Quantum Mechanics FKA081/FIM400

Final Exam January 17 2013

Next review time for the exam: 1 February 15-17 in my room. NB: If you want to come to the review you must collect your exam before at the "Kansli" in Origo 4th floor (Opening hours Mon, Wed, Fri 9:00-11:00. (This info is also available on the course homepage.)

Examinator: Gabriele Ferretti tel. 7723168, 0762293068 ferretti@chalmers.se

Allowed material during the exam:

- The course textbook J.J. Sakurai and Jim Napolitano, Modern Quantum Mechanics Second Edition (2010).
  NB: The old red cover version: J.J. Sakurai, Modern Quantum Mechanics Revised Edition (1994) is *also* allowed.
- A standard calculator.

Write the final answers clearly marked by Ans: ...
and underline them.

You may use without proof any formula in the book.

There is a total of 30 points in this test. The exam counts for 90% of the final grade, (that is $3\times$ points %). The grades are assigned according to the table in the course homepage.

Problem 1

Consider the Hamiltonian $H = H_0 + H'$ where

$$H_0 = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_1 & 0 \\ 0 & 0 & E_2 \end{pmatrix} \quad \text{and} \quad H' = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a & b & 0 \end{pmatrix} \tag{1}$$

where $E_1 < E_2$, H' is a perturbation and a and b are real.
Q1 (*1 points*) What is the degeneracy of the energy levels of H_0?
Q2 (*2 points*) Show that the spectrum is not corrected to first order in perturbation theory.
Q3 (*2 points*) Find the second order correction to the level E_2. (NB: Do not try to do the second order correction for E_1.)

Problem 2

A one dimensional harmonic oscillator of angular frequency ω is in its ground state ($n = 0$) for $t < 0$. For $t > 0$ it is subjected to a potential

$$V(x) = \lambda \mathrm{e}^{-t/\tau} x^4 \tag{2}$$

where x is the coordinate operator and λ a small constant.
Q1 (*2 points*) Discuss what are the transitions $0 \to n$ allowed to first order perturbation theory.
Q2 (*3 points*) Find the probability of finding the oscillator in the $n = 4$ excited state at $t \to +\infty$.

Problem 3

Consider a mixed ensemble of spin 1/2 neutral atoms. The spin ensemble averages of the beam along the **x** and **y** directions are measured to be $[S_x] = 0.4$ and $[S_y] = 0.3$.
Q1 (*1 points*) What other measurement is required to completely specify ρ?
Q2 (*3 points*) What is the maximum value we can expect for $[S_z]$?
Q3 (*3 points*) What can we say about the beam in the case when $[S_z]$ is maximum?

Problem 4

Consider the following statement: "For any three operators A, B and C on some Hilbert space, if A commutes with B and B commutes with C then A commutes with C."
Q1 (*1 points*) Is the statement true or false?
Q2 (*2 points*) If true, give a proof, if false, give a counterexample.

Problem 5

The Hamiltonian of a one-dimensional harmonic oscillator is given by

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2 \tag{3}$$

where for simplicity we set all dimensionfull parameters to one.

Consider the trial wave function

$$\psi(x) = \begin{cases} 1 + (x/a) & \text{for } -a < x < 0 \\ 1 - (x/a) & \text{for } 0 < x < a \\ \quad 0 & \text{for } |x| > a \end{cases} \tag{4}$$

Q1 (*3 points*) Perform the variational calculation and find the energy of the ground state in this approximation.
Q2 (*1 points*) Sketch the form of the trial wave function you would use to compute the energy of the first excited level.

Problem 6

Q1 (*One point for every commutator.*)

Compute the following six commutators of operators in the Hilbert space of a single particle. ($r^2 = x^2 + y^2 + z^2$, $p =$ momentum, $L =$ orbital angular momentum, $S =$ spin.)

$$[L^2, S_x]$$
$$[L^2, xy]$$
$$[p_x, r^2]$$
$$[L_z, p_y]$$
$$[p_x, \sin x]$$
$$[L_z, p^2]$$

Problem 1

Q1: E_1 doubly degenerate, E_2 non degenerate.

