# 1ST HOMEWORK ASSIGNMENT

## –GRAVITATION AND COSMOLOGY (FFM071)– SPRING SEMESTER 2013

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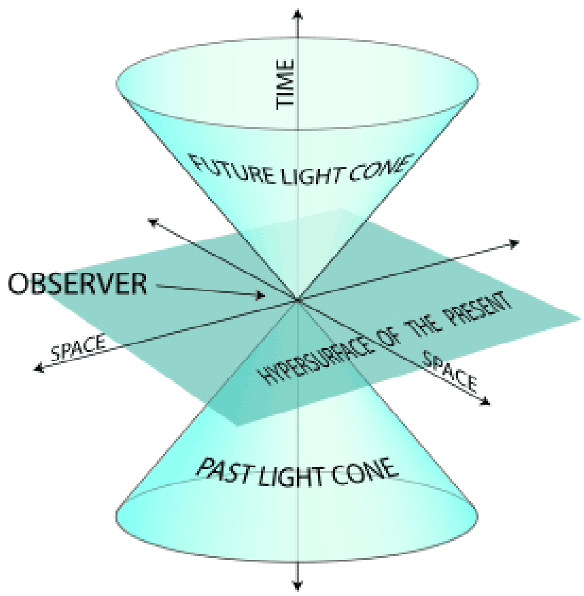



**Instructions:** Please use the conventions of Weinberg for all problems[1], and write out explicitly the formulas you are using. Unless otherwise specified, write all derivations explicitly.

**There are 5 problems, each of which gives a maximum score of 2p.**

**Deadline is Monday 11/2 at 08 am.**

[1]For covariant derivatives you may use Weinberg's notation, e.g. $V_{\mu;\nu}$, or the more standard one $D_\nu V_\mu$.

**Problem 1.** Indicate whether the following statements are true or false. (only answer **true** or **false**; no calculations or motivations needed!) For every wrong answer you get −0.5p (but the minimum score is still 0p).

• The Minkowski metric $\eta_{\alpha\beta}$ is invariant under Lorentz transformations.

• An arbitrary spacetime 4-vector $x^\alpha$ is invariant under Lorentz transformations.

• The first postulate of special relativity states that the outcome of a physical experiment will depend on the coordinate system you use.

• Your worldline lies inside of the lightcone.

• An observer at rest measures the lifetime of a muon in motion to be longer than that of a muon at rest.

• Maxwell's equations are Lorentz invariant but Newton's laws are not.

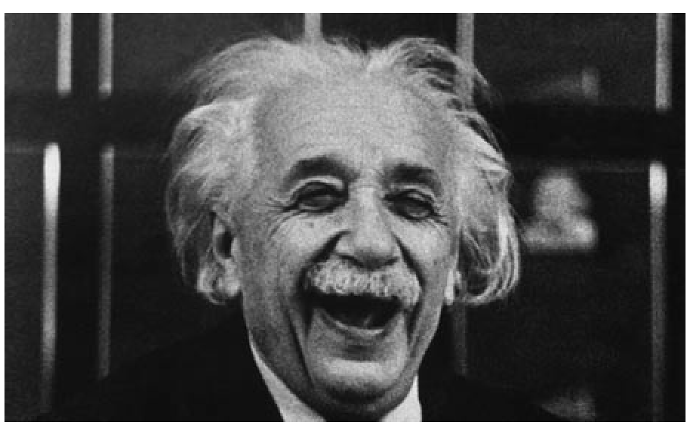
• The affine connection $\Gamma^\lambda_{\mu\nu}$ is a mixed $(1,2)$-tensor with respect to general coordinate transformations.

• The object $\partial_\alpha V^\beta$ does not transform as a tensor under Lorentz transformations.

• The components of the Kronecker delta symbol $\delta^\mu_\nu$ are the same in all coordinate systems.

• When we take the Newtonian limit we may assume that $\mathrm{d}x/\mathrm{d}\tau \ll \mathrm{d}t/\mathrm{d}\tau$.

