Mathematics Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2012–03–05; kl 8.30-12.30.

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 6p. Valid bonus poits will be added to the scores.
Breakings: **3**: 15-20p, **4**: 21-27p och **5**: 28p- For GU students**G**:15-24p, **VG**: 25p-
For solutions and gradings information see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1112/index.html

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1. Prove the interpolation error estimate: $||f-\pi_1 f||_{L_\infty(a,b)} \le C_i(b-a)^2||f''||_{L_\infty(a,b)}$.

2. Prove an a priori and an a posteriori error estimate, in the H^1-norm: $\|u\|_{H^1} := \|u'\|_{L_2(0,1)}$, for the cG(1) finite element method for the following convection-diffusion-absorption problem

$$-u''(x)+2xu'(x)+u(x)=f(x), \quad \text{for} \quad x\in(0,1) \qquad \text{and} \quad u(0)=u(1)=0.$$

3. Consider the heat equation in $\Omega\times[0,T]\subset\mathbb{R}^2\times\mathbb{R}^+$, $(u=u(x,t))$,

$$\dot u-\Delta u=f, \quad t>0; \qquad u(x,t)=0, \quad x\in\partial\Omega, \quad \text{and} \quad u(x,0)=u_0(x), \quad x\in\Omega.$$

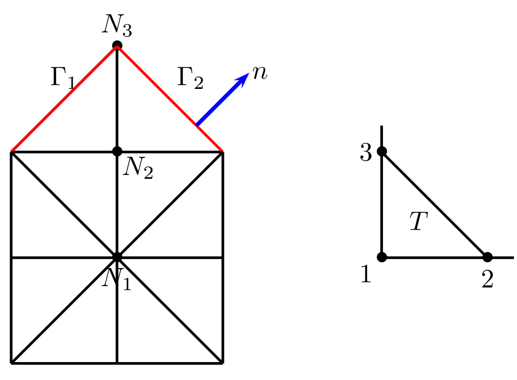
Use the Poincare inequality $(\nabla u,\nabla u)\ge\alpha||u||^2$, $\alpha>0$, and prove the following stability estimate

$$||u(t)||^2+\alpha\int_0^t||u(s)||^2\,ds\le||u_0||^2+\frac{1}{\alpha}\int_0^t||f(s)||^2\,ds, \qquad ||w(s)||^2:=\int_\Omega|w(x,s)|^2\,dx.$$

4. Compute the stiffness and mass matrices as well as load vector for the cG(1) approximation for

$$-\varepsilon\Delta u+u=1, \quad x\in\Omega; \qquad u=0, \quad x\in\partial\Omega\setminus(\Gamma_1\cup\Gamma_2), \quad \nabla u\cdot n=0, \quad x\in\Gamma_1\cup\Gamma_2,$$

where $\varepsilon>0$ and n is the outward unit normal to $\partial\Omega$, (obs! 3 nodes N_1, N_2 and N_3, see Fig.)
Hint: You may first compute the matrices for a standard triangle-element T.



5. Formulate the Lax-Milgram Theorem. Verify the assumptions of the Lax-Milgram Theorem and determine the constants of the assumptions in the case: $I=(0,1)$, $f\in L_2(I)$, $V=H^1(I)$ and

$$a(v,w)=\int_I(uw+v'w')\,dx+v(0)w(0), \quad L(v)=\int_I fv\,dx. \qquad ||w||_V^2=||w||^2_{L_2(I)}+||w'||^2_{L_2(I)}.$$

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TMA372/MMG800: Partial Differential Equations, 2012–03–05; kl 8.30-12.30..
Lösningar/Solutions.

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1. See Lecture Notes or text book chapter 5.

