estimate of the size of our universe.

• The Schwarzschild solution describes a maximally symmetric 4-dimensional spacetime.

• If the universe was contracting, a distant supernova Type IA would appear to be blue-shifted.

• Observations indicate that our universe is currently matter dominated.

• If we would conclude from observations that the total density parameter Ω is equal to 1 then our universe is described by the 4-dimensional Minkowski metric.

• The big bang singularity in the FRW metric is naked and spacelike.

• Most of the energy in the universe is made up of dark matter.

• If you cross the horizon of a charged black hole you are doomed to hit the singularity.

• The FRW metric with $k = 0$ describes a flat 4-dimensional spacetime.

• The general theory of relativity is an accurate description of Nature at all energy scales.

• For each Killing vector there is an associated isometry but the converse is not true.

• The vacuum energy violates the condition the the pressure is positive semi-definite.

• A gravitational plane wave has as many independent components as a light wave.

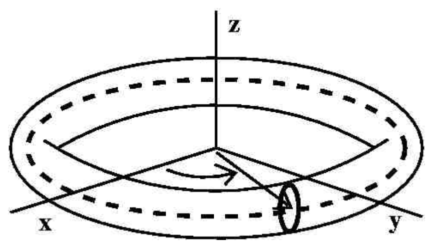
• If I interchange the second and third indices on the Riemann tensor I get back the same tensor with an overall minus sign.

(5p)

Problem 2. Consider a torus T^2 with angular coordinates ϕ_1, ϕ_2. You can think of the torus as a cylinder whose endpoints have been glued together. Let ϕ_1 be the coordinate along the circle which surrounds the hole, and ϕ_2 the coordinate along the circle of the cylinder. We embed the torus into 3-dimensional Euclidean space $\mathbb{R}^3$, and in Cartesian coordinates the embedding is described explicitly by:

$$\begin{aligned} x &= (r_1 + r_2 \cos\phi_2)\, \cos\phi_1, \\ y &= (r_1 + r_2 \cos\phi_2) \sin\phi_1, \\ z &= r_2 \sin\phi_2, \end{aligned} \tag{1}$$

where r_1 is the radius of the circle surrounding the hole of the torus, while r_2 is the radius of the cylinder. We restrict to $r_1 > r_2$.



(a) Find the metric of the torus T^2 using the embedding into $\mathbb{R}^3$ defined by eq. (1).

(b) Compute the Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$ for the metric on T^2 obtained in **(a)**. Comment on the result for different points on the torus.
Note: In this problem you do not have to provide details of your calculations of the components of the affine connection or curvature tensors.

(c) Find all Killing vectors for the metric obtained in **(a)**.

(7p)

Problem 3. A beacon radiating at a fixed frequency ν_0 is released at time $t = 0$ towards a black hole of mass M by an observer situated very far away from the black hole. The observer stays at constant distance from the black hole while the probe is falling. Show that the frequency of the beacon (when it is close to the event horizon) as measured by the observer can be written as $\nu \sim \mathrm{e}^{-t/K}$ for some constant K and relate the constant K to the mass of the black hole. (10p)

Problem 4. Stephen Hawking is orbiting the Earth in Virgin Galactic's first commercial space voyage. To be on the safe side, Richard Branson has chosen the orbit to be perfectly circular and ensured that the clock on-board the spaceship is ticking at the same rate as a clock at rest on the surface of the Earth. Calculate the altitude of Stephen Hawking (you may neglect the rotation of the Earth).

(7p)

Problem 5. Derive the following formulas:

$$D_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda \tag{2}$$

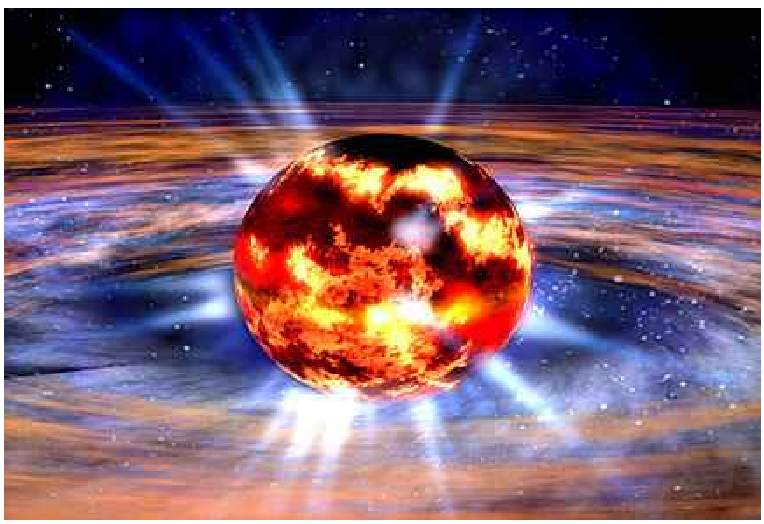
$$\Gamma^\mu_{\mu\lambda} = \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} \tag{3}$$

$$D_\mu V^\mu = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} V^\mu\right) \tag{4}$$

$$D_\mu D^\mu \phi = \frac{1}{\sqrt{g}} \partial_\mu \big(\sqrt{g}\, \partial^\mu \phi\big). \tag{5}$$

(7p)

Problem 6. Something goes wrong with the navigation system in Virgin Galactic's space voyage and the spaceship carries Stephen Hawking out of the solar system toward the center of the Milky Way. Stephen quickly realizes that he must avoid the supermassive black hole and boldly takes the spaceship to manual drive and performs an emergency landing on a big neutron star inside the globular custer *Terzan 5*, close to the heart of the galaxy. Being the fearless scientist that he is, Stephen gets out of the spaceship to take a tour on the surface of the neutron star with his motorized wheel chair (don't worry, he has a suit that protects him from being crushed by the gravitational force). *How far can Stephen drive from his spaceship and still see it?* Assume that the spaceship is 20 meters in height and that the height of Stephen in a wheel chair is 1 meter. The mass of the neutron star is 3 solar masses and its radius is 5 km.



