Quantum Mechanics FKA081/FIM400

Final Exam October 25 2012

Next review time for the exam:
NOVEMBER 23 between 15-17 in my room.

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Allowed material during the exam:

- The course textbook J.J. Sakurai and Jim Napolitano, Modern Quantum Mechanics Second Edition (2010).
  NB: The old red cover version: J.J. Sakurai, Modern Quantum Mechanics Revised Edition (1994) is *also* allowed.
- A standard calculator.

Write the final answers clearly marked by Ans: ...
and underline them.

You may use without proof any formula in the book.

There is a total of 30 points in this test. The exam counts for 90% of the final grade, (that is $3\times$ points %). The grades are assigned according to the table in the course homepage.

Problem 1

A particle can be trapped in one of three identical potential wells on a surface. The three wells are arranged along an equilateral triangle and a particle trapped in one of them has energy $-E_0 < 0$. Denote by $|A\rangle$, $|B\rangle$, $|C\rangle$, the state when the particle is in the well A, B or C.
Q1 (*1 point*) If no tunneling is possible what is the energy and the degeneracy of the ground state?
Q2 (*1 point*) What does the object $|A\rangle\langle B|$ represent?

Now turn on a tunneling effect between the wells and describe it by an Hamiltonian constructed with operators like the ones in Q2 up to an overall constant Δ.
Q3 (*3 points*) Compute the eigenvalues of the system with tunneling.

Problem 2

Two particles of equal mass move in a common one dimensional harmonic oscillator potential $V(x) = m\omega^2 x^2/2$. Write down the energy levels of the system *including their degeneracy* for the three following cases:
Q1 (*2 points*) The particles have no spin and are distinguishable (that is, they are not the same type of particle although they have the same mass).
Q2 (*2 points*) The particles have no spin and are indistinguishable.
Q3 (*2 points*) The particles have spin 1/2, are indistinguishable, and the total spin quantum number of S_z is $m = 1$.

Problem 3

An observable is represented by the matrix

$$\Omega = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{1}$$

A particular state is represented by

$$|\psi\rangle = \mathcal{N} \begin{pmatrix} 1 \\ a \\ 1 \end{pmatrix} \tag{2}$$

for a certain normalization constant $\mathcal{N}$ and a real number a.
Q1 (*1 point*) What are the possible results of a measurement of Ω?
Q2 (*1 point*) What is the expectation value of Ω for the state $|\psi\rangle$?
Q3 (*1 point*) What is the probability that with the system in the state $|\psi\rangle$ the measurement of Ω will yield the largest possible results?
Q4 (*2 points*) Use $|\psi\rangle$ to obtain a variational estimate of the lowest eigenvalue of Ω.

Problem 4

Consider a rigid rotor with Hamiltonian

$$H = \frac{1}{2I}\mathbf{L}^2 \tag{3}$$

where the vector $\mathbf{L}$ is the angular momentum operator and I is a given constant representing the moment of inertia.

Q1 (*1 point*) What are the energy levels of the system and what is their degeneracy?

Suppose the system is perturbed by a small potential

$$V = \lambda \sin\theta \cos\phi \tag{4}$$

where ϕ, θ are the usual polar angles.

Q2 (*2 points*) Show that the first order shift to the energy of the ground state vanishes.

Q3 (*3 points*) Calculate to the second order the shift to the energy of the ground state.

You may use the spherical harmonics given in the book. There is no radial part in this problem, the wave functions are functions of ϕ and θ only.

Problem 5

An electron and a neutron (denoted by e and n) interact with each other and with an external magnetic field $\mathbf{B}$ as follows:

$$H = 2a\mathbf{S}_e \cdot \mathbf{S}_n + \mu_e \mathbf{S}_e \cdot \mathbf{B} + \mu_n \mathbf{S}_n \cdot \mathbf{B} \tag{5}$$

where $\mathbf{S}_e$, $\mathbf{S}_n$ are the spin operators for each particle and μ_e, μ_n their magnetic moments.

Q1 (*1 point*) What are the eigenvalues of the system when $\mathbf{B} = 0$?

Q2 (*1 point*) What is the leading correction to the eigenvalues in Q1 when $\mathbf{B}$ is small but non zero?

Q3 (*1 point*) What are the eigenvalues of the system when $a = 0$?

Q4 (*1 point*) What is the leading correction to the eigenvalues in Q3 when a is small but non zero?

