LÖSNINGAR
INTEGRATIONSTEORI (5p)
(**GU**[$MAF440$] ,**CTH**[$TMV100$])
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Hjälpmedel: Inga.
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1. Suppose

$$f(t) = \int_0^\infty \frac{xe^{-x^2}}{x^2+t^2}dx,\ t > 0.$$

Compute

$$\lim_{t\to 0+} f(t) \text{ and } \int_0^\infty f(t)dt.$$

Finally, prove that f is differentiable.

Solution. Suppose $t_n \downarrow 0$. Then for each $x > 0$, $\frac{xe^{-x^2}}{x^2+t_n^2} \uparrow \frac{1}{x}e^{-x^2}$ and the LMC implies that

$$\int_0^\infty \frac{xe^{-x^2}}{x^2+t_n^2}dx \uparrow \int_0^\infty \frac{1}{x}e^{-x^2}dx = \infty$$

since $e^{-x^2} > \frac{1}{3}\chi_{[0,1]}(x)$ if $x \geq 0$ and

$$\int_0^1 \frac{1}{x}dx = \infty.$$

Hence

$$\lim_{t\to 0+} f(t) = \infty.$$

Furthermore, the Tonelli Theorem yields

$$\int_0^\infty f(t)dt = \int_0^\infty \left\{ \int_0^\infty \frac{xe^{-x^2}}{x^2+t^2}dt \right\} dx$$

$$= \int_0^\infty \left[e^{-x^2} \arctan \frac{t}{x} \right]_{t=0}^{t=\infty} dx = \frac{\pi}{2} \int_0^\infty e^{-x^2}dx = \frac{\pi^{\frac{3}{2}}}{4}.$$

Finally, it is enough to prove that $f(t)$ is differentiable on the interior of any given compact subinterval $[a, b]$ of $]0, \infty[$. To this end, first note that

$$\frac{\partial}{\partial t}\frac{xe^{-x^2}}{x^2+t^2} = -\frac{2txe^{-x^2}}{(x^2+t^2)^2}$$

and

$$\sup_{a \le t \le b} \left| \frac{\partial}{\partial t}\frac{xe^{-x^2}}{x^2+t^2} \right| \le \frac{2bxe^{-x^2}}{(x^2+a^2)^2} \in L^1(m_{0,\infty}).$$

Therefore, by a familiar result (Folland Theorem 2.27 or LN, Example 2.2.1) $f'(t)$ exists for all $a < t < b$ and equals

$$\int_0^\infty \frac{\partial}{\partial t}\frac{xe^{-x^2}}{x^2+t^2}dx = -2t\int_0^\infty \frac{xe^{-x^2}}{(x^2+t^2)^2}dx.$$

2. Suppose μ is a finite positive Borel measure on $\mathbf{R}^n$. (a) Let $(V_i)_{i\in I}$ be a family of open subsets of $\mathbf{R}^n$ and $V = \cup_{i\in I} V_i$. Prove that

$$\mu(V) = \sup_{\substack{i_1,\ldots,i_k \in I \\ k\in\mathbf{N}_+}} \mu(V_{i_1} \cup \ldots \cup V_{i_k}).$$

(b) Let $(F_i)_{i\in I}$ be a family of closed subsets of $\mathbf{R}^n$ and $F = \cap_{i\in I} F_i$. Prove that

$$\mu(F) = \inf_{\substack{i_1,\ldots,i_k \in I \\ k\in\mathbf{N}_+}} \mu(F_{i_1} \cap \ldots \cap F_{i_k}).$$

Solution. (a) Since $V \supseteq V_{i_1} \cup \ldots \cup V_{i_k}$ for all $i_1, \ldots i_k \in I$ and $k \in \mathbf{N}_+$,

$$\mu(V) \ge \sup_{\substack{i_1,\ldots,i_k \in I \\ k\in\mathbf{N}_+}} \mu(V_{i_1} \cup \ldots \cup V_{i_k}).$$

To prove the reverse inequality first note that

$$\mu(A) = \sup_{\substack{K\subseteq A \\ K \text{ compact}}} \mu(K)$$

if $A \in \mathcal{R}_n$. Now first choose $\varepsilon > 0$ and then a compact subset K of $\mathbf{R}^n$ such that

$$\mu(K) > \mu(V) - \varepsilon.$$

Then there are finitely many $i_1, ..., i_k \in I$ such that $V_{i_1} \cup ... \cup V_{i_k} \supseteq K$. Accordingly from this,

$$\mu(V_{i_1} \cup ... \cup V_{i_k}) > \mu(V) - \varepsilon$$

and we get

$$\sup_{\substack{i_1,...,i_k \in I \\ k \in \mathbf{N}_+}} \mu(V_{i_1} \cup ... \cup V_{i_k}) \geq \mu(V).$$

