Quantum Mechanics FKA081/FIM400

Final Exam August 20 2013

Next review time for the exam: 27 September 12-13 in my room. NB: If you want to come to the review you must collect your exam before at the "Kansli" in Origo 5th floor. (This info is also available on the course homepage.)

Examinator: Gabriele Ferretti tel. 7723168, 0739687998 ferretti@chalmers.se

Allowed material during the exam:

- The course textbook J.J. Sakurai and Jim Napolitano, Modern Quantum Mechanics Second Edition (2010).
  NB: The old red cover version: J.J. Sakurai, Modern Quantum Mechanics Revised Edition (1994) is *also* allowed.
- A Chalmers approved calculator (Casio FX82..., Texas TI30..., Sharp ELW5...).

Write the final answers clearly marked by Ans: ...
and underline them.

You may use without proof any formula in the book.

There is a total of 30 points in this test. The exam counts for 90% of the final grade, (that is $3\times$ points %).

Problem 1

A spinless atom of mass m is suspended in a potential $V(r) = \frac{1}{2}m\omega^2 r^2$ where ω is a constant and $r^2 = x^2 + y^2 + z^2$ the distance from the origin.
Q1 (*1 points*) What are the energies of the ground state and the first two exited states?
Q2 (*2 points*) What are the *degeneracies* of the energy levels above? (That is how many independent states are there for each level?)
Q3 (*2 points*) What are the allowed values of $\mathbf{L^2}$ and L_z for such states?

Note: You do not have to solve the 3D Schrödinger equation to answer these questions.

Problem 2

Consider a spinless particle of mass m in a one-dimensional well with potential:

$$V(x) = \begin{cases} 0, & \text{for } 0 \leq x \leq a; \\ \infty, & \text{otherwise.} \end{cases} \tag{1}$$

The first relativistic correction can be described by the Hamiltonian

$$H' = -\frac{p^4}{8m^3c^2} \tag{2}$$

Q1 (*2 points*) Compute the first order correction induced by H' to all the states $|n\rangle$.
Q2 (*2 points*) Given a and m, for which values of n can we trust this approximation?

Problem 3

Consider the same system as in problem 2, namely a (non-relativistic) spinless particle of mass m in a one-dimensional well with potential:

$$V(x) = \begin{cases} 0, & \text{for } 0 \leq x \leq a; \\ \infty, & \text{otherwise.} \end{cases} \tag{3}$$

Suppose at time $t = 0$ the width of the well suddenly doubles (that is $a \to 2a$).
Q1 (*3 points*) If the particle was in the ground state for $t < 0$, what is the probability of finding the particle in some unspecified excited state for $t > 0$? (Use the "sudden approximation".)

Problem 4

Consider the following two observables:

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix} \tag{4}$$

Q1 (*1 points*) Show that they can be simultaneously diagonalized.
Q2 (*2 points*) Find a basis of common eigenvectors $|a, b\rangle$.
Q3 (*2 points*) Let C be another observable that commutes with both A and B. Show that it must have eigenvectors $|a, b\rangle$ and that their eigenvalues uniquely determine C.
Q4 (*2 points*) Why did we need to know that C commutes with both A and B? Would $[A, C] = 0$ be enough?

Problem 5

An ion of a certain atom has $l = 1$ and $s = 0$ and is subjected to a Hamiltonian (a, b real constants)

$$H = a(L_x^2 - L_y^2) + bL_z \tag{5}$$

(This type of Hamiltonian occurs e.g. for ions in a crystals with a magnetic field proportional to b along the z-axis.)
Q1 (*2 points*) Write H as a 3×3 matrix in the basis in which L_z is diagonal.
Q2 (*2 points*) Find and plot the energy levels as a function of b for a fixed.
Q3 (*2 points*) Show that the first order perturbation in b ($b \ll a$) vanishes and comment this result in the light of the exact solution found above.

