Tentamen
MMA110/TMV100 Integration Theory

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Examinator: Johan Jonasson, Matematiska vetenskaper, Chalmers

Telefonvakt: Adam Andersson, telefon: 0703 088 304

Hjälpmedel: Inga hjälpmedel.

This exam together with the hand-in exercises provides the grounds for grading. A total score of 18 is needed for the grade 3 or G, 28 for 4 and 38 for 5 or VG. For 3 or G one also needs at least 6 points at this exam. Solutions will be published on the course url, the day after the exam.

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1. (2p) Give an example of integrable random variables ξ and $\xi_1, \xi_2, \dots$ such that $\xi_n \to \xi$ in L^2 but not a.s. (That $\xi_n \to \xi$ in L^2 means that $\mathbb{E}[|\xi_n - \xi|^2] \to 0$.)

Solution. Use the same example as the example of convergence in probability but not almost surely on the exam of 121025.

2. (4p) Let $(X, \mathcal{M}, \mu)$ be a semifinite measure space; recall that μ semifinite by definition means that whenever $\mu(E) = \infty$, then there exists $F \subset E$ such that $0 < \mu(F) < \infty$. Prove that more generally, for any $0 < c < \infty$, there exists a set $F \subseteq E$ such that $c < \mu(F) < \infty$.

Solution. Suppose to the contrary that $s := \sup\{c : \exists F \subset E : c < \mu(F) < \infty\} < \infty$. Then for each $n = 1, 2, \dots$, there is an $F_n \subset E$ with $s - 1/n < \mu(F_n) \leq s$. Taking $F = \bigcup_n F_n$, we get $\mu(F) = s$. However since $\mu(E \setminus F) = \infty$, there is a $G \subset E \setminus F$ such that $0 < \mu(G) < \infty$, so that $F \cup G \subset E$ and $s < \mu(F \cup G) < \infty$, a contradiction to the definitionc of s.

3. (4p) Let X be an uncountable space and let $\mathcal{M}$ be the collection of all countable subsets of X and their complements. Define the set function μ on $\mathcal{M}$ by letting $\mu(E) = 0$ for all countable E and $\mu(E) = 1$ for all E whose complement is countable. Prove that $\mathcal{M}$ is a σ-algebra and that μ is a measure. Prove also that for any $\mathcal{M}$-measurable function f, there is a constant a such that $f(x) = a$ for all but at most countably many $x \in X$.

Solution. That $X \in \mathcal{M}$ and $\mathcal{M}$ is closed under complements is obvious. If $E_n \in \mathcal{M}$, $n = 1, 2, \dots$, then either E_n is countable for all n, so that $\bigcup_n E_n$ is countable, or E_n^c is countable for some n, in which case $(\bigcup_j E_j)^c \subseteq E_n^c$ is also countable. This proves that $\mathcal{M}$ is a σ-algebra.

If f is $\mathcal{M}$-measurable, then $\{x : f(x) < a\}$ is countable or cocountable for all a. Let $c = \sup\{a : \{x : f(x) < a\} \text{ countable}\}$. Then for any $n = 1, 2, \dots$, $\{x : f(x) < c + 1/n\}$ is cocountable and $\{x : f(x) < c - 1/n\}$ is countable. Hence $f(x) = c$ for all but countably many x.

4. (4p) Let $f : (X, \mathcal{M}, \mu)$ be a finite measure space and $f_1, f_2, \dots$ a sequence of bounded measurable functions and assume that $f_n \to f$ uniformly. Prove that then $\int f_n \, d\mu \to \int f \, d\mu$. Show also that this result fails if one drops the assumption that μ be finite.

Solution. Since $f_n \to f$ uniformly, there is an n_0 such that $n > n_0 \Rightarrow \sup_x |f_n(x) - f(x)| < 1$. Since μ is finite, it follows from the DCT that $\int |f_n - f| \to 0$. Since the f_n's are bounded, $\int f_n$ and $\int f$ are well-defined and we get $\int f_n \to \int f$.

For a counterexample when μ is not finite, consider $(X, \mathcal{M}, \mu) = (\mathbb{R}, \mathcal{B}, m)$ and $f_n = n^{-1}\chi_{[n,\infty)}$.

5. (a) (4p) Let E be a measurable set of real numbers. One says that E har period p ($p > 0$) if $E + p = E$. Suppose that $p_n \to 0$ and that E has period p_n for all n. Prove that then $m(E) = 0$ or $m(E) = \infty$. (where m as usual denotes Lebesgue measure). **Hint:** Pick $a \in R$, let $F(x) = m(E \cap [a, x])$, $x > a$ and show that

$$F(x + p_n) - F(x - p_n) = F(y + p_n) - F(y - p_n)$$

for $y > x > a + p_n$. What does this say about $F'(x)$ if $\mu(E) > 0$?

Solution. By definition $F(x + p_n) - F(x - p_n) = m(E \cap [x - p_n, x + p_n])$ which is independent of x since E has period p_n. This proves the hinted equality, which in turn proves that $F'(x)$ is constant. Hence either $m(E)$ is 0 or ∞.

(b) (2p) Let f be a Lebsgue measurable real function. Suppose that $t_n \to 0$ and that f has period t_n for all n. Prove that there is a constant c such that $f(x) = c$ for a.e. x. **Hint:** Apply (a) to the set $\{x : f(x) > \lambda\}$.

Solution. The set $\{x : f(x) > \lambda\}$ has period t_n by assumption and is therefore of measure 0 or ∞. Hence, with $c = \inf\{a : m\{x : f(x) > a\} = 0\}$, we have $f(x) = c$ a.e.

6. (4p) Let $(X, \mathcal{M}, \mathbb{P})$ be a probability space and let $\xi_1, \xi_2, \ldots$ be a sequence of independent integrable random variables, all having equal expectation: $\mathbb{E}[\xi_n] = v$. Let T be a stopping time, i.e. a positive integer-valued random variable such that $\{x : T(x) > n\} \in \sigma(\xi_1, \ldots, \xi_n)$ for all n. Prove Wald's Theorem, which states that if $\mathbb{E}[T] < \infty$, then

$$\mathbb{E}\Big[\sum_{n=1}^{T} \xi_n\Big] = v\mathbb{E}[T].$$

Hint: Write $\sum_1^T \xi_n = \sum_1^\infty \xi_n \chi_{\{T \geq n\}}$.

Solution. Since $\{T \geq n\} \in \sigma(\xi_1, \ldots, \xi_{n-1})$, we have that $\chi_{\{T \geq n\}}$ and ξ_n are independent, so

$$\mathbb{E}[\xi_n \chi_{\{T \geq n\}}] = v\mathbb{E}[\chi_{\{T \geq n\}}].$$

Using the hint we get

$$\mathbb{E}\Big[\sum_{n=1}^{T} \xi_n\Big] = v\mathbb{E}\Big[\sum_{1}^{\infty} \chi_{\{T \geq n\}}\Big] = v\mathbb{E}[T].$$

Lycka till!
Johan Jonasson