Exam in the course Plasma Physics with Applications, RRY085

2012-10-26, 08:30

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Questions may be asked at 9:30 and 11:30.

Total number of points: 40

Grading: 20p → Mark 3, 26p → Mark 4, 32p → Mark 5

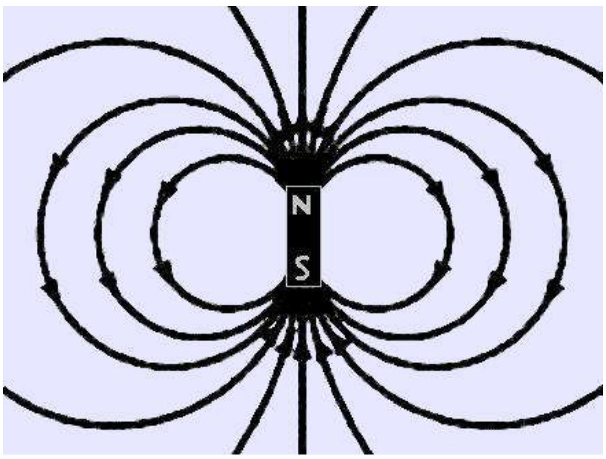
Allowed gadgets: BETA, Physics Handbook, Calculator, one double sided A4 sheet with arbitrary information.

Problem 1: Single Particle Motion (6p)

1. (2p) Under the condition that the Larmor radius is much smaller than the length scale of inhomogeneities in the magnetic field, the first adiabatic invariant

$$\mu = W_{\perp} / B$$

is a constant of motion. Here, $W_{\perp}$ is the kinetic energy in the direction perpendicular to the magnetic field. What does the conservation of μ mean for charged particles moving in the magnetic dipole field below?



2. (2p) A simple expression for the dipole field around a sphere of radius R (e.g. a planet) is given by

$$\mathbf{B}_d(r,\theta) = -B_0\left(\frac{R}{r}\right)^3\left[2\cos(\theta)\hat{r} - \sin(\theta)\hat{\theta}\right].$$

Here, B_0 is a constant and θ is the angle between an axis going through the poles of the dipole and the radius vector $\mathbf{r} = r\,\mathbf{e}_r$, drawn from the center of the dipole to some point outside. If a particle passes the equator, $\theta = \pi/2$, at radius r_0 with perpendicular energy $W_{\perp 0}$ and parallel energy $W_{||0}$, at what angle does the parallel energy reach zero? For simplicity, assume that the parallel energy is very small, so that r can be assumed to be constant.
3. (2p) In addition to the motion addressed in the two questions above, what more motion will the particles perform in the field of this dipole?

Problem 2: Ideal MHD Equilibrium (8p)

1. (4p) Describe briefly the geometry, fields and currents of the Z- and θ-pinches. Discuss how the two configurations pinch the plasma to make the density peak on the cylindrical axis.
2. (2p) Derive the pressure balance equation for the θ-pinch: Use the ideal MHD equilibrium equations

$$\mu_0\mathbf{J} = \nabla\times\mathbf{B} \quad , \qquad \mathbf{J}\times\mathbf{B} = \nabla P \; .$$

3. (2p) Sketch and discuss plausible radial profiles for P, $\mathbf{J}$ and $\mathbf{B}$ in the θ-pinch, assuming that P peaks on the cylindrical axis.

Problem 3: Diffusion (12p)

1. (2p) The diffusion coefficient in weakly ionized gases is roughly $D \approx \upsilon_c \lambda_m^2$, where υ_c is the collision frequency and λ_m the mean free path between collisions. Based on this formula, what would you expect the diffusion coefficient to be in a weakly ionized gas with a strong magnetic field applied?
2. (2p) When the gas becomes ionized to a high degree the diffusion is no longer "free", but instead becomes "ambipolar". What is the reason for this, and what is the effect?
3. (2p) Explain why the electron-ion collision frequency in a plasma is so much higher than the ion-electron collision frequency.
4. (4p) A fully realistic system for creating a weakly ionized gas is as follows: a hollow very long cylinder filled with some gas, begin irradiated by a strong laser or some radioactive material. Neutral molecules and atoms

will be ionized inside the cylinder, and diffuse to the walls where, due to surface processes, they will recombine very rapidly. The electron density can be described using

$$\frac{\partial n}{\partial t} - D\nabla^2 n = S$$

where S is the constant source term due to the ionizing radiation. Solve this differential equation for the electron density as a function of radius in the static case using the boundary condition $n(r = R) = 0$.

