SOLUTIONS
INTEGRATION THEORY (7.5 hp)
(**GU**[$MMA110$] ,**CTH**[$tmv100$])
October 20, 2011, morning, v.
No aids.
Questions on the exam: Richard Lärkäng 0703 - 088304
Each problem is worth 3 points.

1. Let f:$\mathbf{R}\rightarrow\mathbf{R}$ be a Lebesgue integrable function and define

$$g(x)=\int_{-\infty}^{\infty}e^{-|y|}f(x-y)dy \text{ if } x\in\mathbf{R}.$$

Prove that g is a continuous function of bounded variation.

Solution. Put $h(x)=e^{-|x|}$ if $x\in\mathbf{R}$ so that

$$g(x)=\int_{-\infty}^{\infty}h(x-y)f(y)dy.$$

We first prove that the function h is continuous. To this end suppose $(a_n)_{n\in\mathbf{N}_+}$ is a sequence of real numbers which converges to a real number a. Then

$$\mid h(a_n-y)f(y))\mid\in\mid f(y))\mid \text{ if } n\in\mathbf{N}_+ \text{ and } y\in\mathbf{R}$$

and since $f\in\mathcal{L}^1(m)$ by dominated convergence,

$$\lim_{n\rightarrow\infty}g(a_n)=\int_{-\infty}^{\infty}\lim_{n\rightarrow\infty}h(a_n-y)f(y)dy=$$

$$\int_{-\infty}^{\infty}h(a-y)f(y)dy=g(a)$$

and it follows that g is continuous.

We next prove that the function h is of bounded variation. Recall that the total variation function $T_h(x)$ of h at the point x is the supremum of all sums of the type

$$\sum_{i=1}^{n}\mid h(x_i)-h(x_{i-1})\mid$$

where

$$-\infty < x_0 < x_1 < ... < x_n = x < \infty.$$

We claim that h is the difference of two bounded increasing functions. Setting

$$\psi(x) = e^{\min(0,x)}$$

and observing that

$$h(x) = \psi(x) + \psi(-x) - 1$$

the claim above is obvious and

$$C =_{def} \sup T_h < \infty.$$

Moreover, if $-\infty < x_0 < x_1 < ... < x_n < \infty$,

$$\sum_{i=1}^{n} \mid g(x_i) - g(x_{i-1}) \mid =$$

$$\sum_{i=1}^{n} \mid \int_{-\infty}^{\infty} h(x_i - y) f(y) dy - \int_{-\infty}^{\infty} h(x_{i-1} - y) f(y) dy \mid$$

$$\leq \sum_{i=1}^{n} \int_{-\infty}^{\infty} \mid h(x_i - y) - h(x_{i-1} - y) \mid\mid f(y) \mid dy$$

$$\int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} \mid h(x_i - y) - h(x_{i-1} - y) \mid \right) \mid f(y) \mid dy$$

$$\leq \int_{-\infty}^{\infty} C \mid f(y) \mid dy = C \int_{-\infty}^{\infty} \mid f(y) \mid dy < \infty.$$

Hence g is of bounded variation.

2. Let γ_n be the standard Gaussian measure on $\mathbf{R}^n$, that is

$$d\gamma_n(x) = \exp(-\frac{\mid x \mid^2}{2}) \frac{dx}{\sqrt{2\pi}^n}$$

where $\mid x \mid = \sqrt{x_1^2 + ... + x_n^2}$ if $x = (x_1, ..., x_n) \in \mathbf{R}^n$. Find

$$\lim_{k \to \infty} \int_{\mathbf{R}^n} \prod_{i=1}^{n} (1 + \frac{x_i + x_i^2}{4k})^k d\gamma_n(x).$$