(2p)



**Problem 2.** The following is a collection of small problems which concerns simple manipulations with tensors and indices. You don't need to provide derivations here, simply give the answer.

Do any of the following equations fail to make sense for generic[2] coordinate systems? If yes, indicate which ones.

$$\eta_{\alpha\beta}\Phi^{\beta} = \Phi_{\alpha} \tag{1}$$

$$\Lambda^{\alpha_1}{}_{\beta_1}\Lambda^{\alpha_2}{}_{\beta_2}\eta_{\alpha_1\alpha_2}\Lambda^{\beta_1}{}_{\gamma_1}\Lambda^{\beta_2}{}_{\gamma_2}\mathrm{d}x^{\gamma_1}\mathrm{d}x^{\gamma_2} = \eta_{\delta_1\delta_2}\mathrm{d}x^{\delta_1}\mathrm{d}x^{\delta_2} \tag{2}$$

$$V^{\alpha\alpha} = \mathrm{Tr}V \tag{3}$$

$$V_iV^i = V_\alpha V^\alpha \tag{4}$$

$$\mathrm{d}\tau^2 = 2.64 \tag{5}$$

$$T^{\mu\nu} = 196883 \tag{6}$$

$$\partial_\alpha F_{\alpha\beta} = -J_\beta \tag{7}$$

$$D_\nu V_\mu = \partial_\nu V_\mu - \Gamma^{\sigma}_{\mu\nu}V_\rho \tag{8}$$

$$g^{\mu\nu}g_{\nu\rho} = \delta^{\mu}_{\rho} \tag{9}$$

$$T^{\mu\nu} = pg^{\mu\nu} + p + \rho U^\mu U^\nu \tag{10}$$

(2p)

**Problem 3.** Show that Maxwell's equations in four-dimensional Minkowski space can be written as

$$\frac{\partial}{\partial x^\alpha}F^{\alpha\beta} = -J^\beta, \tag{11}$$

$$\epsilon^{\alpha\beta\gamma\delta}\frac{\partial}{\partial x^\beta}F_{\gamma\delta} = 0, \tag{12}$$

[2] "generic" implies that the equations should be sensible for an arbitrary coordinate frame of the type indicated by the index structure. For instance, the components of a 4-vector are generically non-vanishing.

where $F^{\alpha\beta}$ is the electromagnetic field strength tensor and $\epsilon^{\alpha\beta\gamma\delta}$ is the totally antisymmetric Levi-Civita tensor. Use conventions where the Minkowski metric is given by

$$\eta_{\alpha\beta} \doteq \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{13}$$

(2p)

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**Problem 4.**

**(a)** Show that in an arbitrary coordinate frame $x^\mu$ the object $\epsilon^{\mu\nu\rho\sigma}$ defined by

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \mu\nu\rho\sigma \text{ even permutation of } 0,1,2,3 \\ -1 & \mu\nu\rho\sigma \text{ odd permutation of } 0,1,2,3 \\ 0 & \text{two or more indices equal} \end{cases} \tag{14}$$

is a *tensor density* of weight $-1$.

**(b)** Show that

$$\epsilon^{0ijk}\epsilon_{0ijl} = -2\delta^k_l \tag{15}$$

in Minkowski space. (2p)

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**Problem 5.**

**(a)** Calculate the components of the affine connection $\Gamma^\lambda_{\mu\nu}$ for the 2-dimensional hyperbolic space (Poincaré disc model) with metric given by

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2 + r^2\mathrm{d}\phi^2}{(1 - r^2/a^2)^2} \tag{16}$$

**(b)** Calculate the Laplacian $D^\mu D_\mu$ on the 2-sphere $S^2$ and compare it to the standard expression for the 2-dimensional Laplacian in spherical coordinates, as found e.g. in the Mathematics Handbook (Beta). Do the results agree? If, not, why?

(2p)

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