5. (2p) For a rough estimate of the diffusion rate one would use

$$\nabla^2 n \approx -\frac{n(r=0)}{R^2}$$

Based on the solution in the previous question, determine the accurate value for the diffusion rate and compare it with the approximation. How much do they differ?

Problem 4: Kinetic Theory for Fusion Born Alpha Particles (4p)

In fusion experiments, alpha particles are produced whenever D and T fuse. However, the alpha particles are born with high kinetic energy (roughly 3.5 MeV), and are therefore not in thermodynamic equilibrium with the other plasma species. As a result, a proper treatment of fusion born alpha particles requires solving the kinetic equation

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\cdot\frac{\partial f_\alpha}{\partial \mathbf{x}} + \mathbf{a}\cdot\frac{\partial f_\alpha}{\partial \mathbf{v}} = \frac{S}{4\pi v_\alpha^2}\delta\left(v - v_\alpha\right) + \left.\frac{\partial f_\alpha}{\partial t}\right|_{Collisions}$$

where the first term on the right hand side represents a particle source that produces alphas with velocity v_α (δ is the Dirac delta-function) at rate S (particles per m^3) and the second term represents the time rate of change of the alpha distribution due to Coulomb collisions. In a tokamak, it can be shown that the second and third terms on the left hand side do not contribute significantly, and can therefore be averaged out. Moreover, for high energy alpha particles, the collision operator becomes a particularly simple differential operator that describes the slowing down of the alpha particles due to drag on the thermal electrons and ions (all other types of collisions are negligible for alpha particles). The resulting kinetic equation reads

$$\frac{\partial f_\alpha}{\partial t} = \frac{1}{\tau_S}\frac{1}{v^2}\frac{\partial}{\partial v}\left[\left(v^3 + v_c^3\right) f_\alpha(v)\right] + \frac{S}{4\pi v_\alpha^2}\delta\left(v - v_\alpha\right)$$

where τ_S is a constant known as the Spitzer slowing down time (which is the characteristic time it takes for the alphas to slow down and thermalize due to collisional drag).

1. (4p) Derive an expression for the steady state form of the alpha particle distribution, $f_\alpha(v)$, valid for $v < v_\alpha$, assuming that $f_\alpha = 0$ for $v > v_\alpha$!

Hint: Remember how a delta function works,

$$\int_{x_1}^{x_2} f(x)\delta(x-x_0)\,dx = f(x_0)\ ,$$

when $x_1 < x_0 < x_2$.

Problem 5: Cold Plasma Waves (10p)

1. (2p) Use Maxwell's equation to derive the wave equation

$$\vec{\varepsilon}\cdot\mathbf{E}+\frac{c^2}{\omega^2}\mathbf{k}\times(\mathbf{k}\times\mathbf{E})=0$$

for a static and uniform, infinite plasma. Here, the dielectric tensor is defined as

$$\vec{\varepsilon}(\omega,\mathbf{k})=\vec{\mathbf{I}}+i\frac{\vec{\sigma}(\omega,\mathbf{k})}{\varepsilon_0\omega}\ ,$$

and the conductivity tensor relates the Fourier transformed current to the Fourier transformed electric field:

$$\hat{\mathbf{J}}(\omega,\mathbf{k})=\vec{\sigma}(\omega,\mathbf{k})\cdot\hat{\mathbf{E}}(\omega,\mathbf{k})\ .$$

Assume that **E** varies as $\exp\left[i(\mathbf{k}\cdot\mathbf{r}-\omega t)\right]$.

2. (2p) Simplify the wave equation for the two cases of longitudinal (**k** parallel to **E**) and transverse (**k** perpendicular to **E**) waves!
3. (4p) Linearize the two-fluid equations and compute the conductivity tensor for a cold, unmagnetized plasma. Assume that all perturbations vary as $\exp\left[i(\mathbf{k}\cdot\mathbf{r}-\omega t)\right]$ and that there is no equilibrium flow, $\mathbf{v}_{\alpha 0} = 0$. Derive and solve the dispersion relations for both the transverse and the longitudinal wave.
4. (2p) Discuss how the dispersion relations change when finite temperature terms are retained.

