Solutions:
INTEGRATION THEORY (7.5 hp)
(GU$[MMA110]$,**CTH**$[tmv100]$**)**
October 21, 2010, morning, H.
No aids.
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Each problem is worth 3 points.

1. Suppose

$$\mu(A) = \int_A \mid x \mid^n e^{-|x|^n} dx, \; A \in \mathcal{B}(\mathbf{R}^n),$$

where $\mid x \mid = \sqrt{x_1^2 + ... + x_n^2}$ if $x = (x_1, ..., x_n) \in \mathbf{R}^n$. Compute the limit

$$\lim_{\rho \to \infty} \rho^{-n} \ln \mu(\{x \in \mathbf{R}^n ; \mid x \mid \geq \rho\}).$$

Solution. We have

$$\mu(\{x \in \mathbf{R}^n ; \mid x \mid \geq \rho\}) = \int_{\mathbf{R}^n} \chi_{[\rho,\infty[}(\mid x \mid) \mid x \mid^n e^{-|x|^n} dx$$

$$= \sigma_{n-1}(S^{n-1}) \int_\rho^\infty r^{2n-1} e^{-r^n} dr = \frac{1}{n} \sigma_{n-1}(S^{n-1}) \int_{\rho^n}^\infty t e^{-t} dt$$

$$= \frac{1}{n} \sigma_{n-1}(S^{n-1})(\rho^n + 1) e^{-\rho^n}.$$

Hence

$$\lim_{\rho \to \infty} \rho^{-n} \ln \mu(\{x \in \mathbf{R}^n ; \mid x \mid \geq \rho\}) = -1.$$

2. Let μ be a σ-finite positive measure on $(X, \mathcal{M})$ and $(f_n)_{n \in \mathbf{N}}$ a sequence of measurable functions which converges in μ-measure to a measurable function f. Moreover, suppose ν is a finite positive measure on $(X, \mathcal{M})$ such that $\nu << \mu$. Prove that $f_n \to f$ in ν-measure.

Solution. The Radon-Nikodym theorem implies that $d\nu = gd\mu$ for an appropriate non-negative $g \in L^1(\mu)$.

Now, if $\varepsilon > 0$ is given and $k \in \mathbf{N}_+$,

$$0 \leq \nu(\mid f_n - f \mid > \varepsilon) = \int_X \chi_{]\varepsilon,\infty[}(\mid f_n - f \mid) g d\mu$$

$$= \int_{g \leq k} \chi_{]\varepsilon,\infty[}(\mid f_n - f \mid) g d\mu + \int_{g > k} \chi_{]\varepsilon,\infty[}(\mid f_n - f \mid) g d\mu$$

$$\leq k \int_X \chi_{]\varepsilon,\infty[}(\mid f_n - f \mid) d\mu + \int_X \chi_{]k,\infty[}(g) g d\mu$$

$$= k\mu(\mid f_n - f \mid > \varepsilon) + \int_X \chi_{]k,\infty[}(g) g d\mu.$$

Thus

$$0 \leq \limsup_{n \to \infty} \nu(\mid f_n - f \mid > \varepsilon) \leq \int_X \chi_{]k,\infty[}(g) g d\mu.$$

Moreover, $0 \leq \chi_{]k,\infty[}(g) g \leq g \in L^1(\mu)$ and $\chi_{]k,\infty[}(g) g \to 0$ a.e. $[\mu]$ as $k \to \infty$. Hence, by the dominated convergence theorem,

$$\lim_{k \to \infty} \int_X \chi_{]k,\infty[}(g) g d\mu = 0$$

and it follows that

$$\lim_{n \to \infty} \nu(\mid f_n - f \mid > \varepsilon) = 0.$$

Alternative solution. Choose $\varepsilon_0, \varepsilon > 0$. Since $\nu << \mu$ there is a postive number δ such that

$$(E \in \mathcal{M} \text{ and } \mu(E) < \delta) \Rightarrow \nu(E) < \varepsilon_0$$

(Folland, Theorem 3.5). Moreover, since the sequence $(f_n)_{n \in \mathbf{N}}$ converges in μ-measure, there is a postive integer N such that

$$n \geq N \Rightarrow \mu(\mid f_n - f \mid > \varepsilon) < \delta.$$

From the above we conclude that

$$n \geq N \Rightarrow \nu(\mid f_n - f \mid > \varepsilon) < \varepsilon_0.$$

Thus

$$\lim_{n\to\infty} \nu(|\, f_n - f \,| > \varepsilon) = 0.$$

Note that this solution does not use that μ is σ-finite.