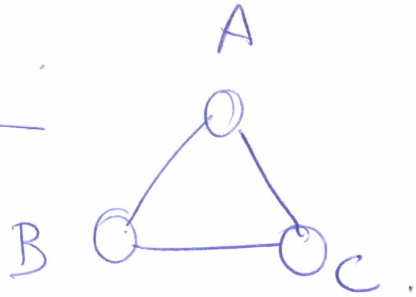
Problem 6

A one dimensional harmonic oscillator is in its ground state for $t < 0$. For $t > 0$ it is subjected to a potential

$$V(x) = -\lambda \mathrm{e}^{-t/\tau} x \tag{6}$$

where x is the coordinate operator and λ a small constant.
Q1 (*2 points*) Using first order time dependent perturbation theory find the probability of finding the oscillator in its first excited state at $t \to +\infty$.
Q2 (*2 points*) Given two arbitrary states $|m\rangle$ and $|n\rangle$, to which order in λ do you expect a transition $m \to n$ to occur and why?

PROBLEM 1.



Q1 $|A\rangle, |B\rangle, |C\rangle$ all have energy $= -E_0$.

$\underbrace{|A\rangle, |B\rangle, |C\rangle}_{\text{degeneracy} = 3.}$ $H_0 = -E_0 \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$.

Q2 $|A\rangle\langle B|$ "moves" a part. from B to A. and annihilates a particle in A or C.

$$H' = \Delta\left(|A\rangle\langle B| + |B\rangle\langle A| + |A\rangle\langle C| + |C\rangle\langle A| + |B\rangle\langle C| + |C\rangle\langle B|\right)$$

Q3: $$H_0 + H' = \begin{pmatrix} -E_0 & \Delta & \Delta \\ \Delta & -E_0 & \Delta \\ \Delta & \Delta & -E_0 \end{pmatrix} \equiv H.$$

$$\det(H - \lambda \mathbb{1}) = \begin{vmatrix} -(E_0+\lambda) & \Delta & \Delta \\ \Delta & -(E_0+\lambda) & \Delta \\ \Delta & \Delta & -(E_0+\lambda) \end{vmatrix} =$$

(set $x = E_0 + \lambda$) $\quad -x^3 + 2\Delta^3 + 3x\Delta^2 = 0$.

one sol. by inspection: $x = -\Delta$ $\therefore$

$$-x^3 + 2\Delta^3 + 3x\Delta^2 = -(x+\Delta)(x^2 - x\Delta + 2\Delta^2)$$

$\Rightarrow x = -\Delta, -\Delta, 2\Delta$.

$\Rightarrow \underline{\lambda = -E_0 - \Delta, \; -E_0 - \Delta, \; -E_0 + 2\Delta}$.

PROBLEM 2.

$$E = \hbar\omega\left(n_1 + n_2 + \tfrac{1}{2} + \tfrac{1}{2}\right) = \hbar\omega(N+1)$$

Q1

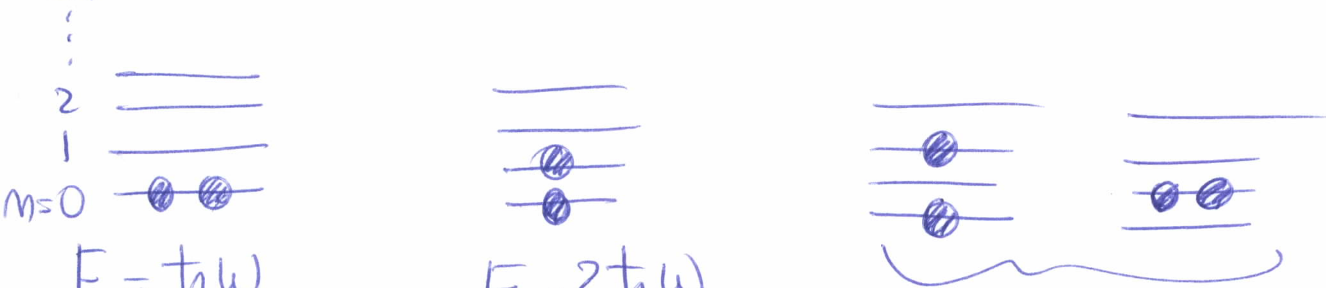
$E = \hbar\omega$, deg = 1

$E = 2\hbar\omega$, deg = 2

$E = 3\hbar\omega$, deg = 3

in general for $E = \hbar\omega(N+1)$, deg $= N$

Q2



$E = \hbar\omega$, deg = 1

$E = 2\hbar\omega$, deg = 1

$E = 3\hbar\omega$, deg = 2

in general for $E = \hbar\omega(N+1)$ $\quad \text{deg} = \begin{cases} \frac{N+2}{2} & N \text{ even} \\ \frac{N+1}{2} & N \text{ odd.} \end{cases}$

