Tentamen
MMA110/TMV100 Integration Theory

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Hjälpmedel: Inga hjälpmedel.

This exam together with the hand-in exercises provides the grounds for grading. A total score of 18 is needed for the grade 3 or G, 28 for 4 and 38 for 5 or VG. For 3 or G one also needs at least 6 points at this exam. Solutions will be published on the course url, the day efter the exam.

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1. (4p) Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable and integrable and let

$$g(x) = \int_{-\infty}^{\infty} e^{-|y|} f(x-y)dy,$$

$x \in \mathbb{R}$. Prove that g is a continuous integrable function.

Solution. By a change of variables, we have that

$$g(x) = \int_{-\infty}^{\infty} e^{-|x-t|} f(t)dt.$$

Let $x_n \to x$. Since $|e^{-|x-t|}f(t)| \le |f(t)|$ and f is integrable, the DCT gives that $g(x_n) \to g(x)$. By Tonelli's Theorem,

$$\int |g(x)|dx = \int \left(e^{-|y|} \int |f(x-y)|dx \right) dy < \infty$$

so g is integrable.

2. (4p) Let $\xi_1, \xi_2, \ldots$ be random variables such that $\mathbb{P}(\xi_n = -3^n) = 2^{-n}$ and $\mathbb{P}(\xi_n = 1) = 1 - 2^{-n}$. Let $S_n = \sum_1^n \xi_j$. Prove that $\mathbb{E}[S_n] \to -\infty$ whereas $S_n \to +\infty$ almost surely.

Solution. We have $\mathbb{E}[\xi_n] = -2^{-n}3^n + 1 - 2^{-n} < 1 - (3/2)^n$ so $\mathbb{E}[S_n] \to -\infty$. However, by Borel-Cantelli, $\xi_n = 1$ almost surely for all but finitely many n. Hence, almost surely, $S_n \to +\infty$.

3. (4p) Compute the limit

$$\lim_n \int_{-\sqrt{n}}^{\sqrt{n}} \left(1 - \frac{x^2}{n}\right)^n (1 + \sqrt{n}x)\, dx.$$

(Hint: Recall that $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.)

Solution. By symmetry, $\int_{-\sqrt{n}}^{\sqrt{n}} \sqrt{n}x(1 + x^2/n)^n dx = 0$, so the limit equals

$$\lim_n \int_{-\infty}^{\infty} \left(1 - \frac{x^2}{n}\right)^n \chi_{[-\sqrt{n},\sqrt{n}]}(x)dx,$$

which by dominated convergence and the hint equals $\sqrt{\pi}$.

4. (4p) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and let f_n, $n = 1, 2, \ldots$ and f be integrable Borel functions on X. One says that $f_n \to f$ *in measure* if for each $\epsilon > 0$, $\lim_n \mu\{x \in X : |f_n(x) - f(x)| > \epsilon\} = 0$ as $n \to \infty$.

(a) Show that $f_n \to f$ in measure whenever $f_n \to f$ a.e. and μ is finite. Show by counterexample that this implication does not extend to σ-finite measures.

(b) Show by counterexample that $f_n \to f$ in measure does not imply that $f_n \to f$ a.e. Show, on the other hand, that if $f_n \to f$ in measure, then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.e.

Solution. For (a), suppose $f_n \to f$ a.e. but $f_n \to f$ in measure fails. Then for some $\epsilon > 0$, $\mu\{|f_n - f| > \epsilon\} \not\to 0$. However since $f_n \to f$ a.e., $\mu(\limsup_n \{x : |f_n(x) - f(x)| > \epsilon\}) = 0$, by continuity of measures (since μ is finite), a contradiction. For the counterexample: let $(X, \mathcal{M}, \mu) = (\mathbb{R}, \mathcal{B}, m)$ and let $f_n = \chi_{[n,\infty)}$.

