Tentamen
MMA110/TMV100 Integration Theory

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This exam together with the hand-in exercises provides the grounds for grading. A total score of 18 is needed for the grade 3 or G, 28 for 4 and 38 for 5 or VG. For 3 or G one also needs at least 6 points at this exam.

Solutions will be published on the course url, the day after the exam.

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1. (4p) Let $f : \mathbb{R} \to \mathbb{R}$ be increasing. Prove that f is $(\mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$-measurable.

Solution. Pick $b \in \mathbb{R}$ and let $y = \sup\{x : f(x) \le b\}$. Then either $f(y) \le b$ and then $f^{-1}(-\infty, b] = (-\infty, y]$ since f is increasing, or $f(y) > b$ in which case $f^{-1}(-\infty, b] = (-\infty, b)$ since f is increasing. Since $\mathcal{B}(\mathbb{R})$ is generated by $(-\infty, b]$, $b \in \mathbb{R}$, the result follows.

2. (4p) Let m be the Lebesgue measure on $[0, 1]$. Prove or give counterexample to the following two statements.

(a) If $m(A) > 0$, then A contains an open interval.

(b) If A is such that $m(A^c) = 0$, then A is dense.

Solution.

(a) False: let e.g. $A = \mathbb{R} \setminus \mathbb{Q}$.

(b) True: If A is not dense, then A^c contains an open interval and has therefore $m(A^c) > 0$.

3. (4p) Let $f : (X, \mathcal{M}, \mu) \to \mathbb{R}$ be nonnegative and integrable and let $c = \int f d\mu > 0$. Show that

$$\lim_n \int n \log\Big(1 + (f/n)^a\Big) d\mu = \begin{cases} \infty, & 0 < a < 1 \\ c, & a = 1 \\ 0, & 1 < a < \infty \end{cases}$$

Solution. Write g_n for the integrated functions. For the cases where $a \ge 1$, we use DCT with af as dominating function; this is obviously an integrable function (which integrates to ac), but it is not equally obvious that it in fact dominates the g_n's. However for x such that $f(x) \ge n$,

$$n \log(1 + (f(x)/n)^a) \le n \log(1 + f(x)/n))^a = an \log(1 + f(x)/n) \le af(x)$$

where the first inequality follows since $a \ge 1$ and the last one since $\log(1 + b) \le b$ for all nonnegative b. For x with $f(x) < n$,

$$n \log(1 + (f(x)/n)^a) \le n \log(1 + f(x)/n) \le f(x) \le af(x)$$

where the first and last inequalities follow since $a \ge 1$. For $a = 1$, we have

$$\lim_n g_n = f$$

so the DCT gives $\lim_n \int g_n = \int f = c$. For $a > 1$, $\lim_n g_n = 0$, so the DCT gives $\lim_n \int g_n = 0$. For $a < 1$, $g_n \to \infty$, so by Fatou's Lemma, $\liminf_n \int g_n = \infty$.

4. (4p) A stronger version of the Dominated Convergence Theorem for finite measures. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and f and $f_1, f_2, \ldots$ integrable functions. One says that $\{f_n\}_{n=1}^{\infty}$ is *uniformly integrable* if $\int_{\{|f_n|>K\}} |f_n| d\mu \to 0$ as $K \to \infty$ uniformly in n. Say that $\{f_n\}$ is dominated if there is an integrable function g such that $|f_n| \le g$ for all n. Show that any dominated sequence of functions is uniformly integrable and find an example that shows that the converse does not hold. Prove finally that if $\{f_n\}$ is uniformly integrable and $f_n \to f$ a.e., then $\lim_n \int f_n d\mu = \int f d\mu$.

Solution. If $\{f_n\}$ is dominated, then $\int_{\{|f_n|>K\}} f_n \le \int_{\{g \ge K\}} g \to 0$ as $K \to \infty$. If $f_n = n\chi_{[0,1/n^2]}$, defined on $([0,1], \mathcal{B}[0,1], m)$, then $\{f_n\}$ is UI but not dominated.