Q3

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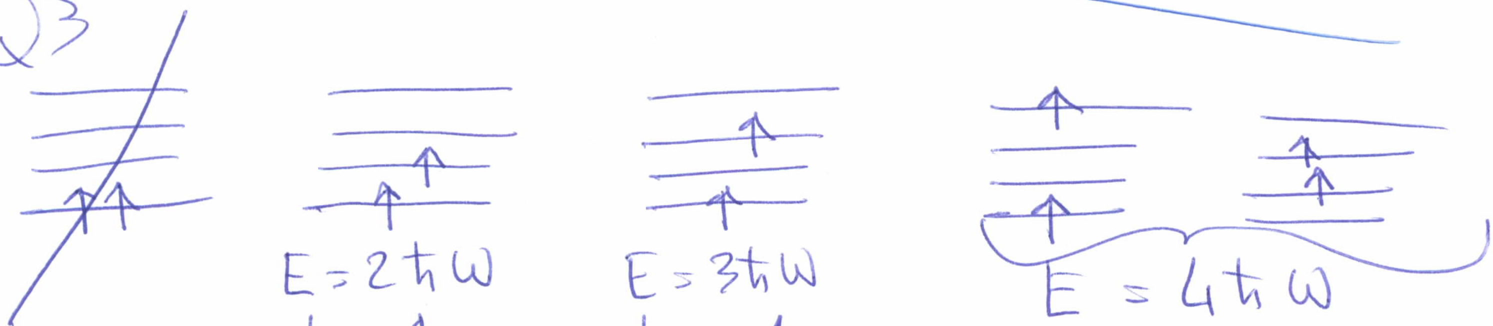

$E = 2\hbar\omega$, deg = 1

$E = 3\hbar\omega$, deg = 1

$E = 4\hbar\omega$, deg = 2

in general, for $E = \hbar\omega(N+1)$ $\quad \text{deg} = \begin{cases} N/2 & N \text{ even} \\ \frac{N+1}{2} & N \text{ odd.} \end{cases}$

PROBLEM 3

Q1 : $\det(\Omega - \lambda \mathbb{1}) = -\lambda^3 + \lambda = 0 \Rightarrow \underline{\lambda = 0, \pm 1.}$

Q2 : $\langle \psi | \psi \rangle = |N|^2 (2 + a^2) = 1 \Rightarrow N = \dfrac{1}{\sqrt{2+a^2}}.$

$$\langle \Omega \rangle = |N|^2 (1\, a\, 1) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ a \\ 1 \end{pmatrix} =$$

$$= \frac{1}{\sqrt{2}(2+a^2)} (1\, a\, 1) \begin{pmatrix} a \\ 2 \\ a \end{pmatrix} = \frac{4a}{\sqrt{2}(2+a^2)} = \underline{\frac{2\sqrt{2}\, a}{2+a^2}.}$$

Q3 . The $|+1\rangle$ eigenstate is found by solving $(\Omega - \mathbb{1})|+1\rangle = 0$ and <u>normalizing</u> :

$$|+1\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}. \quad P(+1) = |\langle +1 | \psi \rangle|^2 =$$

$$= \frac{1}{4(2+a^2)} \left| (1\, \sqrt{2}\, 1) \begin{pmatrix} 1 \\ a \\ 1 \end{pmatrix} \right|^2 = \underline{\frac{(2+\sqrt{2}\,a)^2}{4(2+a^2)}}.$$

Q4 $\dfrac{d\langle \Omega \rangle}{da} = 0$ ← from Q2 $\Rightarrow \dfrac{8 - 4a^2}{(2+a^2)^2} = 0$

$\Rightarrow a = \pm\sqrt{2}$ chosing $a = -\sqrt{2}$ gives

$\underline{\langle \Omega \rangle = -1}$ exact.

PROBLEM 4.

Q1 : Energy $E_\ell^{(0)} = \frac{\hbar^2}{2I}\ell(\ell+1)$ $|\ell, m\rangle$.

Degeneracy : $2\ell+1$.

Q2 : Ground state. $|00\rangle = Y_0^0(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}$.

$$E_0^{(1)} = \langle 00|V|00\rangle = \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta \, \frac{1}{\sqrt{4\pi}} \cdot \lambda \sin\theta \cos\varphi \cdot \frac{1}{\sqrt{4\pi}} = 0.$$

Q3: Write $V = \lambda\sqrt{\frac{2\pi}{3}}\left(Y_1^{-1} - Y_1^1\right)$.