For the counterexample in (b), let $X = [0, 1]$ and $\mu = m$. Let $f_1 = \chi_{(0,1/2)}$, $f_2 = \chi_{(1/2,1)}$, $f_3 = \chi_{(0,1/4)}$, …,$f_6 = \chi_{(3/4,1)}$, $f_7 = \chi_{(0,1/8)}$, …. For the last assertion, let for $k = 1, 2, \ldots$, n_k be such that $n \geq n_k \Rightarrow \mu\{|f_n - f| > 1/k\} < 2^{-k}$. Let $E_k = \{x : |f_{n_k}(x) - f(x)| > 1/k\}$ and $F_j = \bigcup_j^\infty E_k$. Then $\mu(F_1) \leq 1$ so by Borel-Cantelli, $\mu(\limsup_k E_k) = 0$. Hence for a.e. x, $x \in E_k$ for only finitely many k. However, for such x, $f_{n_k}(x) \to f$.

5. (4p) Let m be the Lebesgue measure on $\mathbb{R}$. Construct a measurable set E such that

$$\liminf_{\delta \to 0} \frac{m(E \cap (-\delta, \delta))}{2\delta} = \frac{1}{3}$$

and

$$\limsup_{\delta \to 0} \frac{m(E \cap (-\delta, \delta))}{2\delta} = \frac{2}{3}.$$

Solution. Let $A_n = [2^{-n}, 2^{-(n-1)}]$ and let

$$E = \bigcup_{j=1}^{\infty} A_{2j}.$$

6. (4p) Let ξ be an integrable random variable on $(X, \mathcal{M}, \mathbb{P})$.

(a) Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{M}$ and let η be a $\mathcal{G}$-measurable integrable random variable. Prove that $\mathbb{E}[\xi\eta|\mathcal{G}] = \eta\mathbb{E}[\xi|\mathcal{G}]$ almost surely.

(b) Let η and ϕ be random variables such that $\sigma(\xi, \eta)$ and $\sigma(\phi)$ are independent. Prove that $\mathbb{E}[\xi|\eta, \phi] = \mathbb{E}[\xi|\eta]$ almost surely.

Solution. For (a), let $G \in \mathcal{G}$ be arbitrary. Then if $\eta = \chi_A$ for some $A \in \mathcal{G}$,

$$\int_G \mathbb{E}[\xi\eta|\mathcal{G}]d\mathbb{P} = \int_G \xi\eta d\mathbb{P} = \int_{G\cap A} \xi d\mathbb{P}$$
$$= \int_{G\cap A} \mathbb{E}[\xi|G]d\mathbb{P} = \int_G \eta\mathbb{E}[\xi|\mathcal{G}]d\mathbb{P}$$

where the second and third equalities are by the definition of conditional expectation. By the Monotone Class Theorem and the MCT, the result now holds for all $\mathcal{G}$-measurable η.

In (b), we want to show that $\int_{\{(\eta,\phi)\in C\}} \mathbb{E}[\xi|\eta\phi]d\mathbb{P} = \int_{\{(\eta,\phi)\in C\}} \mathbb{E}[\xi|\eta]d\mathbb{P}$ for all $C \in \mathcal{B}(\mathbb{R}^2)$. By Dynkin's Lemma, it suffices to do this when $C = A \times B$, $A, B \in \mathcal{B}(\mathbb{R})$. However then the left hand side is

$$\mathbb{E}[\xi(\chi_A \circ \eta)(\chi_B \circ \phi)] = \mathbb{E}[\xi(\chi_A \circ \eta)]\mathbb{E}[\chi_B \circ \phi]$$
$$= \mathbb{E}[\mathbb{E}[\xi(\chi_A \circ \eta|\eta]]\mathbb{E}[\chi_B \circ \phi] = \mathbb{E}[(\chi_A \circ \eta\mathbb{E}[\xi|\eta]]\mathbb{E}[\chi_B \circ \phi]$$
$$= \mathbb{E}[\mathbb{E}[\xi|\eta](\chi_A \circ \eta)(\chi_B \circ \phi)]$$

which equals the right hand side. Here the first and fourth equalities follow from independence and the other equalities by the definition of conditional expectation.

Lycka till!
Johan Jonasson