MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2012-09-01 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Oskar Hamlet, 0703-088304

Note:	Write your name and personal number on the cover.
	Write your code on every sheet you hand in.
	Only write on one page of each sheet. Do not use red pen.
	Do not answer more than one question per page.
	State your methodology carefully. Write legibly.
	Questions are not numbered by difficulty.
	Sort your solutions by the order of the questions.
	Mark on the cover the questions you have answered.
	Count the number of sheets you hand in and fill in the number on the cover.
	To pass requires 10 points.

1. Show that the following boundary value problem

$$
\begin{aligned}
&u''(x) + u'(x) + 2\arctan(u^2(\sqrt{x})) = 0, \quad 0 \le x \le 1,\\
&u(0) = u(1) = 1,\\
&u \in C^2([0,1])
\end{aligned}
$$

has a unique solution.

(4p)

2. Consider the linear mapping $S : l^2 \to l^2$ defined by

$$S(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

Is S injective? Is S surjective? Calculate the limits $\lim_{n\to\infty} \|S^n \mathbb{x}\|_{l^2}$, where $\mathbb{x} = (x_1, x_2, x_3, \ldots) \in l^2$, and $\lim_{n\to\infty} \|S^n\|$.

(4p)

3. Set $M = \{f \in L^2(0,1) : \int_0^1 f(x)\,dx = 1\}$. Show that

   (a) M is a closed convex subset of $L^2(0,1)$,

   (b) M has a unique element f_M with minimal norm. Calculate this element.

   Finally, what can be said (concerning the statements (a) and (b) above) if $L^2(0,1)$ is replaced by $L^1(0,1)$?

(4p)

4. Formulate the *Method of continuity.* Prove the statement.

(5p)

5. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $A \in \mathcal{B}(H,H)$. Define the adjoint operator A^*, show that it is a uniquely defined mapping in $\mathcal{B}(H,H)$ and that $\|A^*\| = \|A\|$.

(4p)

6. Let $T : \{\mathbb{x} \in \mathbb{R}^n : \|\mathbb{x}\| \leq 1\} \to \mathbb{R}^n$ be a continuous mapping. Moreover assume that $\langle T(\mathbb{x}), \mathbb{x}\rangle > 0$ for all $\mathbb{x}$ with $\|\mathbb{x}\| = 1$. Here $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{R}^n$ with the induced norm $\|\cdot\|$. Prove[1] that there exists a $\mathbb{x}_0 \in \{\mathbb{x} \in \mathbb{R}^n : \|\mathbb{x}\| \leq 1\}$ such that $T(\mathbb{x}_0) = \mathbb{0}$.

(4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK

[1]Hint: Consider the mapping $G(\mathbb{x}) = \mathbb{x} - \epsilon\, T(\mathbb{x})$ for some properly choosen $\epsilon > 0$.