LÖSNINGAR
INTEGRATIONSTEORI (5p)
(GU$[MAF440]$ **,CTH**$[TMV100]$**)**
Dag, tid: 28 jan 2006
Hjälpmedel: Inga.
Skrivtid: 5 timmar

1. Suppose $f(x) = x\cos(\pi/x)$ if $0 < x < 2$ and $f(x) = 0$ if $x \in \mathbf{R}\setminus]0,2[$. Prove that f is not of bounded variation on $\mathbf{R}$.

Solution. We have

$$\Sigma_{k=1}^{n} \mid f(\frac{1}{k+1}) - f(\frac{1}{k}) \mid = \Sigma_{k=1}^{n} \mid \frac{1}{k+1}\cos(k+1)\pi - \frac{1}{k}\cos k\pi \mid$$

$$= \Sigma_{k=1}^{n}(\frac{1}{k+1} + \frac{1}{k}) = \frac{1}{n+1} + 1 + 2\Sigma_{k=2}^{n}\frac{1}{k} \to \infty \text{ as } n \to \infty.$$

2. Let $(X, \mathcal{M}, \mu)$ be a finite positive measure space and suppose $\varphi(t) = \min(t,1)$, $t \geq 0$. Prove that $f_n \to f$ in measure if and only if $\varphi(\mid f_n - f \mid) \to 0$ in $L^1(\mu)$.

Solution: $\Rightarrow$: For any $\varepsilon > 0$,

$$\int_X \varphi(\mid f_n - f \mid)d\mu \leq \int_{|f_n - f| \leq \varepsilon} \varphi(\mid f_n - f \mid)d\mu$$

$$+ \int_{|f_n - f| > \varepsilon} \varphi(\mid f_n - f \mid)d\mu \leq \int_{|f_n - f| \leq \varepsilon} \varphi(\varepsilon)d\mu + \int_{|f_n - f| > \varepsilon} 1 d\mu$$

$$\leq \varphi(\varepsilon)\mu(X) + \mu(\mid f_n - f \mid > \varepsilon).$$

Thus

$$0 \leq \limsup_{n\to\infty} \int_X \varphi(\mid f_n - f \mid)d\mu \leq \varphi(\varepsilon)\mu(X)$$

and by letting $\varepsilon \downarrow 0$,

$$\lim_{n\to\infty} \int_X \varphi(\mid f_n - f \mid)d\mu = 0.$$

$\Leftarrow$: For any $\varepsilon > 0$,

$$\mu(| f_n - f |> \varepsilon) \leq \mu(\varphi(| f_n - f |) \geq \varphi(\varepsilon))$$

and the Markov inequality gives

$$\mu(| f_n - f |> \varepsilon) \leq \frac{1}{\varphi(\varepsilon)} \int_X \varphi(| f_n - f |) d\mu.$$

Thus $\mu(| f_n - f |> \varepsilon) \to 0$ as $n \to \infty$.

3. Let P denote the class of all Borel probability measures on $[0,1]$ and L the class of all functions $f : [0,1] \to [-1,1]$ such that

$$| f(x) - f(y) |\leq| x - y |, \ x, y \in [0,1] \, .$$

For any $\mu, \nu \in P$, define

$$\rho(\mu, \nu) = \sup_{f \in L} | \int_{[0,1]} f d\mu - \int_{[0,1]} f d\nu | \, .$$

(a) Show that (P, ρ) is a metric space. (b) Compute $\rho(\mu, \nu)$ if μ is linear measure on $[0,1]$ and $\nu = \frac{1}{n}\Sigma_{k=0}^{n-1} \delta_{\frac{k}{n}}$, where $n \in \mathbf{N}_+$ (linear measure on $[0,1]$ is Lebesgue measure on $[0,1]$ restricted to the Borel sets in $[0,1]$).

Solution. (a): (1) Clearly, $\rho(\mu, \nu) \geq 0$ and

$$\rho(\mu, \nu) \leq \mu([0,1]) + \nu([0,1]) = 2 < \infty.$$

Moreover, if $\mu \neq \nu$ there is a compact set $K \subseteq [0,1]$ such that $\mu(K) \neq \nu(K)$. If $f_n(x) = \max(0, 1 - nd(x, K))$, $x \in [0,1]$, then $f_n \downarrow \chi_K$, and the Lebesgue Dominated Convergence Theorem implies that

$$\int_{[0,1]} f_n d\mu \neq \int_{[0,1]} f_n d\nu$$

if n is sufficiently large. But $\frac{1}{n} f_n \in L$, and, hence, if n is large

$$\rho(\mu, \nu) \geq| \int_{[0,1]} \frac{1}{n} f_n d\mu - \int_{[0,1]} \frac{1}{n} f_n d\nu |$$

$$= \frac{1}{n} \mid \int_{[0,1]} f_n d\mu - \int_{[0,1]} f_n d\nu \mid > 0.$$

Thus $\rho(\mu, \nu) > 0$.

(2) Since $\mid t \mid$ is an even function of t, $\rho(\mu, \nu) = \rho(\nu, \mu)$.

(3) If $f \in L$ and $\mu, \nu, \tau \in P$,

$$\mid \int_{[0,1]} f d\mu - \int_{[0,1]} f d\nu \mid$$

$$\leq \mid \int_{[0,1]} f d\mu - \int_{[0,1]} f d\tau \mid + \mid \int_{[0,1]} f d\tau - \int_{[0,1]} f d\nu \mid$$

$$\leq \rho(\mu, \tau) + \rho(\tau, \nu)$$

and we get $\rho(\mu, \nu) \leq \rho(\mu, \tau) + \rho(\tau, \nu)$.

(b) If $f \in L$,

$$\mid \int_{[0,1]} f d\mu - \int_{[0,1]} f d\nu \mid = \mid \int_0^1 f(x) dx - \frac{1}{n} \sum_{k=0}^{n-1} f(\frac{k}{n}) \mid$$

$$= \mid \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} (f(x) - f(\frac{k}{n})) dx \mid$$

$$\leq \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \mid f(x) - f(\frac{k}{n}) \mid dx$$

$$\leq \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \mid x - \frac{k}{n} \mid dx = \frac{1}{2n}$$

where equality occurs if $f(x) = x$. Thus $\rho(\mu, \nu) = \frac{1}{2n}$.

4. Suppose $(X, \mathcal{M}, \mu)$ is a positive measure space and $w : X \to [0, \infty]$ a measurable function. Define

$$\nu(A) = \int_A w d\mu, \ A \in \mathcal{M}.$$

Prove that ν is a positive measure and

$$\int_X f d\nu = \int_X f w d\mu$$

for every measurable function $f : X \to [0, \infty]$.

5. Suppose $f \in L^1_{loc}(m_n)$ and set

$$(A_r f)(x) = \frac{1}{m_n(B(x,r))} \int_{B(x,r)} f(y) dy, \ (x,r) \in \mathbf{R}^n \times]0, \infty[$$

where $B(x,r)$ is the open ball of centre $x \in \mathbf{R}^n$ and radius $r > 0$ (with respect to the Euclidean metric $d(x,y) = \mid x - y \mid$).

(a) Set

$$f^*(x) = \sup_{r>0} \mid (A_r f)(x) \mid, \ x \in \mathbf{R}^n.$$

Prove that

$$\{f^* \geq \lambda\} \in \mathcal{B}(\mathbf{R}^n) \text{ if } \lambda \geq 0.$$

(b) Use the (Wiener) Maximal Theorem to prove that

$$\lim_{r \to 0+} (A_r f)(x) = f(x) \text{ a.e. } [m_n].$$