If $\{f_n\}$ is UI the note first that f must be integrable, since $\sup_n \int |f_n| \le 1 + K\mu(X) < \infty$ for K chosen so that $\int_{\{|f_n|>K\}} |f_n| \le 1$ for all n. Hence $\int |f| \le 1 + K\mu(X)$ by Fatou's Lemma.

Now pick K so large that $\int_{\{|f_n|>K\}} f_n < \epsilon$ for all n and $\int_{\{|f|>K\}} f < \epsilon$. By the DCT with the constant K as dominating function (which works since μ is finite), we have

$$\limsup_n \left| \int f_n - \int f \right| \le \epsilon + \limsup_n \left| \int_{\{|f_n| \le K\}} f_n - \int_{\{|f| \le K\}} f \right| = \epsilon.$$

5. (4p) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and let f_n, $n = 1, 2, \ldots$ and f be integrable Borel functions on X. One says that $f_n \to f$ *in measure* if for each $\epsilon > 0$, $\lim_n \mu\{x \in X : |f_n(x) - f(x)| > \epsilon\} = 0$ as $n \to \infty$.

(a) Show that $f_n \to f$ in measure whenever $f_n \to f$ a.e. and μ is finite. Show by counterexample that this implication does not extend to σ-finite measures.

(b) Show by counterexample that $f_n \to f$ in measure does not imply that $f_n \to f$ a.e. Show, on the other hand, that if $f_n \to f$ in measure, then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.e.

Solution. For (a), suppose $f_n \to f$ a.e. but $f_n \to f$ in measure fails. Then for some $\epsilon > 0$, $\mu\{|f_n - f| > \epsilon\} \not\to 0$. However since $f_n \to f$ a.e., $\mu(\limsup_n \{x : |f_n(x) - f(x)| > \epsilon\}) = 0$, by continuity of measures (since μ is finite), a contradiction. For the counterexample: let $(X, \mathcal{M}, \mu) = (\mathbb{R}, \mathcal{B}, m)$ and let $f_n = \chi_{[n,\infty)}$.

For the counterexample in (b), let $X = [0,1]$ and $\mu = m$. Let $f_1 = \chi_{(0,1/2)}$, $f_2 = \chi_{(1/2,1)}$, $f_3 = \chi_{(0,1/4)}$, $\ldots$,$f_6 = \chi_{(3/4,1)}$, $f_7 = \chi_{(0,1/8)}$, ... For the last assertion, let for $k = 1, 2, \ldots$, n_k be such that $n \ge n_k \Rightarrow \mu\{|f_n - f| > 1/k\} < 2^{-k}$. Let $E_k = \{x : |f_{n_k}(x) - f(x)| > 1/k\}$ and $F_j = \bigcup_j^\infty E_k$. Then $\mu(F_1) \le 1$ so by Borel-Cantelli, $\mu(\limsup_k E_k) = 0$. Hence for a.e. x, $x \in E_k$ for only finitely many k. However, for such x, $f_{n_k}(x) \to f$.

6. (4p) Find an example of two measures μ and ν on the same measurable space $(X, \mathcal{M})$ such that $\nu \ll \mu$, but no measurable function f such that $\nu(E) = \int_E f d\mu$, $E \in \mathcal{M}$ exists.

Solution. Let $(X, \mathcal{M}) = (\mathbb{R}, \mathcal{B})$ and let ν be Lebesgue measure and μ counting measure. Then obviously $\nu \ll \mu$. Assume there was a measurable function f such that $\int_E f d\mu = \nu(E)$ for all $E \in \mathcal{B}$. Now taking $E = \{x\}$ gives that we must have $f(x) = 0$ since $\mu\{x\} = 1$ and $\nu\{x\} = 0$. This holds for all $x \in \mathbb{R}$, so f is identically 0, a contradiction.

Lycka till!
Johan Jonasson