Problem 6

Consider a three levels system with eigenstates $|0\rangle$, $|1\rangle$, $|2\rangle$ of energies $E_0 < E_1 < E_2$ subjected a weak time-dependent Hamiltonian $H'(t)$ that has the following matrix elements:

$$\langle n|H'(t)|m\rangle = \lambda_{mn} e^{-t^2/T^2} \tag{6}$$

Q1 (*1 points*) What conditions must the matrix λ satisfy in order for $H'(t)$ to be acceptable as Hamiltonian?
Q2 (*1 points*) What conditions must the matrix λ satisfy in order for $H'(t)$ to be treated as a perturbation?
Q3 (*2 points*) Assume that the system is in the ground state $|0\rangle$ when $t \ll -T$. Compute the probability of finding the system in $|m \neq 0\rangle$ for $t \gg T$.
Q4 (*1 points*) Evaluate the order of magnitude of the probability of transition $|0\rangle \to |1\rangle$ for $E_1 - E_0 = 1$ eV, $\lambda_{01} = 10^{-2}$ eV, $T = 10^{-16}$ s. Use $\hbar \approx 6 \times 10^{-15}$ eV s and make all the appropriate numerical approximations.

PROBLEM 1

$$H = \frac{\mathbb{P}^2}{2m} + \frac{1}{2} m \omega^2 \mathbb{r}^2 = H_x + H_y + H_z$$

where $H_x = \frac{P_x^2}{2m} + \frac{1}{2} m\omega^2 x^2$ etc ...

So the system is equivalent to 3 one-dim. harmonic oscillator of eigenvalues/vectors:

$$H_x |n_x\rangle = \hbar\omega \left(n_x + \frac{1}{2}\right) \quad \text{etc} \ldots$$

Q1: $|n_x, n_y, n_z\rangle$ has energy $\hbar\omega\left(N + \frac{3}{2}\right)$

where $N = n_x + n_y + n_z = 0, 1, 2, \ldots$

Q2:

$E = \frac{7}{2}\hbar\omega$ $\underline{\text{DEG} = 6}$ $\begin{cases} n_x = 2, n_y = n_z = 0 & \text{etc.. (3 cases)} \\ n_x = n_y = 1, n_z = 0 & \text{etc.. (3 cases)} \end{cases}$

$E = \frac{5}{2}\hbar\omega$ $\underline{\text{DEG} = 3}$ $n_x = 1, n_y = n_z = 0$ etc.. (3 cases)

$E = \frac{3}{2}\hbar\omega$ $\underline{\text{NON DEG}}$. $n_x = n_y = n_z = 0$.

Q3: $\mathbb{L}^2$ and L_z commute with H and with each other so we can also write the eigenfunct. as: $|N, \ell, m\rangle$

$\left(\mathbb{L}^2 |N \ell m\rangle = \hbar^2 \ell.(\ell+1) |N \ell m\rangle \right.$
$\left. L_z |N \ell m\rangle = \hbar m |N \ell m\rangle \right)$.

The ground state is non degenerate $\Rightarrow \underline{\underline{\ell = m = 0}}$.

The N=1 state contains a m=1 wave fct:

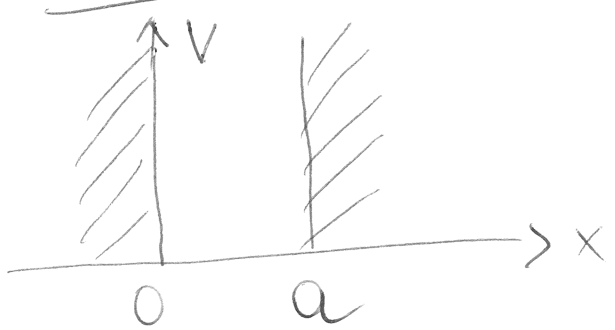
$L_z = -i\hbar \frac{\partial}{\partial \varphi}$, $\quad |n_x=1, n_y=0, n_z=0\rangle +$
$\quad\quad i|n_x=0, n_y=1, n_z=0\rangle \sim e^{i\varphi}$.

hence all 3 N=1 states have $\underline{\underline{\ell=1}}$
(and $m = \pm 1, 0$).

Similarly the 6 N=2 states can be written as $\ell=2$ $(m=-2, \dots +2)$
and $\ell = 0$ $(m=0)$
states

(No $\ell=1$ because $\ell=2,0$ already exhausts all 6 states).

