MPWPS, year 1, study period 1, academic year 2012/2013

Exam
Electromagnetic Waves and Components (RRY 036), 22/10 2012
Department of Earth and Space Sciences

Teachers:
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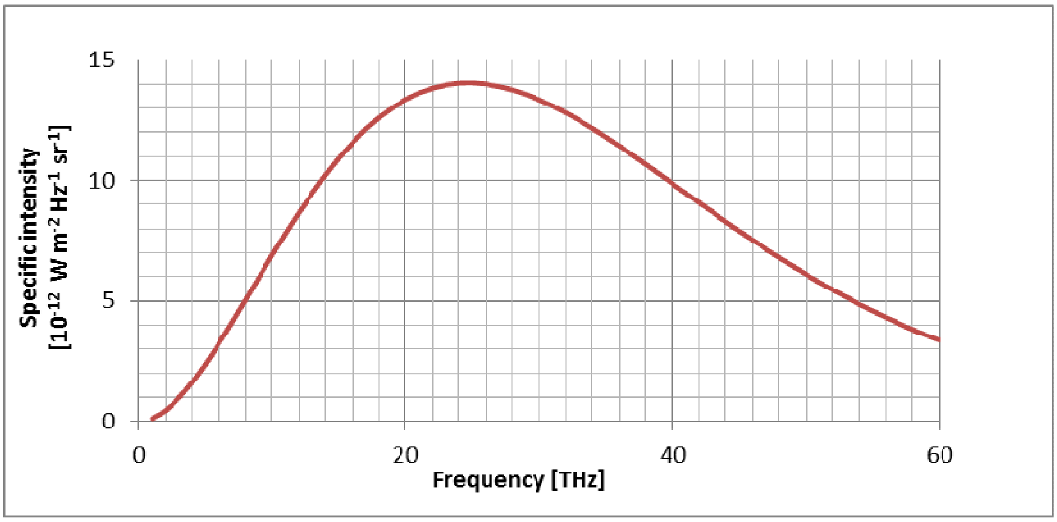
On the exam you may use:
- Chalmers-approved calculator,
- Formulas in Electromagnetic waves (E. Palmberg 2012),
- Formulae and constants for blackbody radiation, excitation of two-level systems and radiative transfer (A.Heikkilä 2012),
- Physics Handbook, Beta, etc,
- Dictionary (not electronic).

Grade limits:
Grade 3(=pass): 20 points
Grade 4: 30 points
Grade 5: 40 points
A maximum of 50 points can be achieved on the exam.

Remember: Give full solutions to the problems you hand in, i.e. explain and motivate your answers carefully! Be careful with units! When drawing graphs, indicate clearly the quantity on each axis, and give the scale.

1. A black-body radiator is formed as a sphere with a radius of 5 cm. Its spectrum is shown in the Figure below. The scale of the frequency axis is THz, and the scale of the specific intensity axis is 10^{-12} W m^{-2} Hz^{-1} sr^{-1}. Estimate the total power emitted by this radiator. (2p)



2. Molecular clouds in the Milky Way are the sites of star formation. In these clouds the gas is mainly composed of molecular hydrogen and atomic helium, mixed with very small amounts of other molecular (or atomic) species. However, these "rare species" are important since they cool the gas through their radiation (and hence assist the star formation process) and serve as measurement probes of the cloud's physical conditions.

Observations of radiation from the molecule ^{13}CO (a rarer isotopomer of carbon monoxide with carbon-13 instead of carbon-12) from one such cloud shows a maximum brightness temperature of 8,0 K at the frequency 110,2 GHz. At frequencies outside this peak, the brightness temperature drops to 3,0 K. The optical depth has been estimated using other observations to 0,30. The spontaneous emission rate for the 110,2 GHz transition is $A_{ul} \approx 6{,}3 \cdot 10^{-8}$ s^{-1}, statistical weights $g_l = 1$ and $g_u = 3$, and a collision coefficient $c_{ul} \approx 3{,}3 \cdot 10^{-11}$ cm^3/s. The density of collision partners (mainly H_2) is $3 \cdot 10^4$ cm^{-3}.