Formulas that you might need

Maxwells equations (which you should btw know by heart…):

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}\,, \quad \nabla \cdot \mathbf{B} = 0\,, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,, \quad \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$

Plasma frequency for species α:

$$\omega_{p\alpha}^2 = \frac{n_\alpha q_\alpha^2}{m_\alpha \varepsilon_0}$$

Fluid equations for species α:

$$m_\alpha \frac{\partial n_\alpha}{\partial t} + \nabla \cdot \left(n_\alpha \mathbf{v}_\alpha \right) = 0$$

$$n_\alpha m_\alpha \left(\frac{\partial \mathbf{v}_\alpha}{\partial t} + \left(\mathbf{v}_\alpha \cdot \nabla \right) \mathbf{v}_\alpha \right) = q_\alpha n_\alpha \left(\mathbf{E} + \mathbf{v}_\alpha \times \mathbf{B} \right) - \nabla P_\alpha$$

$$\frac{d}{dt}\left(\frac{P_\alpha}{n_\alpha^{\gamma_\alpha}} \right) = 0$$

Useful vector relation:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Cylindrical coordinates:

$$\nabla f = \frac{\partial f}{\partial R}\hat{R} + \frac{1}{R}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R}\frac{\partial}{\partial R}\left(R A_R \right) + \frac{1}{R}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \left(\frac{1}{R}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right)\hat{R} + \left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R} \right)\hat{\phi} + \frac{1}{R}\left(\frac{\partial}{\partial R}\left(R A_\phi \right) - \frac{\partial A_R}{\partial \phi} \right)\hat{z}$$

Spherical coordinates:

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$$

(1,1)

Problem 1

1) The conservation of μ states that the perpendicular energy, $W_\perp$, grows when the field, B, becomes stronger. Since the total energy is conserved, and the field does no work, this means that the parallel energy will decrease as the particle moves into a stronger field. In a dipole field the field strength increases at the poles. This will in turn mean that particles having energy in a certain range can be trapped in an orbit going up and down between the two poles.

2) Conservation of energy: (1, 2)

$$W_{\perp 0} + W_{\parallel 0} = W_{\perp} + W_{\parallel}$$

but we seek the point where $W_{\parallel} = 0$

so

$$W_{\perp 0} + W_{\parallel 0} = W_{\perp} \qquad (1)$$

Futhermore μ is constant so

$$\mu = \frac{W_{\perp}}{B} = \frac{W_{\perp 0}}{B_{\theta = \frac{\pi}{2}}} \qquad (2)$$

The field at $\theta = \frac{\pi}{2}$ is

$$\bar{B}_{\theta = \frac{\pi}{2}} = -B_0 \left(\frac{R}{r}\right)^3 \left(2\cos\frac{\pi}{2}\,\hat{r} - \sin\frac{\pi}{2}\,\hat{\theta}\right)$$

$$= B_0 \left(\frac{R}{r}\right)^3 \hat{\theta}$$

But we assume $r \simeq \text{constant} = r_0$, so

$$B_{\theta = \frac{\pi}{2}} = B_0 \left(\frac{R}{r_0}\right)^3 \qquad (3)$$

Combining (1), (2) & (3) gives

$$B(W_{\parallel} = 0) = \left(1 + \frac{W_{\parallel 0}}{W_{\perp 0}}\right) B_0 \left(\frac{R}{r_0}\right)^3 \qquad (4)$$

The magnitude of the field is (1,3)

$$|\bar{B}| = B = B_0 \left(\frac{R}{r_0}\right)^3 \sqrt{4\cos^2\theta + \sin^2\theta} =$$

$$= B_0 \left(\frac{R}{r_0}\right)^3 \sqrt{3\cos^2\theta + 1}$$

combine with (4)

$$\Rightarrow B_0 \left(\frac{R}{r_0}\right)^3 \sqrt{3\cos^2\theta + 1} = \left(1 + \frac{W_{110}}{W_{\perp 0}}\right) B_0 \left(\frac{R}{r_0}\right)^3$$

$$\Leftrightarrow \quad 3\cos^2\theta + 1 = \left(1 + \frac{W_{110}}{W_{\perp 0}}\right)^2$$