$$E_0^{(2)} = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} \frac{|\langle \ell m|V|00\rangle|^2}{E_0^{(0)} - E_\ell^{(0)}} = \frac{|\langle 11|V|00\rangle|^2 + |\langle 1{-1}|V|00\rangle|^2}{-E_1^{(0)}}$$

$$\langle 11|V|00\rangle = \int d\varphi \, d\cos\theta \, Y_1^{1*} \lambda\sqrt{\frac{2\pi}{3}}\left(Y_1^{-1} - Y_1^1\right)\frac{1}{\sqrt{4\pi}} =$$

$$= -\frac{\lambda}{\sqrt{6}}$$

similarly $\langle 1,-1|V|00\rangle = \frac{\lambda}{\sqrt{6}}$

$$\Rightarrow E_0^{(2)} = \frac{\lambda^2/6 + \lambda^2/6}{-\frac{\hbar^2 \cdot 2}{2I}} = -\frac{\lambda^2 I}{3\hbar^2}.$$

PROBLEM 5 (set $\hbar = 1$ for convenience).

Q1: $H_0 = 2a\, \vec{S}_e \cdot \vec{S}_n = 2a \cdot \frac{1}{2}\left(S(S+1) - \frac{3}{4} - \frac{3}{4}\right)$

$$= \begin{cases} -\frac{3}{2}a & \text{for } S=0 \\ \frac{1}{2}a & \text{for } S=1 \end{cases}$$

Q2 $|S=0, m=0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$

$|S=1, m=1\rangle = |++\rangle$

$|S=1, m=0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |+-\rangle\right)$

$|S=1, m=-1\rangle = |--\rangle$.

$H' = B\left(\mu_e S_{ez} + \mu_n S_{nz}\right)$

(using rotational invariance).

$\langle 00| H' |00\rangle = \frac{B}{2}\left(\langle +-| - \langle -+|\right)\left(\frac{\mu_e - \mu_n}{2}|+-\rangle - \frac{-\mu_e + \mu_n}{2}|-+\rangle\right) =$

$= \frac{B}{4}\left(\mu_e - \mu_n + \mu_n - \mu_e\right) = 0$

similarly $\langle 10| H' |10\rangle = 0$

also: $\langle 11| H' |11\rangle = B\left(\mu_e \frac{1}{2} + \mu_n \frac{1}{2}\right) = \frac{B}{2}(\mu_e + \mu_n)$

$\langle 1-1| H' |1-1\rangle = -\frac{B}{2}(\mu_e + \mu_n)$.

Q3: $H_0 = B\left(\mu_e S_{ez} + \mu_n S_{nz}\right)$.

eigenvalues: $\frac{B}{2}\left(\pm \mu_e \pm \mu_n\right)$. for $|\underset{m_e}{\pm}, \underset{m_n}{\pm}\rangle$.

Q4 $H' = 2a\left(\underbrace{\frac{S_{e+}S_{n-} + S_{e-}S_{n+}}{2}}_{\text{does Not contribute}} + S_{ez}S_{nz}\right)$

$\langle m_e m_n | H' | m_e m_n \rangle = 2a\, m_e m_n$.

two possible values: $\pm \frac{a}{2}$.

PROBLEM 6

Q1: $$c^{(1)}(t) = -\frac{i}{\hbar}\int_0^t e^{i\omega_0 t'}\left(-\lambda e^{-t'/\tau}\underbrace{\langle 1|x|0\rangle}_{\sqrt{\frac{\hbar}{2m\omega_0}}}\right)dt' =$$

$$= \frac{i\lambda}{\sqrt{2m\omega_0\hbar}}\int_0^t e^{(i\omega_0-\frac{1}{\tau})t'}dt =$$

$$= \frac{i\lambda}{\sqrt{2m\omega_0\hbar}}\cdot\frac{1}{i\omega_0-\frac{1}{\tau}}\left[e^{(i\omega_0-\frac{1}{\tau})t'}\right]_0^t \to \frac{i\lambda}{\sqrt{2m\omega_0\hbar}}\cdot\frac{-1}{i\omega_0-\frac{1}{\tau}}$$

as $t\to+\infty$. SINCE $e^{-t/\tau}\to 0$ as $t\to+\infty$.

$$P = |c^1(\infty)|^2 = \frac{\lambda^2}{2m\omega_0\hbar}\cdot\frac{1}{\omega_0^2+\frac{1}{\tau^2}}.$$

Q2: Since to go from $|m\rangle \to |n\rangle$

I need at least $x^{|m-n|}$ a

reasonable guess is $\lambda^{2|m-n|}$.

Also eg: $n=0$, $m=3$:

$$\begin{array}{ll} 3 & P\propto\lambda^2 \\ 2 & P\propto\lambda^2 \\ 1 & P\propto\lambda^2 \\ 0 & \end{array} \Bigg\} P\propto\lambda^6.$$