Estimate the kinetic temperature of the gas in this cloud. Treat ^{13}CO as a two-level system. You may neglect the influence of absorption and stimulated emission. (4p)

3. Assume time harmonic fields with angular frequency ω in a source free lossy medium with **J**=σ**E** and where the medium parameters $\mu=\mu_0$ and $\varepsilon=\varepsilon_d$ are real constants.
a) Derive the wave equation (Helmholtz equation) for the magnetic field phasor $\underline{\mathbf{H}}(\mathbf{r})$ starting from Maxwell's equations in phasor form. (5p)
b) In free space ($\varepsilon=\varepsilon_0$, $\mu=\mu_0$, and $\sigma=0$) a solution to the wave equation for the magnetic field phasor is given by:

$$\underline{\vec{H}}(\vec{r}) = \left((1+j)\hat{x} + \sqrt{2}e^{j\pi/4}\hat{z}\right)e^{-j\beta y}$$

Determine the corresponding electric field **E**(**r**,t) and the polarisation of the wave (linear, circular or elliptical). Give reasons for your answer. (5p)

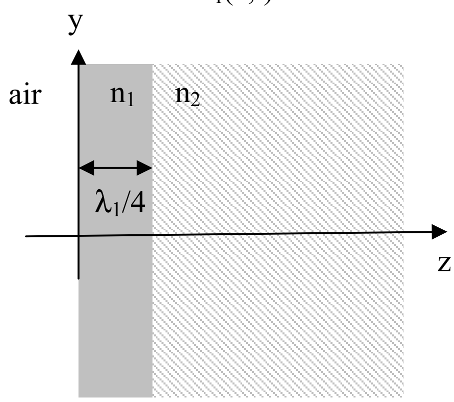
4. The parameters of moist earth at a frequency of f=1 MHz are $\varepsilon=4\varepsilon_0$, $\mu=\mu_0$, and $\sigma=0.1$ S/m. Assuming that the electric field of a uniform plane wave with f=1 MHz propagating in the z-direction is $\underline{\mathbf{E}}=\hat{y}3\text{x}10^{-2}$ V/m at z=0, find:

a) The wavelength inside the earth and in vacuum. (2p)
b) The field **E**(z,t) and the distance through which the wave must travel before the magnitude of the electric field reduces to $1.104\text{x}10^{-2}$ V/m. (4p)

5. Consider the propagation of a uniform monochromatic plane wave in the ionosphere, modelled as a collisionless plasma with refractive index $n^2=1-\omega_p^2/\omega^2$ where f_p=8 MHz.
a) What is the difference in arrival time between a flash of light (f=600 THz) and a simultaneous radio pulse (f=10 MHz) seen through the plasma along a path of 100 km? (5p)
b) Assume that the plasma frequency f_p increases with height above the earth. Make a sketch to show how a ray of light propagates through the ionosphere in that case. Explain the result! (2p)

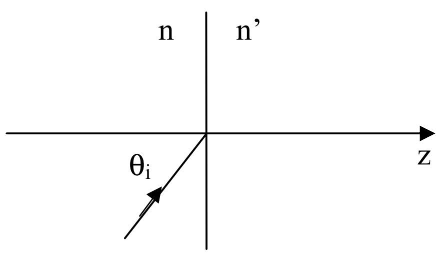
6. A uniform plane wave travelling in air (refractive index n=1) is incident normally on a dielectric medium with refractive index n_2=2.2. The incident field is given by $\mathbf{E}_i=\hat{y}E_0\cos(\omega t-kz)$ V/m. The reflections can be eliminated by placing another dielectric slab with refractive index n_1, $\lambda_1/4$ thick, between air and the original dielectric medium.

a) Determine the refractive index of the $\lambda_1/4$ slab to accomplish this. (1p)
b) A dielectric with the calculated refractive index does not exist. Use the table below to choose the appropriate medium with refractive index n_1 for the $\lambda_1/4$ slab to best reduce reflections. (1p)
c) Calculate the reflected electric field $\mathbf{E}_r$(z,t) in air with the chosen medium n_1. (5p)