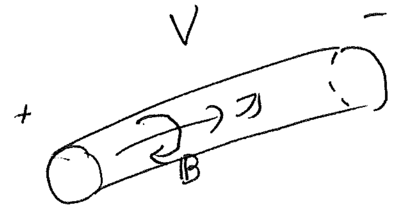
$$\cos^2\theta = \frac{1}{3}\left\{\left(1 + \frac{W_{110}}{W_{\perp 0}}\right)^2 - 1\right\}$$

$$\theta = \begin{cases} \arccos\left\{\sqrt{\frac{1}{3}\left(\left(1 + \frac{W_{110}}{W_{\perp 0}}\right)^2 - 1\right)}\right\} \\ \pi - \arccos\left\{\sqrt{\frac{1}{3}\left(1 + \frac{W_{110}}{W_{\perp 0}}\right)^2 - 1}\right\} \end{cases}$$

3) There is a gradient of the field in the $\hat{r}$ and $\hat{\theta}$ -directions. (1,4)
This will lead to so-called Grad-B drift and Curvature drift.
If the dipole is in fact a very massive object, like a planet, we will also have a drift caused by the gravitational field.

Problem 2: Ideal MHD Equilibrium

7. Z-pinch : A current is driven in a cylindrical plasma via an applied voltage. The current, which runs along the plasma column, induces a magnetic field in the azimuthal direction.



If we adopt cylindrical coords, then

$$\mathbb{J} = J_z(r)\,\hat{z}$$

$$\mathbb{B} = B_\theta(r)\,\hat{\theta}$$

and

$$\vec{\nabla} P = \mathbb{J} \times \mathbb{B} = -J_z B_\theta\,\hat{r} = \frac{dP}{dr}\,\hat{r}$$

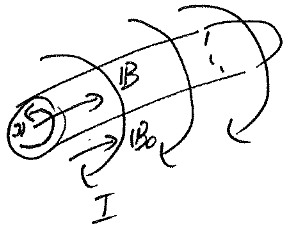
$\Longrightarrow$ Confinement in the radial direction if $J_z, B_\theta > 0$!

Note: $B_\theta(r=0) = 0$

$$\Longrightarrow \left.\frac{dP}{dr}\right|_{r=0} = 0$$

$\longrightarrow$ Density peak on axis!

θ - pinch : A current is driven azimuthally around a cylindrical plasma column. In the absence of plasma, the induced magnetic field is



$$\mathbb{B}_0 = B_0 \hat{z}$$

with B_0 const. When the plasma is there, the field $\mathbb{B}_0$ gives rise to an ambipolar, azimuthal plasma current that induces a field that counteracts $\mathbb{B}_0$. The total $\mathbb{B} = B_z(r)\hat{z}$ inside the plasma is therefore lower than $\mathbb{B}_0$. In total, in the plasma column,

$$\mathbb{B} = B_z(r)\hat{z}$$

$$\mathbb{J} = -J_\theta(r)\hat{\theta}$$

which gives

$$\vec{\nabla} P = \mathbb{J} \times \mathbb{B} = -J_\theta B_z \hat{r} = \frac{dP}{dr}\hat{r}$$

$$\Rightarrow \frac{dP}{dr} < 0$$

⇝ Confinement in radial direction!

Note: $J_\theta(r=0) = 0$

$$\Rightarrow \left.\frac{dP}{dr}\right|_{r=0} = 0$$

⇝ Density peaks on axis!

2. Let's derive a general expression for the radial pressure balance in cylindrically symmetric plasma columns, and only then specialize to the case of a θ-pinch!

Eqs :

$$\mathbb{J} \times \mathbb{B} = \vec{\nabla} P \,, \quad \mu_0 \mathbb{J} = \vec{\nabla} \times \mathbb{B} \,, \quad \vec{\nabla} \cdot \mathbb{B} = 0$$

Cylindrical symmetry

$$\longrightarrow \frac{\partial}{\partial \theta} = 0$$

Assumption : Infinite columns

$$\longrightarrow \frac{\partial}{\partial z} = 0$$

Hence $\mathbb{J} = \mathbb{J}(r)$, $\mathbb{B} = \mathbb{B}(r)$ and $P = P(r)$, and

$$\frac{\partial}{\partial r} = \frac{d}{dr}$$

Note : $P = P(r)$ gives

$$\vec{\nabla} P = \frac{dP}{dr} \hat{r}$$

so that

$$(\mathbb{J} \times \mathbb{B}) \cdot \mathbb{B} = 0 = \mathbb{B} \cdot \vec{\nabla} P = \frac{dP}{dr} B_r$$