2. We multiply the differential equation by a test function $v \in H_0^1(I)$, $I = (0,1)$ and integrate over I. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

$$\int_I (u'v' + 2xu'v + uv) = \int_I fv, \quad \forall v \in H_0^1(I). \tag{1}$$

A *Finite Element Method* with $cG(1)$ reads as follows: Find $U \in V_h^0$ such that

$$\int_I (U'v' + 2xU'v + Uv) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I), \tag{2}$$

where

$$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I,\ v(0) = v(1) = 0\}.$$

Now let $e = u - U$, then (1)-(2) gives that

$$\int_I (e'v' + 2xe'v + ev) = 0, \quad \forall v \in V_h^0. \tag{3}$$

A posteriori error estimate: We note that using $e(0) = e(1) = 0$, we get

$$\int_I 2xe'e = \int_I x\frac{d}{dx}(e^2) = (xe^2)|_0^1 - \int_I e^2 = -\int_I e^2, \tag{4}$$

so that using variational formulation (1) to replace the terms involving continuous solution u and the finite element method (2) to insert the interpolant $\pi_h e$ of the error we can compute

$$\begin{aligned}
\|e\|_{H^1}^2 &= \int_I e'e' = \int_I (e'e' + 2xe'e + ee) \\
&= \int_I ((u-U)'e' + 2x(u-U)'e + (u-U)e) = \{v = e \text{ in } (1)\} \\
&= \int_I fe - \int_I (U'e' + 2xU'e + Ue) = \{v = \pi_h e \text{ in } (2)\} \\
&= \int_I f(e - \pi_h e) - \int_I \Big(U'(e-\pi_h e)' + 2xU'(e - \pi_h e) + U(e - \pi_h e)\Big) \\
&= \{P.I. \text{ on each subinterval}\} = \int_I \mathcal{R}(U)(e - \pi_h e),
\end{aligned} \tag{5}$$

where $\mathcal{R}(U) := f - 2xU' - U$, (for approximation with piecewise linears, $U'' \equiv 0$, on each subinterval). Thus (5) implies that

$$\begin{aligned}
\|e\|_{H^1}^2 &\le \|h\mathcal{R}(U)\|\,\|h^{-1}(e - \pi_h e)\| \\
&\le C_i \|h\mathcal{R}(U)\|\,\|e'\| \le C_i \|h\mathcal{R}(U)\|\,\|e\|_{H^1},
\end{aligned}$$

where C_i is an interpolation constant, and hence we have with $\|\cdot\| = \|\cdot\|_{L_2(I)}$ that

$$\|e\|_{H^1} \le C_i \|h\mathcal{R}(U)\|.$$

A priori error estimate: We use (4) and write

$$\begin{aligned}
\|e\|_{H^1}^2 &= \int_I e'e' = \int_I (e'e' + 2xe'e + ee) \\
&= \int_I \Big(e'(u-U)' + 2xe'(u-U) + e(u-U) \Big) = \{v = U - \pi_h u \text{ in}(3)\} \\
&= \int_I \Big(e'(u-\pi_h u)' + 2xe'(u-\pi_h u) + e(u-\pi_h u) \Big) \\
&\leq \|(u-\pi_h u)'\| \|e'\| + 2\|u-\pi_h u\| \|e'\| + \|u-\pi_h u\| \|e\| \\
&\leq \{\|(u-\pi_h u)'\| + 3\|u-\pi_h u\|\} \|e\|_{H^1} \\
&\leq C_i\{\|hu''\| + \|h^2 u''\|\} \|e\|_{H^1},
\end{aligned}$$

where in the last step we used Poincare inequality. This gives that

$$\|e\|_{H^1} \leq C_i\{\|hu''\| + \|h^2 u''\|\},$$

which is the a priori error estimate.