Material	n_1
MgF_2	1.38
Polystyrene	1.60
PbF_2	1.73

7. Unpolarised light at optical frequencies (f=600 THz) is incident from vacuum, at an incident angle θ_i=60° on a dielectric medium with refractive index n'=1.6. Calculate the degree of polarisation of the reflected wave. The degree of polarisation can be defined as: $\delta=(I_{TE}-I_{TM})/(I_{TE}+I_{TM})$, where I is the intensity (time averaged Poynting vector) of the reflected wave. Explain the result! What happens with δ in the case of normal incidence? (6p)



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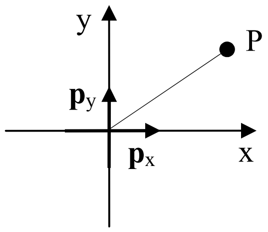
8. Two Hertzian dipoles located at x=y=z=0 are excited $\pi/2$ out of phase with dipole moment phasors $\underline{p}_x=p_0$ and $\underline{p}_y$, respectively, with $\underline{p}_y=j\underline{p}_x$. a) Calculate the electric field **E**(r,t) and the magnetic fields **H**(r,t) at a point P in the x-y plane for z=0 (radiation fields). b) Determine the polarisation of the wave at point P. c) Calculate the Poynting vector **P** in the x-y plane for z=0! (8p)



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Electromagnetic waves and components

(1) $P = A \cdot I \approx 4\pi r^2 \cdot 5{,}67 \cdot 10^{-8} \cdot T^4$

T is estimated by reading out the freq. of the peak specific intensity:

$\nu_{peak} \approx 25\,THz = 2{,}5 \cdot 10^{13}\,Hz$

$T \approx \frac{\nu_{peak}}{5{,}88 \cdot 10^{10}} \approx 425\,K$

radius $r = 5 \cdot 10^{-2}\,m$

$\Rightarrow \underline{\underline{P \approx 58\,W}}$

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(2) T_{kin} is obtained from $\frac{C_{lu}}{C_{ul}} = \frac{g_u}{g_l} \cdot e^{-h\nu_{ul}/kT_{kin}}$

Set up the rate-equations and assume statistical equilibrium

$$\frac{N_u}{N_l} = \frac{C_{lu} \cdot n}{C_{ul} \cdot n + A_{ul}} \Rightarrow \frac{C_{lu}}{C_{ul}} = \frac{N_u}{N_l}\left(1 + \frac{A_{ul}}{n \cdot C_{ul}}\right)$$

Also $\frac{N_u}{N_l} = \frac{g_u}{g_l} \cdot e^{-h\nu_{ul}/kT_{ex}}$ Calculate T_{ex} using the solution to radiative transfer eq.

$$T_b = T_{bg}\, e^{-\tau_\nu} + T_{ex} \cdot (1 - e^{-\tau_\nu}) \Rightarrow T_{ex} \approx 22{,}29\,K$$

$$\Rightarrow \frac{N_u}{N_l} \approx 2{,}366 \Rightarrow \frac{C_{lu}}{C_{ul}} \approx 2{,}517 \Rightarrow T_{kin} \approx 30{,}1 \approx \underline{\underline{30K}}$$

③ a) $\nabla \times \vec{H} = \vec{J} + j\omega \vec{D}$

$$\nabla \times \vec{H} = \sigma \vec{E} + j\omega \varepsilon_d \vec{E} = (\sigma + j\omega \varepsilon_d)\vec{E}$$

$$\nabla \times \nabla \times \vec{H} = (\sigma + j\omega \varepsilon_d)\, \nabla \times \vec{E}$$

$$-\nabla^2 \vec{H} + \cancel{\nabla(\nabla \cdot \vec{H})} = (\sigma + j\omega \varepsilon_d)(-j\omega \mu_0)\vec{H} = j\omega \varepsilon_d \left(1 + \frac{\sigma}{j\omega \varepsilon_d}\right)(-j\omega\mu_0)\vec{H}$$

$$\underline{\nabla^2 \vec{H} = -\omega^2 \mu_0 \varepsilon_c \vec{H}} \quad , \quad \underline{\varepsilon_c = \varepsilon_d \left(1 + \frac{\sigma}{j\omega \varepsilon_d}\right)}$$

b) $\vec{H}(\vec{r}) = \left\{ \underbrace{(1+j)}_{=\sqrt{2}\, e^{j\pi/4}} \hat{x} + \sqrt{2}\, e^{j\pi/4} \hat{z} \right\} e^{-j\beta y}$