$$\longrightarrow B_r = 0$$

In total, then,

$$\mathbb{B} = B_\theta(r)\, \hat{\theta} + B_z(r)\, \hat{z}$$

This gives

$$\mathbb{J} = \frac{1}{\mu_0} \left[-\frac{dB_z}{dr} \hat{\theta} + \frac{1}{r} \frac{d}{dr} (r B_\theta) \hat{z} \right]$$

and

$$\nabla P = \mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0}\left[-\frac{dB_z}{dr}\hat{\theta} + \frac{1}{r}\frac{d}{dr}(rB_\theta)\hat{z}\right] \times \left[B_\theta \hat{\theta} + B_z \hat{z}\right] = -\frac{1}{\mu_0}\left[B_z \frac{dB_z}{dr}\hat{r} + \frac{B_\theta}{r}\frac{d}{dr}(rB_\theta)\hat{r}\right] = \frac{dP}{dr}\hat{r}$$

That is

$$\frac{dP}{dr} = -\frac{1}{2\mu_0}\left[\frac{dB_z^2}{dr} + \frac{dB_\theta^2}{dr}\right] - \frac{B_\theta^2}{\mu_0 r}$$

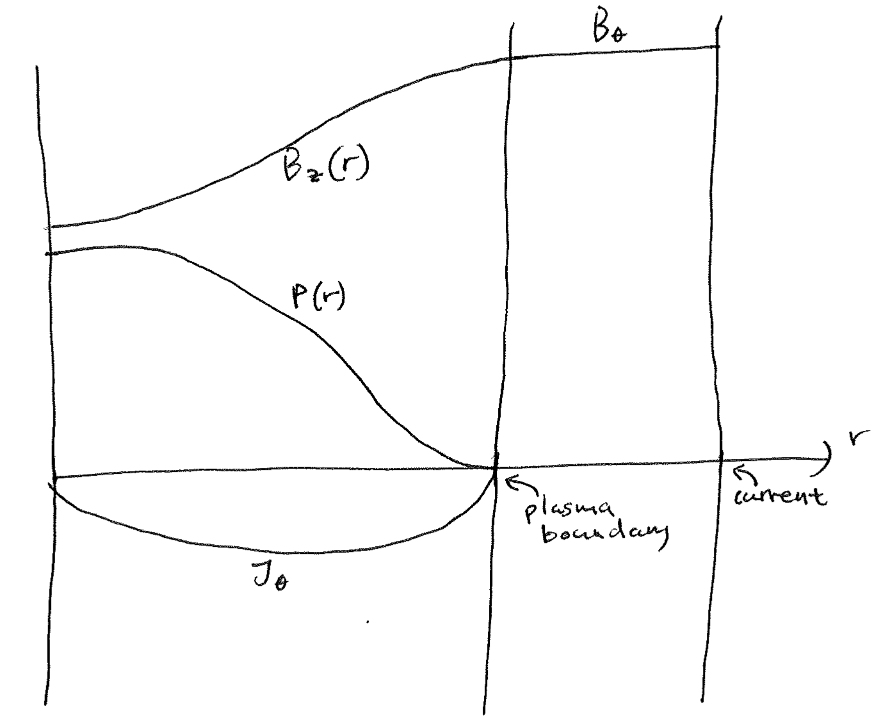
which gives

$$\frac{d}{dr}\left[P + \frac{B^2}{2\mu_0}\right] = -\frac{B_\theta^2}{\mu_0 r}$$

<u>θ- pinch</u> : $B_\theta = 0$, $B = B_z$

$$\Rightarrow \boxed{\frac{d}{dr}\left[P + \frac{B_z^2}{2\mu_0}\right] = 0}$$

<u>3. Profiles</u> :



Problem 3 (3,1)

1) If the magnetic field can be considered as strong it means that the particles are bound to gyrate around the fiel lines. Thus, the maximum distance that the particle center of mass can move in a collision is $2r_L$, where r_L is the larmor radius. Instead of λ_m we should then use r_L

$$\Rightarrow \quad D_B \simeq \nu_c (2r_L)^2 \simeq \nu_c r_L^2$$

2) The reason is that electrons diffuse much faster than ions, which leads to charge separation and a resulting field. The effect is that the electrons are slowed down, and the ions speed up, in their respective diffusion rates.