3. Multiply the differential equation by $u(x,t)$, integrate over the space domain. Then using the Green's formula and the Poincare inequality we get

$$(f,u) = (\dot{u}, u) + (\nabla u, \nabla u) \geq \frac{1}{2}\frac{d}{dt}\|u\|^2 + \alpha\|u\|^2.$$

Now

$$\Big|(\frac{1}{\sqrt{2\varepsilon}} f, \sqrt{2\varepsilon} u)\Big| \leq \frac{1}{2}\Big(\frac{1}{2\varepsilon}\|f\|^2 + 2\varepsilon\|u\|^2\Big) = \frac{1}{4\varepsilon}\|f\|^2 + \varepsilon\|u\|^2.$$

With $\varepsilon = \alpha/2$ we get

$$\frac{1}{\alpha}\|f\|^2 + \alpha\|u\|^2 \geq \frac{d}{dt}\|u\|^2 + 2\alpha\|u\|^2.$$

Integrating in time yields

$$\|u(t)\|^2 - \|u_0\|^2 + \alpha \int_0^t \|u(s)\|^2\, ds \leq \frac{1}{\alpha}\int_0^t \|f(s)\|^2\, ds.$$

4. Let V be the linear function space defined by

$$V_h := \{v : v \text{ is continuous in } \Omega, \;\; v = 0, \;\; \text{on } \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)\}.$$

Multiplying the differential equation by $v \in V$ and integrating over Ω we get that

$$-(\Delta u, v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Now using Green's formula we have that

$$\begin{aligned}
-(\Delta u, \nabla v) &= (\nabla u, \nabla v) - \int_{\partial\Omega} (n\cdot\nabla u) v\, ds \\
&= (\nabla u, \nabla v) - \int_{\partial\Omega\setminus(\Gamma_1\cup\Gamma_2)} (n\cdot\nabla u) v\, ds - \int_{\Gamma_1\cup\Gamma_2} (n\cdot\nabla u) v\, ds \\
&= (\nabla u, \nabla v), \qquad \forall v \in V.
\end{aligned}$$

Thus the variational formulation is:

$$(\nabla u, \nabla v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Let V_h be the usual finite element space consisting of continuous piecewise linear functions satisfying the boundary condition $v = 0$ on $\partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)$: The $cG(1)$ method is: Find $U \in V_h$ such that

$$(\nabla U, \nabla v) + (U, v) = (f, v) \qquad \forall v \in V_h$$

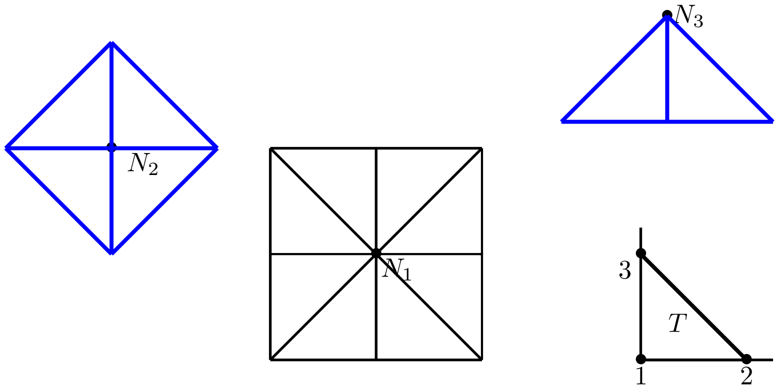
Making the "Ansatz" $U(x) = \sum_{j=1}^{3} \xi_j \varphi_j(x)$, where φ_i are the standard basis functions, we obtain the system of equations

$$\sum_{j=1}^{3} \xi_j \Big(\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx + \int_\Omega \varphi_i \varphi_i \, dx \Big) = \int_\Omega f \varphi_j \, dx, \quad i = 1, 2, 3,$$

or, in matrix form,

$$(S + M)\xi = F,$$

where $S_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix, $M_{ij} = (\varphi_i, \varphi_j)$ is the mass matrix, and $F_i = (f, \varphi_i)$ is the load vector.