$\vec{k} = \beta \hat{y}$,

$$\vec{E} = \eta_0 \vec{H} \times \hat{y} = \eta_0 \sqrt{2}\, e^{j\pi/4} (\hat{x} \times \hat{y} + \hat{z} \times \hat{y})\, e^{-j\beta y}$$

$$= \eta_0 \sqrt{2}\, e^{j\pi/4} (\hat{z} - \hat{x})\, e^{-j\beta y}$$

$$\underline{\vec{E}(y,t)} = \mathrm{Re}\left\{ \vec{E}(y)\, e^{j\omega t} \right\} =$$

$$= \underline{\eta_0 \sqrt{2} \cos(\omega t - \beta y + \pi/4)\, \hat{z}}$$

$$\underline{- \eta_0 \sqrt{2} \cos(\omega t - \beta y + \pi/4)\, \hat{x}}$$

Same amplitude and phase of $\hat{z}$ and $\hat{x}$ component $\Rightarrow$ linear polarisation

4. We find that $\frac{\sigma}{\omega \varepsilon} = \frac{0.1}{2\pi \cdot 10^6 \cdot 4 \cdot 8.85 \cdot 10^{-12}} \gg 1$

$\Rightarrow$ good conductor

The propagation vector is $\vec{k} = k_c \hat{z}$

where $k_c = \beta - j\alpha$

$$\alpha = \beta = \sqrt{\frac{\omega \mu \sigma}{2}} = \sqrt{\frac{2\pi \cdot 10^6 \cdot 0.1 \cdot 4\pi \cdot 10^{-7}}{2}} = \underline{0.628 \ m^{-1}}$$

a/ Wavelength $\lambda = \frac{2\pi}{\beta} = \underline{10 \ m}$

In vacuum: $f \cdot \lambda_0 = c_0 \Rightarrow \lambda_0 = \frac{3 \cdot 10^8}{10^6} = \underline{300 \ m}$

b/ $\vec{E}(z=0) = \hat{y}\, 3 \cdot 10^{-2}$ V/m

$\vec{E}(z) = \hat{y}\, 3 \cdot 10^{-2} e^{-jk_c z} = \hat{y}\, 3 \cdot 10^{-2} e^{-\alpha z} e^{-j\beta z}$ V/m

$E(z,t) = \mathrm{Re}\left\{ \vec{E}(z) e^{j\omega t} \right\} =$

$$= \underline{\hat{y}\, 3 \cdot 10^{-2} e^{-\alpha z} \cos(\overset{2\pi \cdot 10^6}{\omega t} - \overset{\text{as above}}{\beta z}) \quad \text{V/m}}$$

$$\frac{E(z)}{E(0)} = e^{-\alpha z} = \frac{1.104}{3} = 0.368$$

$\Rightarrow -\alpha z = \ln 0.368 = -1 \Rightarrow$

$\Rightarrow z = \delta = \frac{1}{\alpha} = \underline{1.59 \ m}$ (skin depth)

5.
$$n^2 = 1 - \omega_p^2/\omega^2$$

Propagation : $k = (n k_0) = \omega\sqrt{\mu_0 \varepsilon} = \frac{\omega}{c_0} n$

$$k = \frac{\omega}{c_0}\sqrt{1-\frac{\omega_p^2}{\omega^2}} = \frac{1}{c_0}\sqrt{\omega^2-\omega_p^2}$$