(3,2)

3) This is simply a question of what we mean by collision frequency. For an electron to suffer a change in momentum, due to a collision with an ion, that is as large as the actual momentum of the electron, it only takes one collision. Hence the collision frequency for the electron is simply equal to the number of times per second an electron collides with an ion.

The ions being much heavier need to suffer thousands of collisions with electrons to suffer significant momentum change, and the corresponding collision frequency is much lower, and not equal to the number of times per second the ion actually collides with electrons.

4) We solve the equation in the (3,3) static case, i.e. $\frac{\partial n}{\partial t} = 0$

So
$$\nabla^2 n = -\frac{S}{D}$$

but in cylindrical geometry
$$\nabla^2 n = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 n}{\partial \phi^2} + \frac{\partial^2 n}{\partial z^2}$$

but the cylinder is very long so we neglect $\frac{\partial^2 n}{\partial z^2}$, and there is no reason for any dependence on ϕ, so $\frac{\partial^2 n}{\partial \phi^2} = 0$

$$\Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n}{\partial r}\right) = -\frac{S}{D}$$

$$\Leftrightarrow \quad \frac{\partial}{\partial r}\left(r\frac{\partial n}{\partial r}\right) = -\frac{S}{D}r$$

$$r\frac{\partial n}{\partial r} = -\frac{S}{2D}r^2 + C_1$$

$$\frac{\partial n}{\partial r} = -\frac{S}{2D}r + \frac{C_1}{r}$$

$$n = -\frac{S}{4D}r^2 + C_1 \ln r + C_2$$

(3,4)

Now the solution must be finite at $r=0$

$\Rightarrow \quad C_1 = 0$

And it is zero at $r=R$

$$\Rightarrow \quad -\frac{S}{4D}R^2 + C_2 = 0 \quad \Leftrightarrow \quad C_2 = \frac{SR^2}{4D}$$

$$\Rightarrow \quad n(r) = -\frac{S}{4D}r^2 + \frac{SR^2}{4D} =$$

$$= \frac{S}{4D}\left(R^2 - r^2\right)$$

5) We calculate $\nabla^2 n$ from the solution in (4)

$$\Rightarrow \quad \nabla^2 n = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{S}{4D}(R^2-r^2)\right)\right]$$

$$= \frac{S}{4D}\left\{ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial(-r^2)}{\partial r}\right)\right\} =$$

$$= \frac{S}{4D}(-4) = -\frac{S}{D}$$

The approximation $\nabla^2 n \simeq -\frac{n(r=0)}{R^2}$

gives us $n(r=0) = \frac{SR^2}{4D}$

and $\nabla^2 n \simeq -\frac{SR^2}{4DR^2} = -\frac{S}{4D}$

The solutions differ by a factor $\frac{1}{4}$

Problem 4: Kinetic Theory for Fusion Born Alpha Particles

1. Solve

$$\frac{\partial f_\alpha}{\partial t} = \frac{1}{\tau_s}\frac{1}{v^2}\frac{\partial}{\partial v}\left[\left(v^3+v_c^3\right)f_\alpha\right] + \frac{S}{4\pi v_\alpha^2}\delta(v-v_\alpha)$$

in steady state, subject to

$$f_\alpha(v>v_\alpha) = 0$$

Solution: $\frac{\partial f_\alpha}{\partial t} = 0$

$$\Rightarrow \frac{\partial}{\partial v}\left[\left(v^3+v_c^3\right)f_\alpha\right] = -\frac{S\tau_s}{4\pi v_\alpha^2}v^2\delta(v-v_\alpha)$$

Integrate:

$$\int_{v<v_\alpha}^{\infty}\frac{\partial}{\partial v'}\left[\left(v'^3+v_c^3\right)f_\alpha\right]dv' =$$

$$= \left[\left(v'^3+v_c^3\right)f_\alpha\right]_{v<v_\alpha}^{\infty} = -\left(v^3+v_c^3\right)f_\alpha =$$

$$= \int_{v<v_\alpha}^{\infty} -\frac{S\tau_s}{4\pi v_\alpha^2}v'^2\delta(v'-v_\alpha)\,dv' =$$

$$= -\frac{S\tau_s}{4\pi}$$

$$\Rightarrow \boxed{f_\alpha(v) = \frac{S\tau_s}{4\pi\left(v^3+v_c^3\right)}}$$

Problem 5: Cold Plasma Waves

1. Maxwell's eqs:

$$\vec{\nabla} \times \mathbb{E} = - \frac{\partial \mathbb{B}}{\partial t} \qquad (1)$$

$$\vec{\nabla} \times \mathbb{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbb{E}}{\partial t} + \mu_0 \mathbb{J} \qquad (2)$$

Note: The divergence relations are not needed!

