TMS 165/MSA350 Stochastic Calculus Part I

Written Exam Wednesday 27 April 2011 8.30 am–1.30 pm

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AIDS: None.

GRADES: 12000 points (40%) out of the full score 30000 points to pass the exam.

MOTIVATIONS: All answers/solutions must be motivated.

Througout this exam $B = \{B(t)\}_{t\geq 0}$ is a Brownian motion. And Good Luck to you all!

Task 1. The quadratic variation $[B,B](t)$ of B over the interval $[0,t]$, defined by

$$[B,B](t) = \lim_{\max_{1\leq i\leq n} t_i - t_{i-1}\downarrow 0} \sum_{i=1}^{n} (B(t_i) - B(t_{i-1}))^2$$

for increasingly fine partitions $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = t$ of $[0,t]$, is shown to be $[B,B](t) = t$ by establishing that the mean of the above sum is always t, while its variance is zero in the limit. In a similar manner, find the quadratic covariation $[B_1, B_2](t)$ between two independent Brownian motions B_1 and B_2. **(5000 points)**

Task 2. I is known that there exists a continuous martingale $X = \{X(t)\}_{t\geq 0}$ that has Brownian motion univariate marginal distribution, that is, $X(t)$ is $\mathrm{N}(0,t)$-distributed for each t, but that is not a Brownian motion. Use the fact that martingales have uncorrelated increments to prove that X cannot be a Gaussian process. **(5000 points)**

Task 3. Solve the SDE

$$dX(t) = 6\,X(t)^{1/2}\,dt + 4\,X(t)^{3/4}\,dB(t), \quad X(0) = x_0 > 0. \qquad \textbf{(5000 points)}$$

Task 4. Find a solution $p(y,t)$ to the PDE

$$\frac{1}{2}\frac{\partial^2 p}{\partial y^2} - \frac{\partial p}{\partial y} - \frac{\partial p}{\partial t} = 0 \quad \text{for } (y,t) \in \mathbb{R}\times(0,\infty)$$

that is a probability density function as a function of y for any given t. **(5000 points)**

Task 5. Describe how one for a two-dimensional standardized Gaussian random variable (X,Y) with probability density function

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2 - 2\,xy + y^2}{2\,(1-\rho^2)}\right\} \quad \text{for } x,y \in \mathbb{R}$$

(where $-1 < \rho < 1$ is a constant) can make X and Y independent unit mean exponential distributed by means of a change of probability measure. **(5000 points)**

Task 6. Write a short but still packed with substantial content essay on the topic of numerical solution of SDE. **(5000 points)**

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Solutions to Written Exam Wednesday 27 April 2011

Task 1. We have $\mathbf{E}\{\sum_{i=1}^n (B_1(t_i)-B_1(t_{i-1}))\,(B_2(t_i)-B_2(t_{i-1}))\} = \sum_{i=1}^n \mathbf{E}\{(B_1(t_i)-B_1(t_{i-1}))\,(B_2(t_i)-B_2(t_{i-1}))\} = \sum_{i=1}^n \mathbf{E}\{B_1(t_i)-B_1(t_{i-1})\}\,\mathbf{E}\{B_2(t_i)-B_2(t_{i-1})\} = 0$, while $\mathbf{Var}\{\sum_{i=1}^n (B_1(t_i)-B_1(t_{i-1}))\,(B_2(t_i)-B_2(t_{i-1}))\} = \sum_{i=1}^n \mathbf{Var}\{(B_1(t_i)-B_1(t_{i-1}))$ $(B_2(t_i)-B_2(t_{i-1}))\} = \sum_{i=1}^n \mathbf{E}\{(B_1(t_i)-B_1(t_{i-1}))^2 (B_2(t_i)-B_2(t_{i-1}))^2\} = \sum_{i=1}^n \mathbf{E}\{(B_1(t_i)$ $-B_1(t_{i-1}))^2\}\,\mathbf{E}\{(B_2(t_i)-B_2(t_{i-1}))^2\} = \sum_{i=1}^n (t_i-t_{i-1})^2 \leq \max_{1\leq i\leq n}(t_i-t_{i-1}) \sum_{i=1}^n (t_i-t_{i-1}) = \max_{1\leq i\leq n}(t_i-t_{i-1})\,t \to 0$ as $\max_{1\leq i\leq n} t_i - t_{i-1} \downarrow 0$. Hence $[B_1,B_2](t)=0$.

Task 2. Assume that X is Gaussian. Then the uncorrelated increments of X are in fact independent, so that X is an independent increment process. From the fact that $\mathrm{N}(0,t) =_D X(t) = (X(t)-X(s)) + X(s)$ for $0<s<t$, where $X(t)-X(s)$ and $X(s) =_D \mathrm{N}(0,s)$ are independent Gaussian we see that $X(t)-X(s) =_D \mathrm{N}(0,t-s)$. This means that X is Brownian motion, which is a contardiction.

Task 3. By means of Itô's formula we see that $X(t) = (B(t)+x_0^{1/4})^4$ solves the SDE.

Task 4. The PDE is the Kolmogorov forward equation for the SDE $dX(t) = dt + dB(t)$, so that p is the transition density function for Brownian motion with unit drift, that is,

$$p(y,t) = \frac{1}{\sqrt{2\pi\,(t-s)}}\,\exp\biggl\{-\frac{(x-y-t+s)^2}{2\,(t-s)}\biggr\} \quad \text{for any } \ x,y\in\mathbb{R} \ \text{ and } \ 0<s<t.$$

Task 5. If (X,Y) is standard Gaussian distributed under the probability measure $\mathbf{P}$ on the sample space Ω, then (X,Y) independent unit mean exponential distributed under the probability measure $\mathbf{Q}$ given by

$$\mathbf{Q}(A) = \int_A \mathrm{e}^{-(X+Y)}\,\frac{1}{f(X,Y)}\,d\mathbf{P}$$

for measurable subsets A of Ω.

Task 6. See Stig Larsson's lecture notes.