We first compute the mass and stiffness matrix for the reference triangle T. The local basis functions are

$$\begin{aligned} \phi_1(x_1, x_2) &= 1 - \frac{x_1}{h} - \frac{x_2}{h}, & \nabla\phi_1(x_1, x_2) &= -\frac{1}{h} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \phi_2(x_1, x_2) &= \frac{x_1}{h}, & \nabla\phi_2(x_1, x_2) &= \frac{1}{h} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \phi_3(x_1, x_2) &= \frac{x_2}{h}, & \nabla\phi_3(x_1, x_2) &= \frac{1}{h} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Hence, with $|T| = \int_T dz = h^2/2$,

$$\begin{aligned} m_{11} &= (\phi_1, \phi_1) = \int_T \phi_1^2 \, dx = h^2 \int_0^1 \int_0^{1-x_2} (1 - x_1 - x_2)^2 \, dx_1 dx_2 = \frac{h^2}{12}, \\ s_{11} &= (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2 \, dx = \frac{2}{h^2}|T| = 1. \end{aligned}$$

Alternatively, we can use the midpoint rule, which is exact for polynomials of degree 2 (precision 3):

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1^2 \, dx = \frac{|T|}{3} \sum_{j=1}^{3} \phi_1(\hat{x}_j)^2 = \frac{h^2}{6} \Big(0 + \frac{1}{4} + \frac{1}{4} \Big) = \frac{h^2}{12},$$

where $\hat{x}_j$ are the midpoints of the edges. Similarly we can compute the other elements and obtain

$$m = \frac{h^2}{24} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad s = \frac{1}{2} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrices M and S from the local ones m and s:

$$
\begin{aligned}
M_{11} &= 8m_{22} = \frac{8}{12}h^2, & S_{11} &= 8s_{22} = 4,\\
M_{12} &= 2m_{12} = \frac{1}{12}h^2, & S_{12} &= 2s_{12} = -1,\\
M_{13} &= 0, & S_{13} &= 2s_{23} = 0,\\
M_{22} &= 4m_{11} = \frac{4}{12}h^2, & S_{22} &= 4s_{11} = 4,\\
M_{23} &= 2m_{12} = \frac{1}{12}h^2, & S_{23} &= 2s_{12} = -1,\\
M_{33} &= 2m_{22} = \frac{2}{12}h^2, & S_{33} &= 2s_{22} = 1.
\end{aligned}
$$

The remaining matrix elements are obtained by symmetry $M_{ij} = M_{ji}$, $S_{ij} = S_{ji}$. Hence,

$$
M = \frac{h^2}{12}\begin{bmatrix} 8 & 1 & 0\\ 1 & 4 & 1\\ 0 & 1 & 2\end{bmatrix}, \quad S = \varepsilon\begin{bmatrix} 4 & -1 & 0\\ -1 & 4 & -1\\ 0 & -1 & 1\end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} (1,\varphi_1)\\ (1,\varphi_2)\\ (1,\varphi_3)\end{bmatrix} = \begin{bmatrix} 8\cdot\frac{1}{3}\cdot\frac{1}{2} = \frac{4}{3}\\ 4\cdot\frac{1}{3}\cdot\frac{1}{2} = \frac{2}{3}\\ 2\cdot\frac{1}{3}\cdot\frac{1}{2} = \frac{1}{3}\end{bmatrix}.
$$

5. For the formulation of the Lax-Milgram theorem see the book, Chapter 21.
As for the given case: $I = (0,1)$, $f \in L_2(I)$, $V = H^1(I)$ and

$$
a(v,w) = \int_I (uw + v'w')\,dx + v(0)w(0), \quad L(v) = \int_I fv\,dx,
$$

it is trivial to show that $a(\cdot,\cdot)$ is bilinear and $b(\cdot)$ is linear. We have that

$$
(6) \qquad a(v,v) = \int_I v^2 + (v')^2\,dx + v(0)^2 \geq \int_I (v)^2\,dx + \frac{1}{2}\int_I (v')^2\,dx + \frac{1}{2}v(0)^2 + \frac{1}{2}\int_I (v')^2\,dx.
$$