Pulse propagates with group velocity:

$$V_g = \frac{d\omega}{dk} = \frac{1}{\frac{dk}{d\omega}} = c_0\sqrt{1-\frac{\omega_p^2}{\omega^2}} \quad , \omega_p = 2\pi\cdot 8\cdot 10^6 \text{ rad/s}$$

a) $f_1 = 10\ MHz \; ; \; T_1 = \frac{\ell}{V_{g1}}$

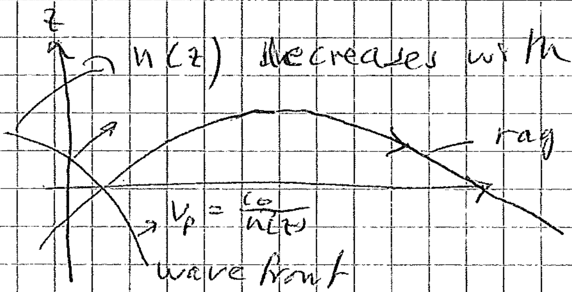
$f_2 = 600\cdot 10^{12}\ Hz \; ; \; T_2 = \frac{\ell}{V_{g2}} \; , V_{g2} \simeq c_0$

$$\Delta T = T_1 - T_2 = \ell\left(\frac{1}{V_{g1}} - \frac{1}{c_0}\right)$$

$$V_{g1} = c_0\sqrt{1-\left(\frac{8\cdot 10^6}{10\cdot 10^6}\right)^2} = 0{,}6\, c_0$$

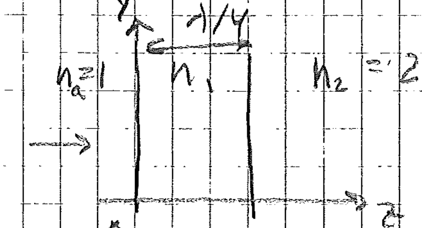
$$\Rightarrow \Delta T = \frac{100\cdot 10^3}{3\cdot 10^8}\left(\frac{1}{0.6} - 1\right) = \underline{2.2\cdot 10^{-4}\ s}$$

b)



Phase velocity of wave front varies with z $\Rightarrow$ ray bends

(6)



$n_2 = 2.2 = n_b$

$E_i = \hat{y}\, E_0 \cos(\omega t - kz)$

a) $n_1 = \sqrt{n_a n_b} = \sqrt{2.2} = \underline{1.48}$

b) Use $\underline{n_1 = 1.38}$ (closest value)

c) $\Gamma_1 = ?$ Propagation / matching of Z:

$$Z_2 = \eta_2 \quad , \quad Z_1 = \frac{\eta_1^2}{Z_2} = \frac{\eta_1^2}{\eta_2} = \left(\eta = \frac{\eta_0}{n}\right)$$

$$= \frac{\eta_0\, n_2}{n_1^2}$$

$$\underline{\Gamma_1} = \frac{Z_1 - \eta_0}{Z_1 + \eta_0} = \frac{n_2 - n_1^2}{n_2 + n_1^2} = \frac{2.2 - 1.38^2}{2.2 + 1.38^2} = \underline{0.07}$$

$$\Gamma_1(0) = \frac{E_-}{E_+} \Rightarrow E_- = 0.07\, E_+$$

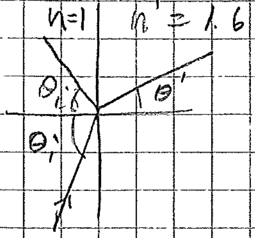
$z = 0$: $\quad E_+ = E_0 \Rightarrow E_- = 0.07\, E_0$

$$\vec{E}_-(z) = \hat{y}\, 0.07\, E_0\, e^{+jkz} \quad , \quad \frac{\omega}{k} = c_0$$

$$\underline{\vec{E}_r(z,t) = \vec{E}_-(z,t) = \hat{y}\, 0.07\, E_0 \cos(\omega t + kz)}$$

Small amplitude reflected wave, not completely reflectionless since $n_1 \neq 1.48$

(7)



$n\sin\theta_i = n'\sin\theta' \Rightarrow \theta' = 32.77^\circ$

Incoming field is unpolarised

$\Rightarrow$ 50% TE + 50% TM

$f = 600 \cdot 10^{12}$ Hz

Calculate reflection coefficients:

TE-case:

$$\rho_{TE} = \frac{n\cos\theta - n'\cos\theta'}{n\cos\theta + n'\cos\theta'} = \frac{\cos 60^\circ - \overbrace{1.6\cos 32.77^\circ}^{1.345}}{\cos 60^\circ + 1.6\cos 32.77^\circ}$$

$$= -\frac{0.845}{1.845} = -0.458$$

TM-case:

$$\rho_{TM} = \frac{n\cos\theta' - n'\cos\theta}{n\cos\theta' + n'\cos\theta} = \frac{\overbrace{\cos 32.77}^{0.8} - 1.6\cos 60^\circ}{\cos 32.77 + 1.6\cos 60^\circ} =$$

$$= \frac{0.041}{1.641} = 0.025$$

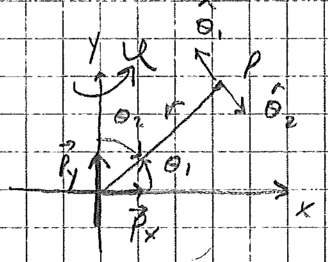
Intensities $\sim |\rho|^2$; $P_{refl} = \frac{1}{2\eta_0}|E_i \cdot \rho|^2$

$$\delta = \frac{|\rho_{TE}|^2 - |\rho_{TM}|^2}{|\rho_{TE}|^2 + |\rho_{TM}|^2} = \frac{0.209}{0.210} = 0.994$$

The degree of polarisation is nearly 1 (100%), corresponding to polarised waves! The reason is that we are close to the Brewster angle where $\rho_{TM} = 0$; $\tan\theta_B = \frac{n'}{n} \Rightarrow \theta_B = 58^\circ$, shows that reflection can produce polarised waves.

For normal incidence $\rho_{TE} = \rho_{TM} \Rightarrow \delta = 0$, reflected wave is unpolarised (as incoming field)

8.



$\hat{\varphi}$ into page

Time harmonic dipoles ; $p_x = p_0$, $p_y = j p_x$

a) Use formulas :

$$\vec{E}(r,\theta,\varphi) = -\hat{\theta}_2 \frac{k^2 p_y \sin\theta_2}{4\pi\varepsilon_0 r} e^{-jkr} - \hat{\theta}_1 \frac{k^2 p_x \sin\theta_1}{4\pi\varepsilon_0 r} e^{-jkr}$$

In the x-y plane, $\hat{\theta}_2 = -\hat{\theta}_1$, $\sin\theta_2 = \cos\theta_1$

$$\Rightarrow \vec{E} = -\frac{k^2 p_0}{4\pi\varepsilon_0 r} e^{-jkr} \left[\hat{\theta}_1 \sin\theta_1 - j\hat{\theta}_1 \cos\theta_1 \right]$$

$$\Rightarrow \underline{\vec{E}(r,\theta,\varphi,t)} = -\frac{k^2 p_0}{4\pi\varepsilon_0 r} \left\{ \cos(\omega t - kr)\sin\theta_1 \hat{\theta}_1 + \cos(\omega t - kr - \pi/2)\cos\theta_1 \hat{\theta}_1 \right\}$$

$$\vec{H}(r,\theta,\varphi) = -\hat{\varphi} \frac{k\omega p_y \sin\theta_2}{4\pi r} e^{-jkr} + \hat{\varphi} \frac{k\omega p_x \sin\theta_1}{4\pi r} e^{-jkr} \quad ; \; p_y = j p_x$$

$$\Rightarrow \underline{\vec{H}(r,\theta,\varphi,t)} = +\hat{\varphi} \frac{k\omega p_0}{4\pi r} \left\{ \cos(\omega t - kr)\sin\theta_1 - \cos(\omega t - kr - \pi/2)\cos\theta_1 \right\}$$

b) $\vec{E}$ - field in $\hat{\theta}_1$ direction $\Rightarrow$ linear polarisation

c) $$P = \frac{1}{2\eta_0} |\vec{E}|^2 = \frac{1}{2\eta_0} \left(\frac{k^2 p_0}{4\pi\varepsilon_0 r} \right)^2 \Big| \underbrace{\sin\theta_1 - j\cos\theta_1}_{=1} \Big|^2$$

$$= \underline{\frac{1}{2\eta_0} \left| \frac{k^2 p_0}{4\pi\varepsilon_0 r} \right|^2} \quad \text{at } z = 0$$