PROBLEM 2



$$\psi_n(x) = \mathcal{N} \cdot \sin\left(\frac{n\pi x}{a}\right) \qquad n = 1, 2, 3 \ldots$$

($\mathcal{N}$ is a normalization constant that you don't need ...).

$$p^2 = -\hbar^2 \frac{\partial^2}{\partial x^2} \quad \Rightarrow \quad P^2 \psi_n = n^2 \left(\frac{\hbar \pi}{a}\right)^2 \psi_n .$$

$$H_0 = \frac{P^2}{2m} \quad \Rightarrow \quad H_0 \psi_n = n^2 \cdot \frac{1}{2m} \left(\frac{\hbar \pi}{a}\right)^2 \psi_n .$$

Q1:

First order non deg. pert. th.

$$E_n^{(1)} = \frac{\langle \psi_n | H' | \psi_n \rangle}{\langle \psi_n | \psi_n \rangle} = -\frac{1}{8m^3c^2} \frac{\langle \psi_n | P^4 | \psi_n \rangle}{\langle \psi_n | \psi_n \rangle}$$

$$= -\frac{1}{8m^3c^2} \left(n^2 \left(\frac{\hbar \pi}{a}\right)^2 \right)^2 \cancel{\frac{\langle \psi_n | \psi_n \rangle}{\langle \psi_n | \psi_n \rangle}} \to 1$$

Q2: We can trust this approx. up to $|E_n^{(1)}| \lesssim |E_n^{(0)}|$ =>

$$\frac{1}{8m^3c^2} n^4 \left(\frac{\hbar\pi}{a}\right)^4 \lesssim \frac{1}{2m} n^2 \left(\frac{\hbar\pi}{a}\right)^2$$

$$\Rightarrow \quad n^2 \lesssim 4m^2c^2\left(\frac{a}{\hbar\pi}\right)^2$$

Note that for $ma \lesssim \frac{\hbar}{c}$ this is never allowed.

PROBLEM 3

Q1 : for $t<0$: $\psi^{<}_{gs} = \sqrt{\frac{2}{a}} \sin \frac{\pi x}{a}$ · FOR $(0<x<a)$

for $t>0$ $\psi^{>}_{gs} = \sqrt{\frac{2}{2a}} \sin \frac{\pi x}{2a}$. FOR $(0<x<2a)$

The probability that the part. stays in the ground state is:

$$P_{gs} = \left| \langle \psi^{>}_{gs} | \psi^{<}_{gs} \rangle \right|^2$$

NOT 2a since $\psi^{<}_{gs} = 0$ for $x > a$.

$$\langle \psi^{>}_{gs} | \psi^{<}_{gs} \rangle = \frac{\sqrt{2}}{a} \int_0^a \sin \frac{\pi x}{2a} \cdot \sin \frac{\pi x}{a} \, dx =$$

$$= \frac{4\sqrt{2}}{3\pi}$$

$$P_{gs} = \left(\frac{4\sqrt{2}}{3\pi} \right)^2 \simeq 0.36.$$

The probability of being in some excited state is $P_{ex.} = 1 - P_{gs} = 0.64$

64%

PROBLEM 4.

Q1: $[A, B] = 0$ ok.

Q2. $$\begin{vmatrix} 3-a & 0 & -1 \\ 0 & 2-a & 0 \\ -1 & 0 & 3-a \end{vmatrix} = (3-a)^2(2-a) - (2-a) = 0.$$

$$a = 2, 2, 4.$$

Non deg. eigenvector: $|a=4\rangle = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow |a=4\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{vmatrix} 3-b & 0 & 1 \\ 0 & 2-b & 0 \\ 1 & 0 & 3-b \end{vmatrix} = (3-b)^2(2-b) - (2-b) = 0$$

$b = 2, 2, 4$ as well.

Non deg. eigenvector: $|b=4\rangle = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$:

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow |b=4\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

The two eigenvectors above must be common eigenvectors of both A and B:

$$A|b=4\rangle = \underset{NB}{\underline{2}}|b=4\rangle \qquad B|a=4\rangle = \underset{NB}{\underline{2}}|a=4\rangle$$

The last eigenvector is $\perp$ to both:

$$|a=b=2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Summarizing: $|ab\rangle$:

$$|2,4\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad |2,2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |4,2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Q3: Consider $|2,4\rangle$.

$$A \cdot C|2,4\rangle = C \cdot A|2,4\rangle = 2C|2,4\rangle$$

$$B \cdot C|2,4\rangle = C \cdot B|2,4\rangle = 4C|2,4\rangle$$

This means that $C|2,4\rangle$ is an eigenvector of A and B with $a=2$, $b=4$. But since $|2,4\rangle$ is non degenerate:

$\therefore$ $C|2,4\rangle = c_1|2,4\rangle$ for some number c_1.