3. Suppose $f{:}[0,1] \to \mathbf{R}$ is a continuous function. Find

$$\lim_{n\to\infty} n \int_0^1 f(x) e^{-(n\min(x,1-x))^2} dx.$$

Solution. We have

$$n \int_0^{\frac{1}{2}} f(x) e^{-(n\min(x,1-x))^2} dx = n \int_0^{\frac{1}{2}} f(x) e^{-(nx)^2} dx$$

$$= \int_0^{\frac{n}{2}} f(\frac{t}{n}) e^{-t^2} dt = \int_0^{\infty} \chi_{[0,\frac{n}{2}]}(t) f(\frac{t}{n}) e^{-t^2} dt.$$

Here for each $n \in \mathbf{N}$, the function

$$g_n(t) = \chi_{[0,\frac{n}{2}]}(t) f(\frac{t}{n}) e^{-t^2},\ t \geq 0$$

is non-negative and bounded by the function

$$(\max |\, f \,|) e^{-t^2},\ t \geq 0$$

which is Lebesgue integrable on $[0,\infty[$. Since

$$\lim_{n\to\infty} g_n(t) = f(0) e^{-t^2},\ t \geq 0$$

the dominated convergence theorem implies that

$$\lim_{n\to\infty} n \int_0^{\frac{1}{2}} f(x) e^{-(n\min(x,1-x))^2} dx = f(0) \int_0^{\infty} e^{-t^2} dt = f(0) \frac{\sqrt{\pi}}{2}.$$

Furthermore,

$$n \int_{\frac{1}{2}}^1 f(x) e^{-(n\min(x,1-x))^2} dx = n \int_0^{\frac{1}{2}} f(1-x) e^{-(n\min(x,1-x))^2} dx$$

and it follows from the above that

$$\lim_{n\to\infty} n \int_{\frac{1}{2}}^{1} f(x)e^{-(n\min(x,1-x))^2}dx = f(1)\frac{\sqrt{\pi}}{2}.$$

Thus

$$\lim_{n\to\infty} n \int_{0}^{1} f(x)e^{-(n\min(x,1-x))^2}dx = \frac{\sqrt{\pi}}{2}(f(0)+f(1)).$$

4. Suppose $f :]a,b[\times X \to \mathbf{R}$ is a function such that $f(t,\cdot) \in L^1(\mu)$ for each $t \in]a,b[$. Moreover, assume $\frac{\partial f}{\partial t}$ exists and there is a $g \in L^1(\mu)$ such that

$$\mid \frac{\partial f}{\partial t}(t,x) \mid \leq g(x) \text{ for all } (t,x) \in]a,b[\times X.$$

Prove that the function

$$F(t) = \int_X f(t,x)d\mu(x),\ t \in]a,b[,$$

is differentiable and

$$F'(t) = \int_X \frac{\partial f}{\partial t}(t,x)d\mu(x) \text{ if } t \in]a,b[.$$

5. (a) Suppose $f{:}\mathbf{R}^n \to [0,\infty]$ and that $\{x \in \mathbf{R}^n;\ f(x) > \alpha\}$ is an m_n-null set for every $\alpha > 0$. Prove that $f(x) = 0$ a.e. $[m_n]$. (b) Suppose $f \in L^1_{loc}(m_n)$ and define

$$A_r f(x) = \frac{1}{m_n(B(x,r))}\int_{B(x,r)} f(y)dy,\ (x,r) \in \mathbf{R}^n \times]0,\infty[$$

where $B(x,r)$ is the open ball of centre $x \in \mathbf{R}^n$ and radius $r > 0$ (with respect to the Euclidean metric $d(x,y) = \mid x-y \mid$).

Use the maximal theorem to prove that

$$\lim_{r\to 0+} A_r f(x) = f(x) \text{ a.e. } [m_n].$$