Thus

$$\mu(V) = \sup_{\substack{i_1,...,i_k \in I \\ k \in \mathbf{N}_+}} \mu(V_{i_1} \cup ... \cup V_{i_k}).$$

b) Since μ is a finite measure, by Part (a)

$$\mu(F) = \mu(\mathbf{R}^n) - \mu(F^c)$$

$$= \mu(\mathbf{R}^n) - \sup_{\substack{i_1,...,i_k \in I \\ k \in \mathbf{N}_+}} \mu(F_{i_1}^c \cup ... \cup F_{i_k}^c)$$

$$= \inf_{\substack{i_1,...,i_k \in I \\ k \in \mathbf{N}_+}} (\mu(\mathbf{R}^n) - \mu(F_{i_1}^c \cup ... \cup F_{i_k}^c))$$

$$= \inf_{\substack{i_1,...,i_k \in I \\ k \in \mathbf{N}_+}} \mu((F_{i_1}^c \cup ... \cup F_{i_k}^c)^c)$$

$$= \inf_{\substack{i_1,...,i_k \in I \\ k \in \mathbf{N}_+}} \mu(F_{i_1} \cap ... \cap F_{i_k}).$$

3. Suppose f and g are real-valued absolutely continuous functions on the compact interval $[a, b]$. Show that the function $h = \max(f, g)$ is absolutely continuous and $h' \leq \max(f', g')$ a.e. $[m_{a,b}]$ ($m_{a,b}$ denotes Lebesgue measure on $[a, b]$).

Solution. If $(A_i)_{1 \leq i \leq 2}$ and $(B_i)_{1 \leq i \leq 2}$ are sequences of real numbers

$$A_i \leq B_i + \mid A_i - B_i \mid$$

$$\leq B_i + \max_{1\leq i\leq 2} \mid A_i - B_i \mid \leq \max_{1\leq i\leq 2} B_i + \max_{1\leq i\leq 2} \mid A_i - B_i \mid$$

and, hence,

$$\max_{1\leq i\leq 2} A_i \leq \max_{1\leq i\leq 2} \mid A_i - B_i \mid + \max_{1\leq i\leq 2} B_i$$

and

$$\max_{1\leq i\leq 2} A_i - \max_{1\leq i\leq 2} B_i \leq \max_{1\leq i\leq 2} \mid A_i - B_i \mid .$$

Thus, by interchanging A_i and B_i,

$$\mid \max_{1\leq i\leq 2} A_i - \max_{1\leq i\leq 2} B_i \mid \leq \max_{1\leq i\leq 2} \mid A_i - B_i \mid .$$

Next choose $\varepsilon > 0$. Then there exists a $\delta > 0$ such that

$$\Sigma_{k=1}^n \mid f(a_k) - f(b_k) \mid < \varepsilon/2$$

and

$$\Sigma_{k=1}^n \mid g(a_k) - g(b_k) \mid < \varepsilon/2$$

if $n \in \mathbf{N}_+$ and $]a_i, b_i[$, $i = 1, ..., n$, are mutually disjoint subintervals of $[a, b]$. Thus, for such intervals

$$\Sigma_{k=1}^n \mid h(a_k) - h(b_k) \mid$$

$$\leq \Sigma_{k=1}^n \max(\mid f(a_k) - f(b_k) \mid, \mid g(a_k) - g(b_k) \mid)$$

$$\leq \Sigma_{k=1}^n (\mid f(a_k) - f(b_k) \mid + \mid g(a_k) - g(b_k) \mid) < \varepsilon$$

and it follows that h is absolutely continuous.

As above it follows that

$$\max_{1\leq i\leq 2} A_i - \max_{1\leq i\leq 2} B_i \leq \max_{1\leq i\leq 2}(A_i - B_i).$$

Therefore, for each $x \in]a, b[$ and $\omega \in]0, b - x[$,

$$h(x + \omega) - h(x) \leq \max(f(x + \omega) - f(x), g(x + \omega) - g(x))$$

and

$$\frac{h(x + \omega) - h(x)}{\omega} \leq \max(\frac{f(x + \omega) - f(x)}{\omega}, \frac{g(x + \omega) - g(x)}{\omega}).$$

Since f, g, and h are absolutely continuous, by letting $\omega \downarrow 0$, we get $h'(x) \leq \max(f'(x), g'(x))$ for $m_{a,b}$-almost all $x \in [a, b]$.

4. Suppose $(X, \mathcal{M}, \mu)$ is a positive measure space. (a) If $f_n \to f$ in measure and $f_n \to g$ in measure, show that $f = g$ a.e. $[\mu]$. (b) If $f_n \to f$ in L^1, show that $f_n \to f$ in measure.

5. (Lebesgue's Dominated Convergence Theorem) Suppose $(X, \mathcal{M}, \mu)$ is a positive measure space and $f_n : X \to \mathbf{R}$, $n \in \mathbf{N}_+$, measurable functions such that

$$| f_n(x) | \leq g(x), \text{ all } x \in X \text{ and } n \in \mathbf{N}_+$$

where $g \in \mathcal{L}^1(\mu)$. Moreover, suppose the limit $\lim_{n\to\infty} f_n(x)$ exists and equals $f(x)$ for every $x \in X$.

Prove that $f \in \mathcal{L}^1(\mu)$,

$$\lim_{n\to\infty} \int_X | f_n - f | \, d\mu = 0$$

and

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$