Further

$$
v(x) = v(0) + \int_0^x v'(y)\,dy, \quad \forall x \in I
$$

implies

$$
v^2(x) \leq 2\Big(v(0)^2 + (\int_0^x v'(y)\,dy)^2\Big) \leq \{C-S\} \leq 2v(0)^2 + 2\int_0^1 v'(y)^2\,dy,
$$

so that

$$
\frac{1}{2}v(0)^2 + \frac{1}{2}\int_0^1 v'(y)^2\,dy \geq \frac{1}{4}v^2(x), \quad \forall x \in I.
$$

Integrating over x we get

$$
(7) \qquad \frac{1}{2}v(0)^2 + \frac{1}{2}\int_0^1 v'(y)^2\,dy \geq \frac{1}{4}\int_I v^2(x)\,dx.
$$

Now combining (6) and (7) we get

$$
\begin{aligned}
a(v,v) &\geq \frac{5}{4}\int_I v^2(x)\,dx + \frac{1}{2}\int_I (v')^2(x)\,dx\\
&\geq \frac{1}{2}\Big(\int_I v^2(x)\,dx + \int_I (v')^2(x)\,dx\Big) = \frac{1}{2}||v||_V^2,
\end{aligned}
$$

so that we can take $\kappa_1 = 1/2$. Further

$$
\begin{aligned}
|a(v,w)| &\le \Big|\int_I vw\,dx\Big| + \Big|\int_I v'w'\,dx\Big| + |v(0)w(0)| \le \{C-S\} \\
&\le ||v||_{L_2(I)}||w||_{L_2(I)} + ||v'||_{L_2(I)}||w'||_{L_2(I)} + |v(0)||w(0)| \\
&\le \Big(||v||_{L_2(I)} + ||v'||_{L_2(I)}\Big)\Big(||w||_{L_2(I)} + ||w'||_{L_2(I)}\Big) + |v(0)||w(0)| \\
&\le \sqrt{2}\Big(||v||^2_{L_2(I)} + ||v'||^2_{L_2(I)}\Big)^{1/2} \cdot \sqrt{2}\Big(||w||^2_{L_2(I)} + ||w'||^2_{L_2(I)}\Big)^{1/2} + |v(0)||w(0)| \\
&\le \sqrt{2}||v||_V\sqrt{2}||w||_V + |v(0)||w(0)|.
\end{aligned}
$$

Now we have that

$$
v(0) = -\int_0^x v'(y)\,dy + v(x), \quad \forall x \in I, \tag{8}
$$

and by the Mean-value theorem for the integrals: $\exists \xi \in I$ so that $v(\xi) = \int_0^1 v(y)\,dy$. Choose $x = \xi$ in (8) then

$$
\begin{aligned}
|v(0)| &= \Big| -\int_0^\xi v'(y)\,dy + \int_0^1 v(y)\,dy\Big| \\
&\le \int_0^1 |v'|\,dy + \int_0^1 |v|\,dy \le \{C-S\} \le ||v'||_{L_2(I)} + ||v||_{L_2(I)} \le 2||v||_V,
\end{aligned}
$$

implies that

$$
|v(0)||w(0)| \le 4||v||_V||w||_V,
$$

and consequently

$$
|a(u,w)| \le 2||v||_V||w||_V + 4||v||_V||w||_V = 6||v||_V||w||_V,
$$

so that we can take $\kappa_2 = 6$. Finally

$$
|L(v)| = \Big|\int_I fv\,dx\Big| \le ||f||_{L_2(I)}||v||_{L_2(I)} \le ||f||_{L_2(I)}||v||_V,
$$

taking $\kappa_3 = ||f||_{L_2(I)}$ all the conditions in the Lax-Milgram theorem are fulfilled.

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