Home examination, Gravitation & Cosmology, 2008

To be handed in January 23, 2009

The maximum score for home assignments (10+10 points) and home examination (80 points) is 100 points. 50 points is minimum requirement (see the course web page for Chalmers and GU grading).

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1. The electromagnetic field strength tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Check that the derivatives can be replaced with covariant derivatives. Find the action for the electromagnetic field, and show that its variation implies Maxwell's equations. Find the energy-momentum tensor. Show that it contains the known expressions for the energy density and Poynting vector in terms of the **E** and **B** fields.
(10 points)

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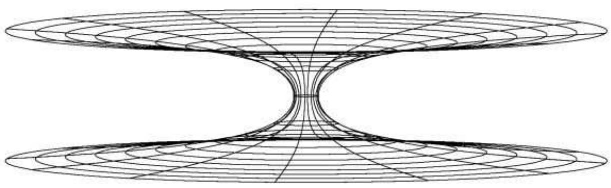
2. Consider the space-time geometry

$$ds^2 = -dt^2 + dr^2 + (r^2 + a^2)d\Omega^2 \ ,$$

where $d\Omega^2$ is the maximally symmetric metric on S^2. Similar geometries have been proposed to describe "wormholes", tunnels between different regions of space-time. The "radial coordinate" r can take negative as well as positive values. Calculate the affine connection and investigate whether or not there are time-like geodesics traversing the wormhole (*i.e.*, going from large positive r to large negative r or vice versa). Calculate the Riemann tensor. Find all isometries of the metric. Find the energy-momentum tensor needed in order to make the geometry a solution to Einstein's equations. Are there any problems (in principle, not technical) with engineering such a solution?
(12 points)



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3. Three-dimensional anti-de Sitter space (AdS_3) is described by the metric

$$ds^2 = -du^2 - dv^2 + dx^2 + dy^2 \ ,$$

where the coordinates are confined to the hyperboloid

$$-u^2 - v^2 + x^2 + y^2 = -b^2 \ .$$

Change coordinates according to

$$\begin{aligned} u &= \sqrt{b^2 + r^2} \cos \frac{t}{b} \ , \\ v &= \sqrt{b^2 + r^2} \sin \frac{t}{b} \ , \\ x &= r \cos \phi \ , \\ y &= r \sin \phi \end{aligned}$$

(check that this defines a valid set of coordinates on the hyperboloid). Using *e.g.* the metric in the new coordinates, calculate the proper distance from the origin $r = 0$ to spatial infinity $r \to \infty$. Also find the coordinate time needed for a photon to travel this distance.
(8 points)

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4. A beacon radiating at a fixed frequency ν_0 is released at time $t = 0$ towards a black hole of mass M by an observer situated very far away from the black hole. The observer stays at a constant distance from the black hole while the probe is falling. Show that the frequency of the beacon (when it is close to the event horizon) as measured by the observer can be written as $\nu \sim e^{-\frac{t}{K}}$ for some constant K and relate the constant K to the mass of the black hole.
(10 points)

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5. Einstein originally introduced the cosmological constant with the purpose of modifying his equations to allow for solutions describing a static universe. Consider the equations governing the time evolution of a maximally symmetric space:

$$\begin{aligned} \frac{\ddot{a}}{a} &= \frac{\Lambda}{3} - \frac{4\pi G}{3}(\varrho + 3p) \ , \\ \left(\frac{\dot{a}}{a}\right)^2 &= \frac{\Lambda}{3} + \frac{8\pi G}{3}\varrho - \frac{k}{a^2} \ . \end{aligned}$$

Assume that ϱ and p describe "ordinary" matter (with $\varrho > 0$ and $w \geq 0$). When are there static solutions? For a dust-dominated universe, what is the relation between a and ϱ? Are the solutions stable?
(12 points)

6. Consider the two-dimensional metric

$$ds^2 = e^{\phi(x,y)}(dx^2 + dy^2) \ .$$

Calculate the affine connections and the Riemann tensor for a general function $\phi(x, y)$. Show that the choice

$$e^{\phi(x,y)} = \frac{1}{(1 + \frac{x^2+y^2}{4})^2}$$

describes the unit sphere. Calculate, in this case, the quantity

$$\chi = -\frac{1}{2\pi} \int_{\mathbb{R}^2} d^2x \sqrt{g} R$$

(the minus sign is due to Weinberg's sign convention for the curvature). χ is called the Euler characteristic of the surface. If the function ϕ is changed by the addition of a function that falls off very rapidly as $x^2 + y^2 \to \infty$, how does χ change?
(10 points)

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7. The Weyl tensor is defined as

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{2}{D-2}(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}) + \frac{2}{(D-1)(D-2)} g_{\mu[\rho}g_{\sigma]\nu}R \ .$$

Check that $C_{\mu\rho\nu}{}^{\rho} = 0$. How many algebraically independent components does such a tensor have? If two metrics g and $\tilde{g}$ are conformally equivalent, *i.e.*, if $\tilde{g}_{\mu\nu}(x) = e^{2\phi(x)}g_{\mu\nu}(x)$, how are the corresponding Weyl tensors related?
(10 points)

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8. When one observes distant luminous objects, their emitted light becomes redshifted due to the expansion of the universe. We assume that both the emitting object and the observer are at rest in the standard coordinates. The redshift is commonly defined by the parameter

$$z = \frac{\lambda - \lambda_0}{\lambda_0} \ ,$$

where λ_0 is the emitted wavelength and λ the observed one. Show that it can be expressed in terms of the scale factor of the universe at the times of emission and observation as

$$z = \frac{a(t)}{a(t_0)} - 1 \ .$$

Is it correct to interpret the redshift as a Doppler shift corresponding to the relative velocity of emitter and observer due to the expansion of the universe? What is the value of z for the cosmic background radiation? Also, find information of the value for the most distant observed galaxies.
(8 points)