Q2: The eigenvectors corresponding to E_1 are:

$|E_1,1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $|E_1,2\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$.

and: $\langle E_1, i|H'|E_1, j\rangle = 0$ $i,j = 1$ or 2.

(Note: $H'|E_1,1\rangle = \begin{pmatrix}0\\0\\a\end{pmatrix}$, $H'|E_1,2\rangle = \begin{pmatrix}0\\0\\b\end{pmatrix}$).

$\Rightarrow$ No 1st order correction to E_1.

The eigenvector of E_2 is $|E_2\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$

and similarly: $\langle E_2|H'|E_2\rangle = 0$.

$\Rightarrow$ No 1st order correction to E_2.

Q3: $$E_2^{(2)} = \sum_{q=1,2} \frac{|\langle E_1, q|H'|E_2\rangle|^2}{E_2 - E_1} =$$

$$= \frac{1}{E_2-E_1}\left\{ \left|(1\,0\,0)\,H'\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2 + \left|(0\,1\,0)\,H'\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2 \right\} =$$

$$= \frac{1}{E_2-E_1}\left\{ \left|(1\,0\,0)\begin{pmatrix}a\\b\\0\end{pmatrix}\right|^2 + \left|(0\,1\,0)\begin{pmatrix}a\\b\\0\end{pmatrix}\right|^2 \right\} =$$

$$= \frac{|a|^2 + |b|^2}{E_2 - E_1}$$

PROBLEM 2

Q1: Transition prob. amplitude:

$$C_m^{(1)} = -\frac{i}{\hbar}\int_0^t e^{im\omega t'} \cdot \lambda e^{-t/\tau} \langle m|x^4|0\rangle.$$

$$|m\rangle = \frac{1}{\sqrt{m!}}(a^\dagger)^m|0\rangle. \qquad x^4 = \left(\frac{\hbar}{2m\omega}\right)^2 (a+a^\dagger)^4$$

Note: $(a+a^\dagger)^4 = \cancel{a^4} + a^3a^\dagger + \cancel{a^2a^\dagger a} + \cancel{aa^\dagger a^2} + \cancel{a^\dagger a^3} +$
$+ a^2a^{\dagger 2} + aa^\dagger aa^\dagger + a^\dagger a^2 a^\dagger + \cancel{aa^{\dagger 2}a} + \cancel{a^\dagger aa^\dagger a}$
$+ \cancel{a^{\dagger 2}a^2} + \cancel{a^{\dagger 3}a} + a^{\dagger 2}aa^\dagger + a^\dagger aa^{\dagger 2} + aa^{\dagger 3} + a^{\dagger 4}$
(16 terms.).

I can immediately drop the 8 terms with a to the right because $a|0\rangle = 0$

The term $a^3a^\dagger$ also gives zero on $|0\rangle$:

$a^3a^\dagger|0\rangle = a^2[a,a^\dagger]|0\rangle = a^2|0\rangle = 0$.

The terms $a^2a^{\dagger 2}$, $aa^\dagger aa^\dagger$ and $a^\dagger a^2 a^\dagger$ contain the same amount of a's and $a^\dagger$'s and give zero on $\langle m| \;\text{''}\; |0\rangle$ for $m>0$.

The terms $a^{\dagger 2}aa^\dagger$, $a^\dagger aa^{\dagger 2}$, $aa^{\dagger 3}$ give non zero only for $n=2$ and $a^{\dagger 4}$ requires $n=4$: Only $0 \to 2$ or $0 \to 4$ allowed