Take curl of (1)

$$\vec{\nabla} \times (\vec{\nabla} \times \mathbb{E}) = \vec{\nabla} \times \left(- \frac{\partial \mathbb{B}}{\partial t} \right) =$$

$$= - \frac{\partial}{\partial t} \left(\vec{\nabla} \times \mathbb{B} \right) = - \frac{\partial}{\partial t} \left(\mu_0 \varepsilon_0 \frac{\partial \mathbb{E}}{\partial t} + \mu_0 \mathbb{J} \right) =$$

$$= \mu_0 \varepsilon_0 \frac{\partial^2 \mathbb{E}}{\partial t^2} - \mu_0 \frac{\partial \mathbb{J}}{\partial t}$$

Note: Def. of speed of light is

$$c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

This gives

$$- \frac{\partial^2 \mathbb{E}}{\partial t^2} - c^2 \vec{\nabla} \times \left(\vec{\nabla} \times \mathbb{E} \right) - \frac{1}{\varepsilon_0} \frac{\partial \mathbb{J}}{\partial t} = 0 \qquad (3)$$

Fourier transform: This is equivalent to assuming that

$$\mathbb{E}, \mathbb{J} \sim e^{i(\mathbb{k} \cdot \mathbb{r} - \omega t)}$$

In either case

$$\frac{\partial}{\partial t} \longrightarrow -i\omega$$

$$\vec{\nabla} \longrightarrow i\mathbb{k}$$

We get from ③ that

$$\omega^2 \hat{\mathbb{E}} + c^2 \mathbb{k} \times (\mathbb{k} \times \hat{\mathbb{E}}) + i \frac{\omega}{\varepsilon_0} \hat{\mathbb{J}} = 0, \qquad ④$$

where $\hat{\mathbb{E}}$ and $\hat{\mathbb{J}}$ are the Fourier transfs of $\mathbb{E}$ & $\mathbb{J}$.

Introduce conductivity tensor (it is really a matrix)

$$\hat{\mathbb{J}}(\omega, \mathbb{k}) = \overleftrightarrow{\sigma}(\omega, \mathbb{k}) \cdot \hat{\mathbb{E}}(\omega, \mathbb{k})$$

Then ④ becomes

$$\hat{\mathbb{E}} + \frac{c^2}{\omega^2} \mathbb{k} \times (\mathbb{k} \times \hat{\mathbb{E}}) + i \frac{\overleftrightarrow{\sigma}}{\omega \varepsilon_0} \cdot \hat{\mathbb{E}} = 0$$

This can be written as

$$\boxed{\overleftrightarrow{\varepsilon}(\omega, \mathbb{k}) \cdot \hat{\mathbb{E}} + \frac{c^2}{\omega^2} \mathbb{k} \times (\mathbb{k} \times \hat{\mathbb{E}}) = 0}$$

where the dielectric tensor is

$$\overleftrightarrow{\varepsilon}(\omega, \mathbb{k}) = \mathbb{1} + i \frac{\overleftrightarrow{\sigma}(\omega, \mathbb{k})}{\omega \varepsilon_0}$$

2. <u>Longitudinal wave</u>: $\mathbb{k} \times \hat{\mathbb{E}} = 0$

$$\Longrightarrow \boxed{\overleftrightarrow{\varepsilon}(\omega, \mathbb{k}) \cdot \hat{\mathbb{E}} = 0}$$

<u>Transverse wave</u>: $\mathbb{k} \cdot \hat{\mathbb{E}} = 0$

$$\rightsquigarrow \mathbb{k} \times (\mathbb{k} \times \hat{\mathbb{E}}) = \mathbb{k} \cancel{(\mathbb{k} \cdot \hat{\mathbb{E}})} - \hat{\mathbb{E}} k^2 = -\hat{\mathbb{E}} k^2$$

$$\Longrightarrow \left[\overleftrightarrow{\varepsilon}(\omega, \mathbb{k}) - \frac{c^2 k^2}{\omega^2} \mathbb{1} \right] \cdot \hat{\mathbb{E}} = 0$$

