Home assignment 1, Gravitation & Cosmology, 2009

To be handed in Friday, November 20

1. Construct the Laplacian (in the case of euclidean signature) / the wave operator (in the case of Lorentzian signature), acting on a scalar, from covariant derivatives, and give an explicit form in terms of derivatives and the affine connection.

   As an alternative construction (we stick to Lorentzian signature here), one may consider the action for a massless scalar field,

   $$S = -\tfrac{1}{2}\int d^4x\sqrt{-g}\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

   (g denotes the determinant of $g_{\mu\nu}$), whose variation will give the wave equation. Show that this gives the same expression as the construction with covariant derivatives.

   Calculate the Laplacian on S^2 and compare it to the known explicit expression for the square of the angular momentum in three dimensions (used *e.g.* when separating variables in the solution of the wave function of the hydrogen atom).

   (If you have no experience of variational principles, it suffices to show that the operator in question acting on a scalar ϕ can be written as $\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi)$.)

2. Show that the tensor $T^{\mu\nu}$, defined from the action for a scalar field in the previous exercise as $T^{\mu\nu}(x) = \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}(x)}$, satisfies $D_\nu T^{\mu\nu} = 0$. Give a physical interpretation of $T^{\mu\nu}$ and of the differential equation.

   (Again, if you have not seen a functional derivative: You may think of $\frac{\delta S}{\delta g_{\mu\nu}(x)}$ as $\frac{\partial\mathscr{L}}{\partial g_{\mu\nu}}$, where $S = \int d^4x\mathscr{L}$.)

3. As an example of a space with constant negative curvature, take the simplest example, the two-dimensional "Poincaré disk". Its metric can be written as

   $$ds^2 = \frac{dr^2 + r^2 d\phi^2}{(1-\frac{r^2}{a^2})^2}\ ,$$

   where $0 \leq r < a$ and $0 \leq \phi < 2\pi$ (ϕ is an angle). What is the distance from a point with coordinates (r,ϕ) to the "boundary" $r = a$? Calculate the curvature tensor, the Ricci tensor and the curvature scalar. Compare to the corresponding results for a two-dimensional sphere.

4. Show that if $A_{\mu_1 \ldots \mu_p}$ is a totally antisymmetric tensor in D dimensions, the "dual tensor"

$$\star A_{\mu_1 \ldots \mu_{D-p}} = \frac{1}{p! \sqrt{|g|}} g_{\mu_1 \nu_1} \ldots g_{\mu_{D-p} \nu_{D-p}} \varepsilon^{\nu_1 \ldots \nu_D} A_{\nu_{D-p+1} \ldots \nu_D}$$

is indeed a tensor.

Show that the two Maxwell equations with zero on the right hand side (the "Bianchi identities") may be written as $\partial^\beta \star F_{\alpha\beta} = 0$ in Minkowski space.