Q2: First note: $\langle 4 | x^4 | 0 \rangle =$

$$= \left(\frac{\hbar}{2m\omega}\right)^2 \cdot \frac{1}{\sqrt{4!}} \underbrace{\langle 0 | a^4 a^{+4} | 0 \rangle}_{= 4!} = \left(\frac{\hbar}{2m\omega}\right)^2 \cdot \sqrt{4!}$$

$$\Rightarrow c^{(1)}(t) = -\frac{i}{\hbar} \int_0^t dt' \, e^{i\omega 4 t'} \cdot \lambda e^{-t'/\tau} \cdot \left(\frac{\hbar}{2m\omega}\right)^2 \cdot \sqrt{4!}$$

$$= -\frac{i}{\hbar} \lambda \sqrt{\frac{3}{2}} \cdot \left(\frac{\hbar}{m\omega}\right)^2 \cdot \int_0^t dt' \, e^{(i\omega 4 - \frac{1}{\tau})t'} =$$

$$= -\frac{i}{\hbar} \lambda \sqrt{\frac{3}{2}} \left(\frac{\hbar}{m\omega}\right)^2 \cdot \frac{e^{4i\omega t - t/\tau} - 1}{4i\omega - \frac{1}{\tau}}$$

$$P_{0 \to 4} = \lim_{t \to +\infty} |c^{(1)}(t)| = \frac{3\lambda^2 \hbar^2}{2 m^4 \omega^4} \cdot \frac{1}{(4\omega)^2 + \frac{1}{\tau^2}}$$

PROBLEM 3 $(\hbar \equiv 1)$.

Q1: We need something along $\hat{z}$, e.g. $[S_z]$

Q2: Let $[S_z] = x$ unknow.

let $\rho = \begin{pmatrix} a & b-ic \\ b+ic & 1-a \end{pmatrix}$ general 2×2 $\rho = \rho^\dagger$, $\mathrm{tr}\,\rho = 1$.

$$[S_x] = \frac{1}{2}\mathrm{tr}\,\rho\sigma_x = b = 0.4$$

$$[S_y] = \frac{1}{2}\mathrm{tr}\,\rho\sigma_y = c = 0.3.$$

$$[S_z] = \frac{1}{2}\mathrm{tr}\,\rho\sigma_z = \frac{1}{2}(2a-1) = x \Rightarrow a = x + \frac{1}{2}$$

$$\mathrm{tr}\,\rho^2 = 2x^2 + \frac{1}{2} + b^2 + c^2 \leq 1$$

$$\Rightarrow x \leq \sqrt{\frac{1}{4} - \frac{b^2+c^2}{2}} \simeq 0.35$$

Q3. For $x = 0.35$ $\mathrm{tr}\,\rho^2 = 1$

$\Rightarrow$ pure ensemble.

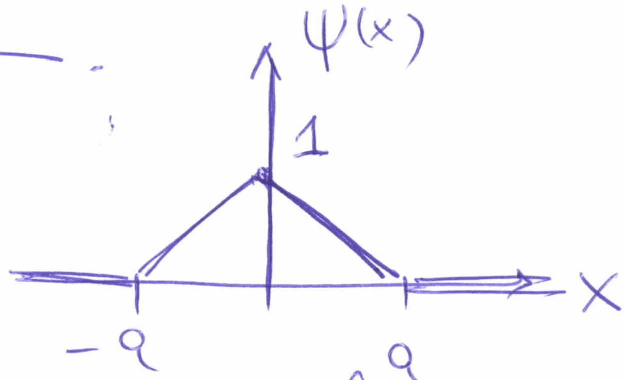
PROBLEM 4

Q1: FALSE

Q2: $A = L_x,\ B = \mathbb{L}^2,\ C = L_y$
(or many other!)

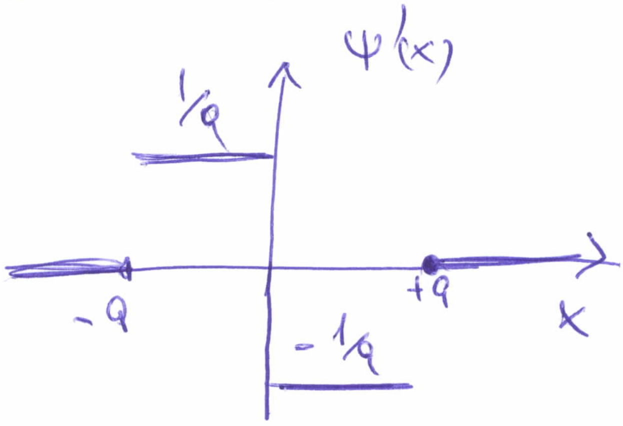
PROBLEM 5

Q1:



$$\|\psi\|^2 = \int dx\, \psi^2(x) = 2\int_0^a \left(1-\frac{x}{a}\right)^2 dx = \frac{2}{3}a.$$

$$\frac{d}{dx}\psi = \begin{cases} 1/a & -a < x < 0 \\ -1/a & 0 < x < a \\ 0 & |x| > a \end{cases}$$



$$\frac{d^2}{dx^2}\psi = \frac{1}{a}\delta(x+a) - \frac{2}{a}\delta(x) + \frac{1}{a}\delta(x-a).$$

$$\Rightarrow \left\langle \frac{1}{2}p^2 \right\rangle = -\frac{1}{2}\int \psi\, \psi''(x)\, dx = -\frac{1}{2}\cdot\left(-\frac{2}{a}\right) = \frac{1}{a}.$$

⑥

$$\langle \tfrac{1}{2}x^2 \rangle = \frac{1}{2}\int x^2 \varphi^2(x)\,dx = \frac{1}{2}\cdot 2\cdot \int_0^a x^2\left(1-\frac{x}{a}\right)^2 dx =$$

$$= \frac{a^3}{30}.$$

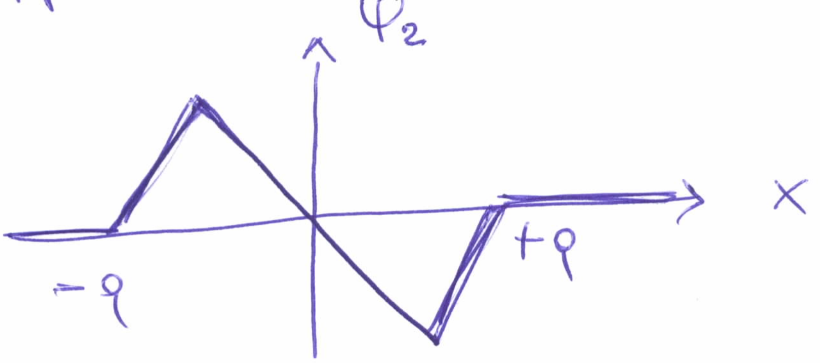
$$E(a) = \frac{\frac{1}{a} + \frac{a^3}{30}}{\frac{2}{3}a} = \frac{3}{2}\frac{1}{a^2} + \frac{a^2}{20}.$$

Minimum for $\frac{dE}{da^2} = 0 \Rightarrow a^2 = \sqrt{30}$.

$$\Rightarrow E_{min} = \frac{3}{2}\cdot\left(\frac{1}{\sqrt{30}}\right) + \frac{\sqrt{30}}{20} \simeq 0{,}548$$

(not bad compared w/ EXACT $= \frac{1}{2}$).

Q2: I need something $\perp$ to φ w/ opposite parity and 1 node, e.g.:



Problem 6

$\hbar = 1$

Q1 : 1) $[\mathbb{L}^2, S_x] = 0$

2) $[\mathbb{L}^2, xy] = [L_x^2, xy] + [L_y^2, xy] + [L_z^2, xy] =$

$= \dots = i\, x L_x z + i\, x z L_x - i L_y z y - z L_y y +$

$+ i L_z y^2 + i y L_z y - i x L_z x - i x^2 L_z = \dots$

(can also be written in term of p's)

3) $[P_x, r^2] = [P_x, x^2] = x[P_x x] + [P_x x]x$

$= -2ix.$

4) $[L_z, P_y] = [\underbrace{\cancel{x P_y}}_{0} - y P_x, P_y] =$

$= -[y, P_y] P_x = -i P_x$

5) $[P_x, \sin x] = -i \frac{d}{dx} \circ \sin x - \sin x \circ \left(-\frac{i\, d}{dx}\right) =$

$= -i \sin' x = -i \cos x.$

6) $[L_z, P^2] = 0$ by rotational invariance