Similarly for $|2,2\rangle$ and $|4,2\rangle$.

$$\Rightarrow C = c_1 |2,4\rangle\langle 2,4| + c_2 |22\rangle\langle 22| + c_3 |42\rangle\langle 42|$$

Q4: $[A, C] = 0$ is NOT enough because $|2,4\rangle$ and $|22\rangle$ are degenerate in the eigenspace of $A = a = 2$.

PROBLEM 5 $(\hbar = 1)$

Q1: $L_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $L_x^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

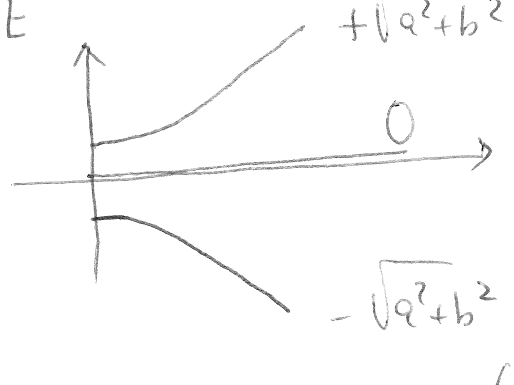
$L_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$, $L_y^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

$L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$$H = a\left(\frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}\right) + b\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} b & 0 & a \\ 0 & 0 & 0 \\ a & 0 & -b \end{pmatrix}$$

Q2: $\begin{vmatrix} b-E & 0 & a \\ 0 & -E & 0 \\ a & 0 & -b-E \end{vmatrix} = 0 \qquad E = 0, \pm\sqrt{a^2+b^2}$



Q3: $H|_{b=0} = a\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

eigenvectors: for $b=0$

$$|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} \qquad |2\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad |3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}.$$

First order perturbation:

$$\langle 1|L_z|1\rangle = \langle 2|L_z|2\rangle = \langle 3|L_z|3\rangle = 0 .$$

Consistent with the exact solution:

$$\sqrt{a^2+b^2} = a\sqrt{1+\frac{b^2}{a^2}} \simeq a + \frac{1}{2}\frac{b^2}{a}$$

No term linear in b.

PROBLEM 6.

Q1: $H'^{\dagger} = H' \Rightarrow \lambda^{\dagger} = \lambda$

Q2: The eigenvalues of λ must be small compared to $E_1 - E_0$, $E_2 - E_1$.

Q3: For $m = 1$ or 2:

$$P(0 \to m) \simeq \frac{|\lambda_{om}|^2}{\hbar^2} \left| \int_{-\infty}^{+\infty} e^{-t^2/T^2} \cdot e^{i\omega_{om}t} \, dt \right|^2 =$$

$$= \frac{|\lambda_{om}|^2}{\hbar^2} \cdot \pi T^2 e^{-\frac{\omega_{om}^2 T^2}{4}}$$

Q4: for $m = 1$ $\quad \omega_{01} = \frac{E_1 - E_0}{\hbar} = \frac{1\,\cancel{eV}}{6 \cdot 10^{-15}\,\cancel{eV} \cdot s}$

$$\Rightarrow \frac{\omega_{01} \cdot T}{2} = \frac{10^{-16}\,\cancel{s}}{2 \cdot 6 \cdot 10^{-15}\,\cancel{s}} \sim 10^{-2}$$

$$\Rightarrow \exp\left(-\left(\frac{\omega_{01} T}{2}\right)^2\right) \simeq 1$$

$$P(0 \to 1) \doteq \frac{10^{-4}\,\cancel{eV^2}}{36 \cdot 10^{-30}\,\cancel{eV^2} \cdot \cancel{s^2}} \cdot 3 \cdot 10^{-32}\,\cancel{s^2} \sim 10^{-5}$$