3. Actually, all we need is the eqns

$$* \quad n_\alpha m_\alpha \left[\frac{\partial \mathbb{V}_\alpha}{\partial t} + (\mathbb{V}_\alpha \cdot \vec{\nabla}) \mathbb{V}_\alpha \right] = q_\alpha n_\alpha \left[\mathbb{E} + \mathbb{V}_\alpha \times \mathbb{B} \right] - \vec{\nabla} P_\alpha$$

We linearize $*$ around a static & uniform equilibrium, i.e. one with

$$\frac{\partial}{\partial t} = \vec{\nabla} = 0,$$

that has no flow,

$$\mathbb{V}_{\alpha 0} = 0 \iff \mathbb{E}_0 = 0$$

We get

$$** \quad m_\alpha n_{\alpha 0} \frac{\partial \mathbb{V}_{\alpha 1}}{\partial t} = q_\alpha n_{\alpha 0} \left[\mathbb{E}_1 + \mathbb{V}_{\alpha 1} \times \mathbb{B}_0 \right] - \vec{\nabla} P_{\alpha 1}$$

Cold assumption: $P_{\alpha 0} = P_{\alpha 1} = 0$

Unmagnetized: $\mathbb{B}_0 = 0$

Thus, $**$ becomes

$$m_\alpha \cancel{n_{\alpha 0}} \frac{\partial \mathbb{V}_{\alpha 1}}{\partial t} = q_\alpha \cancel{n_{\alpha 0}} \mathbb{E}_1$$

$$m_\alpha \frac{\partial \mathbb{V}_{\alpha 1}}{\partial t} = q_\alpha \mathbb{E}_1$$

Fourier transform: $\mathbb{V}_{\alpha 1}, \mathbb{E}_1 \sim e^{i(\mathbb{k} \cdot \mathbb{r} - \omega t)}$

We get

$$-i\omega m_\alpha \hat{\mathbb{V}}_{\alpha 1} = q_\alpha \hat{\mathbb{E}}_1$$

Fourier transformed current

$$\hat{\mathbb{J}} = \sum_{e,i} q_\alpha n_{\alpha 0} \hat{\mathbb{V}}_{\alpha 1} =$$

$$= \frac{i}{\omega} \sum_{e,i} \frac{q_\alpha^2 n_{\alpha 0}}{m_\alpha} \hat{\mathbb{E}}_1 = \left[\text{Quasineutrality: } n_{i0} = n_{e0} \right] =$$

$$= \frac{i e^2 n_{e0}}{\omega}\left[\frac{1}{m_i} + \frac{1}{m_e}\right]\hat{E}_1 = \left[\omega_{pe}^2 = \frac{e^2 n_{e0}}{\varepsilon_0 m_e}\right] =$$

$$= i\,\frac{\varepsilon_0}{\omega}\,\omega_{pe}^2\left[1 + \cancel{\frac{m_e}{m_i}}\right]\hat{E}_1 = i\,\frac{\varepsilon_0\,\omega_{pe}^2}{\omega}$$

Hence

$$\overleftrightarrow{\sigma} = \frac{i\,\varepsilon_0\,\omega_{pe}^2}{\omega}\,\mathbb{1}$$

which gives

$$\overleftrightarrow{\varepsilon}(\omega, \mathbf{k}) = \mathbb{1} + i\,\frac{\overleftrightarrow{\sigma}(\omega, \mathbf{k})}{\varepsilon_0\,\omega} =$$

$$= \left[1 - \frac{\omega_{pe}^2}{\omega^2}\right]\mathbb{1}$$

Longitudinal :

$$\overleftrightarrow{\varepsilon}\cdot\hat{E}_1 = 0 = \left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)\hat{E}_1$$

$$\Longrightarrow \boxed{\omega^2 = \omega_{pe}^2}$$

Transverse :

$$\left[\overleftrightarrow{\varepsilon} - \frac{c^2k^2}{\omega^2}\mathbb{1}\right]\cdot\hat{E}_1 = 0 = \left[1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{c^2k^2}{\omega^2}\right]\hat{E}_1$$

$$\Longrightarrow \boxed{\omega^2 = \omega_{pe}^2 + c^2k^2}$$

4. Longitudinal : Does not change

Transverse : Acquires dispersion

$$\omega^2 = \omega_{pe}^2 + (\dots)^2 k^2$$

Sorry! wrong... Switch these