(12p)

Problem 7. Consider the Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2MG}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2MG}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{6}$$

After changing to Kruskal coordinates

$$u = \sqrt{\left(\tfrac{r}{2GM} - 1\right)}\, e^{r/4GM}\cosh\left(\tfrac{t}{4GM}\right), \qquad v = \sqrt{\left(\tfrac{r}{2GM} - 1\right)}\, e^{r/4GM}\sinh\left(\tfrac{t}{4GM}\right) \tag{7}$$

the Schwarzschild metric can be rewritten as follows

$$\mathrm{d}s^2 = \frac{32G^3M^3}{r}e^{-r/2GM}(-\mathrm{d}v^2 + \mathrm{d}u^2) + r^2\mathrm{d}\Omega^2. \tag{8}$$

Discuss the global structure of the spacetime described by the Kruskal metric (8). Illustrate your discussion with a spacetime diagram. Include in your answer a discussion of how lightcones behave as they approach the horizon. Compare with the original Schwarzschild metric. What are the main benefits of using Kruskal coordinates?

(7p)

Problem 8.

(a) Using the continuity equation, derive how the energy density ρ evolves with the scale factor a, for matter with the general equation of state $p = w\rho$.

(b) Restrict the result of **(a)** to the 3 special cases when the energy density is completely due to dust, radiation or vacuum energy. Compare the behavior in the 3 cases, and explain why they differ.

(b) Derive the form of the time evolution of the scale factor $a(t)$ for matter with the equation of state $p = w\rho$ and restricted to $\rho > 0, p \geq 0$ using the FRW equations and the continuity equation. You may assume the general form $a(t) \sim t^\beta$.

(7p)

Problem 9. A rotating black hole is described by the Kerr metric, which can be written as follows in so called Boyer-Lindquist coordinates:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \Sigma\left(\frac{\mathrm{d}r^2}{\Delta} + \mathrm{d}\theta^2\right) + (r^2+a^2)\sin^2\theta\,\mathrm{d}\phi^2 + \frac{2Mr}{\Sigma}\left(a\sin^2\theta\,\mathrm{d}\phi - \mathrm{d}t\right)^2, \tag{9}$$

where M is the mass and a is a parameter such that $J = Ma$ is the angular momentum, and we have defined the quantities

$$\Delta \equiv r^2 - 2Mr + a^2, \qquad \Sigma \equiv r^2 + a^2\cos^2\theta. \tag{10}$$

In the questions below, please provide motivations and the calculational details that underlie your answer.

(a) Is the Kerr metric static?

(b) Does it have spherically symmetric spacelike slices?

(c) Is the metric asymptotically flat in the same sense as the Schwarzschild metric?

(d) What is the geometry of a massless Kerr black hole?

(e) Show that there is a limit in which Kerr $\to$ Schwarzschild.

(f) Comment on the singularity structure of the Kerr metric. You may find it useful to study the limit in **(d)**. Does the Kerr metric have a horizon?

(g) Find all the Killing vectors of the Kerr metric. Intelligent guesses give partial credit. For full credit, use a method that guarantees you *all* Killing vectors in a systematic manner.

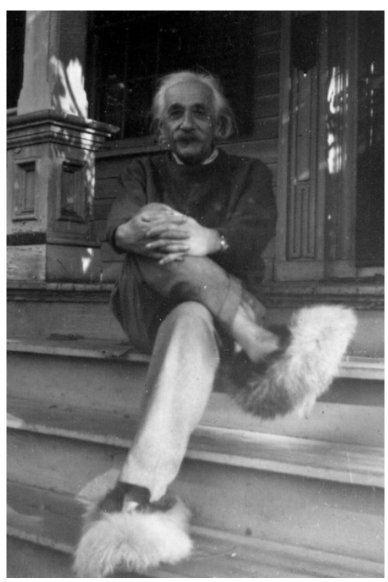
(10p)

Problem 10. The most popular and well-studied solution to the flatness and horizon problems in the standard model of cosmology is *inflation*. The theory of inflation posits that the early universe underwent a period of accelerated expansion, driven by some unknown matter component, typically thought to be a scalar field called the "inflaton field". Another more speculative solution, originating in string theory, is the so called *ekpyrotic universe*. Read pages 1-5 in the paper "Ekpyrotic Non-Gaussianity – A Review" by J. L. Lehners, available at this link:

`http://arxiv.org/pdf/1001.3125.pdf`

and then try to explain the basic ideas behind how the ekpyrotic model solves the flatness and horizon problems. Compare and contrast this with the solutions proposed by inflation. *Note: the answer does not need to contain any mentioning of branes or string theory!*

(8p)



Good Luck!