MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet
Extra Tentamen i
Funktionalanalys ENM, TMA401/ Tillämpad funktionalanalys GU, MMA400,
DATUM 2010-10-27, TID 8.30-13.30
Inga hjälpmedel, förutom penna och linjal, är tillåtna, ej heller räknedosa.
Telefonvakt: Peter Kumlin, 0739-603800

OBS: Ange linje samt personnummer och namn på omslaget.
Ange kod på *varje* inlämnat blad.
Motivera dina svar väl. Det är i huvudsak beräkningarna och motiveringarna som ger poäng, inte svaret. Skriv tydligt.
För godkänt krävs minst 10 poäng sammanlagt.

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1. Prove the existence and uniqueness of solution to the following boundary value problem:

$$\begin{cases} -((1+x)u'(x))' = \arctan u(x), \quad 0 \le x \le 1 \\ u(0) = 1, u(1) = 0, \quad u \in C^2([0,1]) \end{cases}$$

(4p)

2. Let $(E, \|\cdot\|)$ be a Banach space and $T \in \mathcal{B}(E,E)$ with the property $\Sigma_{n=1}^{\infty} \|T^n(x)\| < \infty$ for all $x \in E$. Let $x_0, y \in E$ and set $x_{n+1} = y + T(x_n),\ n = 0, 1, 2, \ldots$. Show that

   (a) $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in E, and that

   (b) there is a unique $x \in E$ such that $x = y + T(x)$.

   Moreover show that if $\|T\| < 1$ then the equation $x = y + T(x)$ has a unique solution $x \in E$ for every $y \in E$.

(4p)

3. Let $(f_n)_{n=1}^{\infty}$ be a sequence in $C([0,1])$ such that $|f_n(x)| \le 1$ for all n. Show that there exists a $g \in C([0,1])$ and a subsequence $(f_{n_k})_{k=1}^{\infty}$ of $(f_n)_{n=1}^{\infty}$ such that

$$\lim_{k\to\infty} \sup_{x\in[0,1]} |\int_0^1 e^{\sin(x^2+y)} f_{n_k}(y)\, dy - g(x)| = 0.$$

(4p)

P.T.O

4. State and prove the Hilbert-Schmidt theorem. Propositions that are used in the proof should be properly stated but need not be proven.

(5p)

5. State and prove the Riesz lemma. Show that the closed unit ball $\{x \in E : \|x\| \leq 1\}$ in the normed space $(E, \|\cdot\|)$ is not compact if E is infinite-dimensional.

(4p)

6. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $T : H \to H$ a linear mapping with the property

$$\langle T(x), y \rangle = \langle x, T(y) \rangle \text{ for all } x, y \in H.$$

Show that T is a bounded linear mapping on H.

(4p)

Information om när tentan är färdigrättad och tid för visning av tentan hos föreläsaren kommer att lämnas på kurshemsidan. När resultaten är registrerade i Ladok kommer ett e-brev.

LYCKA TILL! PK

Solutions to written exam TMA401/MMA400 (2010-10-27)

① See exam 2010-10-20

② $(E, \|\cdot\|)$ Banach space, $T \in \mathcal{B}(E,E)$ where $\sum_{n=1}^{\infty} \|T^n(x)\| < \infty$ for all $x \in E$.

1) $x_0, y \in E$ and set $x_{n+1} = y + T(x_n)$, $n = 0, 1, 2, \ldots$

Show that

a) $(x_n)_{n=1}^{\infty}$ Cauchy sequence in $(E, \|\cdot\|)$

b) there is a unique $x \in E$ s.th. $x = y + T(x)$.

solution: For $n > m$ we obtain

$$\|x_n - x_m\| = \|(y + T(y) + \ldots + T^{n-1}(y) + T^n(x_0)) - (y + \ldots + T^{m-1}(y) + T^m(x_0))\|$$

$$= \|\sum_{k=m}^{n-1} T^k(y) + T^n(x_0) - T^m(x_0)\| \le \sum_{k=m}^{n-1} \|T^k(y)\| + \|T^n(x_0)\| + \|T^m(x_0)\| \to 0, \quad n, m \to \infty$$

since $\sum_{k=1}^{\infty} \|T^k(y)\| < \infty$ and $\sum_{k=1}^{\infty} \|T^k(x_0)\| < \infty$

$(E, \|\cdot\|)$ Banach space implies $x_n \to x$ in E for some $x \in E$. But also $T(x_n) \to T(x)$ since $T \in \mathcal{B}(E,E)$. Hence we get

$$\begin{array}{ccc} x_{n+1} & = & y + T(x_n) \\ \downarrow & \| & \downarrow \\ x & = & y + T(x) \end{array}$$

, i.e. there is a solution to $x = y + T(x)$ for each $y \in E$.

Assume that $x = y + T(x)$ and $\tilde{x} = y + T(\tilde{x})$.

Then $x - T(x) = \tilde{x} - T(\tilde{x})$ and hence $x - \tilde{x} = T(x - \tilde{x})$.

But $\sum_{k=1}^{\infty} \|T^k(x - \tilde{x})\| < \infty$ and $x - \tilde{x} = T^k(x - \tilde{x})$ for all $k \in \mathbb{Z}_+$ $\Rightarrow$ $x - \tilde{x} = 0$, i.e. the solution is unique.

2) $\|T\| < 1$ Show that the equation has a unique solution:

Solution Either observe that $\sum_{k=1}^{\infty} \|T^k x\| \le \sum_{k=1}^{\infty} \|T\|^k \|x\| < \infty$ for all $x \in E$ and use 1) or use Banach's fixed point theorem

③ $(f_n)_{n=1}^{\infty}$ sequence in $C([0,1])$ with $|f_n(x)| \le 1$ all n. Show that there exists a $g \in C([0,1])$ and a subsequence $(f_{n_k})_{k=1}^{\infty}$ of $(f_n)_{n=1}^{\infty}$ such that

$$\lim_{k\to\infty} \sup_{x\in[0,1]} \left| \int_0^1 e^{\sin(x^2+y)} f_{n_k}(y)\,dy - g(x) \right| = 0$$

Solution:

Consider $\left(\int_0^1 e^{\sin(x^2+y)} f_n(y)\,dy \right)_{n=1}^{\infty}$ in $C([0,1])$ and apply the Arzela-Ascoli thm. Straight forward calculations give that the sequence is uniformly bdd and equicontinuous. The conclusion follows

④, ⑤ see textbook

⑥ Compare the proof of the theorem: Every weakly converging sequence in a Hilbert space is bdd.