Solution. Since $e^t \geq 1+t$ for every real t we have for each fixed $i \in \{1, ..., n\}$,

$$1 + \frac{x_i + x_i^2}{4k} \leq e^{\frac{x_i+x_i^2}{4k}}.$$

Moreover, if $k \in \mathbf{N}_+$, then

$$1 + \frac{x_i + x_i^2}{4k} = \frac{1}{4k}((x_i + \frac{1}{2})^2 + 4k - \frac{1}{4}) \geq 0$$

and we conclude that

$$(1 + \frac{x_i + x_i^2}{4k})^k \leq e^{\frac{x_i+x_i^2}{4}}.$$

Thus, for any $k \in \mathbf{N}_+$,

$$0 \leq f_k(x) =_{def} \prod_{i=1}^{n}(1 + \frac{x_i + x_i^2}{4k})^k \leq \prod_{i=1}^{n} e^{\frac{x_i+x_i^2}{4}} =_{def} g(x)$$

where $g \in \mathcal{L}^1(\gamma_n)$ since

$$\int_{\mathbf{R}^n} g(x)d\gamma_n(x) = \int_{\mathbf{R}^n} \prod_{i=1}^{n} e^{\frac{x_i-x_i^2}{4}} \frac{dx}{\sqrt{2\pi}^n} = \{\text{Tonelli's Theorem}\} =$$

$$\prod_{i=1}^{n} \int_{\mathbf{R}} e^{\frac{x_i-x_i^2}{4}} \frac{dx_i}{\sqrt{2\pi}} = \sqrt{2}^n e^{\frac{n}{16}}.$$

Moreover,

$$\lim_{k\to\infty} f_k(x) = g(x)$$

and by dominated convergence we get

$$\lim_{k\to\infty} \int_{\mathbf{R}^n} \prod_{i=1}^{n}(1 + \frac{x_i + x_i^2}{4k})^k d\gamma_n(x) = \int_{\mathbf{R}^n} g(x)d\gamma_n(x) = \sqrt{2}^n e^{\frac{n}{16}}.$$

3. Suppose $\Sigma_1^\infty a_n$ is a positive convergent series and let E be the set of all $x \in [0, 1]$ such that

$$\min_{p\in\{0,...,n\}} \mid x - \frac{p}{n} \mid < \frac{a_n}{n}$$

for infinitely many $n \in \mathbf{N}_+$. Prove that E is a Lebesgue null set.

Solution. For fixed $n \in \mathbf{N}_+$, let E_n be the set of all $x \in [0,1]$ such that

$$\min_{p \in \mathbf{N}_+} \mid x - \frac{p}{n} \mid < \frac{a_n}{n}.$$

Then if $B(x,r) =]x-r, x+r[\,,\ x \in [0,1]\,,\ r > 0$, we have

$$E_n \subseteq \bigcup_{p=0}^{n} B(\frac{p}{n}, \frac{a_n}{n})$$

and

$$m(E_n) \leq (n+1)\frac{2a_n}{n} \leq 4a_n.$$

Hence

$$\sum_1^\infty m(E_n) < \infty$$

and by the Beppo Levi theorem

$$\int_0^1 \sum_1^\infty \chi_{E_n} dm < \infty.$$

Accordingly from this the set

$$F = \left\{ x \in [0,1]\,;\ \sum_1^\infty \chi_{E_n}(x) < \infty \right\}$$

is of Lebesgue measure 1. Since $E \subseteq [0,1] \setminus F$ we have $m(E) = 0$.

4. Suppose $(X, \mathcal{M}, \mu)$ is a positive measure space. (a) If $f{:}X \to \mathbf{R}$ is measurable and $\mu(\mid f \mid > \varepsilon) = 0$ for every $\varepsilon > 0$, show that $f = 0$ a.e. $[\mu]$. (b) If $f_n \to f$ in μ-measure and $f_n \to g$ in μ-measure, show that $f = g$ a.e. $[\mu]$. (c) If $f_n \to f$ in $L^1(\mu)$, show that $f_n \to f$ in μ-measure.

5. Suppose $(X, \mathcal{M}, \mu)$ is a positive measure space and $f_n : X \to [0, \infty]$, $n \in \mathbf{N}_+$, a sequence of measurable functions such that $f_n \leq f_{n+1}, n \in \mathbf{N}_+$, and

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ if } x \in X.$$

Prove that f is measurable and

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$