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INTEGRATIONSTEORI (5p)
(**GU**[$MAF440$],**CTH**[$TMV100$])
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Hjälpmedel: Inga.

1. Suppose

$$f(t) = \lim_{r\to\infty} \int_0^r \frac{\cos x}{t+x} dx,\ t > 0.$$

Show that f is differentiable.

Solution. By partial integration,

$$\int_0^r \frac{\cos x}{t+x} dx = \frac{\sin r}{t+r} + \int_0^r \frac{\sin x}{(t+x)^2} dx$$

and it follows that

$$f(t) = \int_0^\infty \frac{\sin x}{(t+x)^2} dx.$$

Let $a > 0$. It is enough to prove that $f(t)$ is differentiable for $t > a$. We have

$$\sup_{t>a} \left| \frac{\partial}{\partial t} \frac{\sin x}{(t+x)^2} \right| \leq \frac{2}{(a+x)^3},\ x > 0$$

and

$$\int_0^\infty \frac{dx}{(a+x)^3} < \infty.$$

The theorem about interchanging a derivative and an integral now yields

$$f'(t) = \int_0^\infty \frac{\partial}{\partial t} \frac{\sin x}{(t+x)^2} dx = -\int_0^\infty \frac{2\sin x}{(t+x)^3} dx$$

for each $t > a$. This proves that f is differentiable.

2. Suppose $a, b \in \mathbf{R}$ and $a < b$. Show that if f and g are absolutely continuous functions on $[a, b]$, so is their product fg.

Solution. The functions f and g are continuous. Set $M_f = \max_{a\leq x\leq b} \mid f(x) \mid$ and $M_g = \max_{a\leq x\leq b} \mid g(x) \mid$.

Choose $\varepsilon > 0$. There exists a $\delta > 0$ such that, if

$$a \leq a_1 < b_1 \leq a_2 < b_2 \leq ... < a_n < b_n \leq b$$

and

$$\Sigma_{k=1}^n \mid b_k - a_k \mid < \delta$$

then

$$\Sigma_{k=1}^n \mid f(b_k) - f(a_k) \mid < \frac{1}{2}\frac{\varepsilon}{1+M_g}$$

and

$$\Sigma_{k=1}^n \mid g(b_k) - g(a_k) \mid < \frac{1}{2}\frac{\varepsilon}{1+M_f}.$$

But then

$$\Sigma_{k=1}^n \mid f(b_k)g(b_k) - f(a_k)g(a_k) \mid$$
$$= \Sigma_{k=1}^n \mid (f(b_k) - f(a_k))g(b_k) + f(a_k)(g(b_k) - g(a_k)) \mid$$
$$\leq \Sigma_{k=1}^n \mid (f(b_k) - f(a_k))g(b_k) \mid + \Sigma_{k=1}^n \mid f(a_k)(g(b_k) - g(a_k)) \mid$$
$$\leq M_g \Sigma_{k=1}^n \mid f(b_k) - f(a_k) \mid + M_f \Sigma_{k=1}^n \mid g(b_k) - g(a_k) \mid$$
$$< \frac{1}{2}\frac{\varepsilon M_g}{1+M_g} + \frac{1}{2}\frac{\varepsilon M_f}{1+M_f} < \varepsilon.$$

It follows that fg is absolutely continuous.

3. Let $f : [0, \pi] \to \mathbf{R}$ be a continuous function. Compute the limit

$$\lim_{n\to\infty} n \int_0^\pi f(t)e^{-n\sin t}dt.$$

Solution. We have

$$n \int_0^{\frac{\pi}{2}} f(t)e^{-n\sin t}dt = \int_0^{\frac{n\pi}{2}} f(\frac{x}{n})e^{-n\sin\frac{x}{n}}dx.$$

Here

$$\mid \chi_{[0,\frac{n\pi}{2}]}(x) f(\frac{x}{n})e^{-n\sin\frac{x}{n}} \mid \leq (\max \mid f \mid)e^{-\frac{2}{\pi}x}$$

(recall that the concavity of $\sin_{|[0,\frac{\pi}{2}]}$ yields $\sin v \geq \frac{2v}{\pi}$, $0 \leq v \leq \frac{\pi}{2}$). Since

$$\int_0^\infty e^{-\frac{2}{\pi}x} dx < \infty$$

the Lebesgue Dominated Convergence Theorem implies that

$$\lim_{n\to\infty} n \int_0^{\frac{\pi}{2}} f(t)e^{-n\sin t} dt = \int_0^\infty f(0)e^{-x} dx = f(0).$$

Furthermore

$$n \int_{\frac{\pi}{2}}^{\pi} f(t)e^{-n\sin t} dt = n \int_0^{\frac{\pi}{2}} f(\pi - t)e^{-n\sin t} dt$$

and the first part of the solution proves that

$$\lim_{n\to\infty} n \int_{\frac{\pi}{2}}^{\pi} f(t)e^{-n\sin t} dt = f(\pi).$$

From the above we now conclude that

$$\lim_{n\to\infty} n \int_0^{\pi} f(t)e^{-n\sin t} dt = f(0) + f(\pi).$$

4. Let $(X, \mathcal{M})$ be a measurable space and $f : X \to [0, \infty]$ a measurable function. Show that there exist simple measurable functions $\varphi_n : X \to [0, \infty[$, $n \in \mathbf{N}_+$, such that $\varphi_n \uparrow f$.

5. Let $f \in L^1(m_n)$. Use the Maximal Theorem to conclude that

$$\lim_{r\downarrow 0} \frac{1}{m_n(B(x,r))} \int_{B(x,r)} f(y)dy = f(x) \text{ a.e. } [m_n].$$