MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2011-10-19 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Adam Andersson, 0703-088304

Note:	Write your name and personal number on the cover.
	Write your code on every sheet you hand in.
	Only write on one page of each sheet. Do not use red pen.
	Do not answer more than one question per page.
	State your methodology carefully. Write legibly.
	Questions are not numbered by difficulty.
	Sort your solutions by the order of the questions.
	Mark on the cover the questions you have answered.
	Count the number of sheets you hand in and fill in the number on the cover.
	To pass requires 10 points.

1. Show that the following boundary value problem

$$\begin{aligned}&u''(x) + u'(x) + 2\arctan(u^2(\sqrt{x})) = 0, \quad 0 \le x \le 1,\\ &u(0) = u(1) = 1,\\ &u \in C^2([0,1])\end{aligned}$$

has a unique solution.

(4p)

2. Set

$$Tf(x) = \int_0^1 \sinh(x-t) f(t)\,dt, \quad 0 \le x \le 1.$$

Show that T is a linear bounded and compact operator when T is considered as an operator on the Banach spaces

(a) $C([0,1])$
(b) $L^2([0,1])$

respectively (with the standard norms). Also calculate the operator norms.

(4p)

P.T.O

3. Let $T : \{\mathbb{x} \in \mathbb{R}^n : \|\mathbb{x}\| \leq 1\} \to \mathbb{R}^n$ be a continuous mapping. Moreover assume that $\langle T(\mathbb{x}), \mathbb{x}\rangle > 0$ for all $\mathbb{x}$ with $\|\mathbb{x}\| = 1$. Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$ with the induced norm $\|\cdot\|$. Prove[1] that there exists a $\mathbb{x}_0 \in \{\mathbb{x} \in \mathbb{R}^n : \|\mathbb{x}\| \leq 1\}$ such that $T(\mathbb{x}_0) = \mathbb{0}$.

(4p)

4. Formulate the *Method of continuity*. Prove the statement.

(5p)

5. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $A \in \mathcal{B}(H, H)$. Define the adjoint operator A^*, show that it is a uniquely defined mapping in $\mathcal{B}(H, H)$ and that $\|A^*\| = \|A\|$.

(4p)

6. Show that it is impossible to equip $C([0, 1])$ with an inner product in such a way that the norm induced by the inner product is equal to the standard norm $\|f\| = \max_{x \in [0,1]} |f(x)|$ for $f \in C([0, 1])$.

(4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK

[1]Hint: Consider the mapping $G(\mathbb{x}) = \mathbb{x} - \epsilon\, T(\mathbb{x})$ for some properly choosen $\epsilon > 0$.

Lösningsförslag (prel.) till TMA401/MMA400 19/10 2011

1) $\begin{cases} u''(x) + u'(x) + 2\arctan(u^2(\sqrt{x})) = 0, & 0 \le x \le 1 \\ u(0) = u(1) = 1, \ u \in C^2([0,1]) \end{cases}$

Show existence of unique solution

Sol: Set $u(x) = 1 + v(x)$ to obtain homogeneous BC

$(*) \begin{cases} v'' + v' \equiv Lv = -2\arctan((1+v)^2(\sqrt{\cdot})) & \text{on } [0,1] \\ v(0) = v(1) = 0 \end{cases}$

Step 1: Calculation of $g(x,t) = e(x,t)\Theta(x-t) + b_1(t)u_1(x) + b_2(t)u_2(x)$

where $e(x,t) = a_1(t)u_1(x) + a_2(t)u_2(x)$, $e(t,t) = 0$, $e_x'(t,t) = 1$

and $u_1(x) = 1$, $u_2(x) = e^{-x}$, and $g(0,t) = g(1,t) = 0$, $0 < t < 1$.

This gives $a_1(t) = 1$, $a_2(t) = -e^t$, $b_1(t) = \frac{e^t - e}{e-1} = -b_2(t)$

Set $T(v)(x) = \int_0^1 g(x,t)\left(-2\arctan((1+v)^2(\sqrt{t})\right)dt$, $x \in [0,1]$.

Then $T: C([0,1]) \to C([0,1])$, where $C([0,1])$ have the norm $\|f\| = \max_{x\in[0,1]} |f(x)|$. Here $(C([0,1]), \|\cdot\|)$ is a Banach space. Every solution to $(*)$ corresponds to a fixed point for T and vice versa, so it remains to show that T has a unique fixed point. It is enough to prove that T is a contraction on $C([0,1])$ since then Banach's fixed point theorem implies that T has a unique fixed point. For $v, \tilde{v} \in C([0,1])$ we have

$|T(v)(x) - T(\tilde{v})(x)| \le \int_0^1 |g(x,t)| \, |\arctan((1+v(\sqrt{t}))^2 - $
$- \arctan((1+\tilde{v}(\sqrt{t}))^2)| \, dt$

By the mean value theorem $|\arctan((1+a)^2) - \arctan((1+b)^2)| \le$
$\le \max\left\{ \left| \frac{2(1+\xi)}{1+(1+\xi)^4} \right| : \xi \text{ between } a \text{ and } b \right\} |a-b| \le \ldots \le$
$\le \frac{3}{2}|a-b|$

Hence $|T(v)(x) - T(\tilde{v})(x)| \le \frac{2}{3}\int_0^1 |g(x,t)|\,dt \, \|v - \tilde{v}\|$.

Moreover $g(x,t) \le 0$ all $x, t \in [0,1]$ and

$\int_0^1 g(x,t)(-1)\,dt$ satisfies $h''(x) + h'(x) = -1$, $h(0) = h(1) = 0$.

Calculation (see also lecture notes) yields $0 \le h(x) \le \frac{1}{e(e-1)}$

Hence $\|T(v) - T(\tilde{v})\| \leq \frac{3}{2}\frac{1}{e(e-1)} \|v - \tilde{v}\|$ all $v, \tilde{v} \in C([0,1])$.

Here $\frac{3}{2}\frac{1}{e(e-1)} < \frac{3}{4}$ and we have shown that T is a contraction on $(C([0,1]), \|\cdot\|)$.

2) $T(f)(x) = \int_0^1 \sinh(x-t) f(t)\,dt$, $0 \leq x \leq 1$

T bounded linear and compact on

1) $C([0,1])$ follows by Arzela-Ascoli thm

2) $L^2([0,1])$ follows since $\sinh(x-t) \in C([0,1]\times[0,1])$.

$\|T\|_{C([0,1]) \to C([0,1])} = \dots = \int_0^1 \sinh(t)\,dt = \frac{1}{2}(e - 2 + \frac{1}{e})$

$\|T\|_{L^2([0,1]) \to L^2([0,1])}$: Since iT is self-adjoint we have that

$\|T\| = \sup\{|\lambda| : \lambda \text{ eigenvalue for } T\}$

Since $T(f)(x)$ is a linear combination of e^x and e^{-x} an eigenfunction must be given by $ae^x + be^{-x}$.

Calculation of $T(ae^{\cdot} + be^{-\cdot})(x) = \lambda(ae^x + be^{-x})$ gives

$$\frac{a}{2} - \frac{b}{4}\left(\left(\tfrac{1}{e}\right)^2 - 1\right) = \lambda a \quad , \quad -\frac{a}{4}(e^2-1) - \frac{b}{2} = \lambda b$$

and hence there exists a non trivial solution iff

$$\left(\tfrac{1}{2} - \lambda\right)\left(-\tfrac{1}{2} - \lambda\right) - \tfrac{1}{4}\left(\left(\tfrac{1}{e}\right)^2 - 1\right)\tfrac{1}{4}(e^2-1) = 0$$

i.e. $\lambda^2 = \frac{1}{4}\left[1 + \frac{1}{4}\left(\left(\tfrac{1}{e}\right)^2 - 1\right)(e^2-1)\right]$

i.e. $\lambda = \pm\frac{1}{2}\sqrt{1 + \frac{1}{4}\left(\left(\tfrac{1}{e}\right)^2 - 1\right)(e^2-1)}$

$$\|T\|_{L^2([0,1]) \to L^2([0,1])} = \frac{1}{2}\sqrt{\frac{3}{2} - \frac{1}{4}\left(e^2 + \left(\tfrac{1}{e}\right)^2\right)}$$

3) $T : B \equiv \{x \in \mathbb{R}^n : \|x\| \leq 1\} \to \mathbb{R}^n$ continuous
$\langle T(x), x\rangle > 0$ all $\|x\| = 1$
$\Rightarrow$

there exists $x_0 \in B$ s.th. $T(x_0) = 0$.

$\{x \in \mathbb{R}^n \ \|x\| = 1\}$ is a compact set in $\mathbb{R}^n$ and $\langle T(x), x\rangle > 0$ on $\|x\| = 1$ implies that $\langle T(x), x\rangle \geq K > 0$ all $\|x\| = 1$. Then there is a $R \in (0,1)$ s.th

$\langle T(x), x\rangle \geq \frac{K}{2}$ all $\|x\| \in [R, 1]$.

Set $G(x) = x - \varepsilon T(x)$ for some $\varepsilon > 0$ to be chosen

$\|G(x)\|^2 = \|x - \varepsilon T(x)\|^2 = \|x\|^2 - 2\varepsilon \langle T(x), x\rangle + \varepsilon^2 \|T(x)\|^2$

$T: B \to \mathbb{R}^n$ continuous and B compact implies

$\|T(x)\| \le M$ for all $x \in B$ for some $M > 0$.

$\|x\| \in [0, R]$: $\|G(x)\|^2 \le \|x\|^2 + 2\varepsilon M \|x\| + \varepsilon^2 M^2 \le$

$$\le R^2 + 2\varepsilon M R + \varepsilon^2 M^2$$

$\|x\| \in [R, 1]$: $\|G(x)\|^2 = \|x\|^2 - 2\varepsilon \frac{K}{2} + \varepsilon^2 M^2 \le 1 - \varepsilon K + \varepsilon^2 M^2$

Now choose $\varepsilon > 0$ so small such that

$$\max\left(R^2 + 2\varepsilon M R + \varepsilon^2 M^2,\ 1 - \varepsilon K + \varepsilon^2 M^2\right) \le 1$$

Then $G: B \to B$ is continuous. Brouwer's fixed point theorem implies the existence of an $x_0 \in B$ s.th. $G(x_0) = x_0$.

This implies $\varepsilon T(x_0) = 0$ i.e. $T(x_0) = 0$.

4&5) See textbook + lecture notes

6) Use the fact that a norm induced by an inner product satisfies the //-law. Set e.g. $f(x) = x$ and $g(x) = 1 - x$ and calculate $\|f \pm g\|$, $\|f\|$, $\|g\|$. Here

$$\|f+g\|^2 + \|f-g\|^2 \ne 2(\|f\|^2 + \|g\|^2).$$