Chalmers/Gothenburg University
Mathematical Sciences

EXAM

TMA947/MAN280
OPTIMIZATION, BASIC COURSE

Date:	11–12–12
Time:	House V, morning
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Adam Wojciechowski (0703-088304)
Result announced:	12–01–09 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \geq 1, \\ & x_1 - x_2 \geq -2, \\ & x_1 \geq 0, \\ & x_2 \geq 0. \end{aligned}$$

(2p) a) Solve this problem using phase I (so that you begin with a unit matrix as the first basis) and phase II of the simplex method.

Aid: utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(1p) b) If an optimal solution exists, use your calculations to decide if it unique. If the problem is unbounded, use your calculations to specify the direction of unboundedness.

(3p) Question 2

(modeling)

An airplane has a route that takes it from city 1 to city n by going from city i to $i+1$, where $i = 1, \ldots, n-1$. Let

- w_i be the weight, excluding fuel, of the plane on flight from city i to $i+1$, $i = 1, \ldots, n-1$,
- c_i be the cost of fuel per unit weight at city i, $i = 1, \ldots, n$,
- K_i be the maximum amount of fuel that can be purchased in city i, $i = 1, \ldots, n$,

- M the maximum weight of fuel that can be loaded into the plane.

Let z_i be the variables denoting the total combined weight of the plane, including fuel, at takeoff from city i, $i = 1, \dots, n-1$. Assume that the amount of fuel (in weight units) needed to fly from city i to $i+1$, $i = 1, \dots, n-1$, is $\alpha_i z_i$, where α_i are given positive constants. Formulate a linar program that determines how much fuel one should buy at each city, such that the total fual cost for completing the trip is minimized.

Question 3

(interior penalty methods)

Consider the problem to

$$\begin{aligned} &\text{minimize } f(\boldsymbol{x}) := (x_1 - 2)^4 + (x_1 - 2x_2)^2, \\ &\text{subject to } g(\boldsymbol{x}) := x_1^2 - x_2 \leq 0. \end{aligned}$$

We attack this problem with an interior penalty (barrier) method, using the barrier function $\phi(s) = -s^{-1}$. The penalty problem is to

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \;\; f(\boldsymbol{x}) + \nu \hat{\chi}_S(\boldsymbol{x}), \tag{1}$$

where $\hat{\chi}_S(\boldsymbol{x}) = \phi(g(\boldsymbol{x}))$, for a sequence of positive, decreasing values of the penalty parameter ν.

We repeat a general convergence result for the interior penalty method below.

Theorem 1 (convergence of an interior point algorithm) *Let the objective function $f : \mathbb{R}^n \to \mathbb{R}$ and the functions g_i, $i = 1, \dots, m$, defining the inequality constraints be in $C^1(\mathbb{R}^n)$. Further assume that the barrier function $\phi : \mathbb{R}_- \to \mathbb{R}_+$ is in C^1 and that $\phi'(s) \geq 0$ for all $s < 0$.*

Consider a sequence $\{\boldsymbol{x}_k\}$ of points that are stationary for the sequence of problems (1) with $\nu = \nu_k$, for some positive sequence of penalty parameters $\{\nu_k\}$ converging to 0. Assume that $\lim_{k \to +\infty} \boldsymbol{x}_k = \hat{\boldsymbol{x}}$, and that LICQ holds at $\hat{\boldsymbol{x}}$. Then, $\hat{\boldsymbol{x}}$ is a KKT point of the problem at hand.

In other words,

$$\left.\begin{array}{r} \boldsymbol{x}_k \textit{ stationary in } (1) \\ \boldsymbol{x}_k \to \hat{\boldsymbol{x}} \textit{ as } k \to +\infty \\ \textit{LICQ holds at } \hat{\boldsymbol{x}} \end{array}\right\} \implies \hat{\boldsymbol{x}} \textit{ stationary in our problem.}$$

(1p) a) Does the above theorem apply to the problem at hand and the selection of the penalty function?

(2p) b) Implementing the above-mentioned procedure, the first value of the penalty parameter was set to $\nu_0 = 10$, which is then divided by ten in each iteration, and the initial problem (1) was solved from the strictly feasible point $(0, 1)^{\mathrm{T}}$. The algorithm terminated after six iterations with the following results: $\boldsymbol{x}_6 \approx (0.94389, 0.89635)^{\mathrm{T}}$, and the multiplier estimate [given by $\nu_6 \phi'(g(\boldsymbol{x}_6))$] $\hat{\mu}_6 \approx 3.385$. Confirm that the vector $\boldsymbol{x}_6$ is close to being a KKT point. Are the KKT point(s) globally optimal? Why/Why not?

Question 4

(Lagrangian duality)

Consider the quadratic problem

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \end{aligned} \tag{1}$$

where $\boldsymbol{Q}$ is a symmetric matrix.

(1p) a) Assume that $\boldsymbol{Q}$ is positive definite. Construct the Lagrangian dual problem by relaxing all the constraints and show that the dual problem itself is a quadratic problem.

Hint: An explicit solution to the problem $\min_{x \in X} L(\boldsymbol{x}, \boldsymbol{\mu})$ can be found for each $\boldsymbol{\mu}$.

(1p) b) Is the dual function always strictly concave if $\boldsymbol{Q}$ is positive definite? If so, provide a proof. If not, provide a counter example.

(1p) c) Consider the following properties:

i) the primal is a convex problem;

ii) the dual is a convex problem;

iii) the dual objective function q is differentiable;

iv) the duality gap is zero (i.e. $q^* = f^*$).

Which of these hold when $\boldsymbol{Q}$ is positive definite? Which properties do the primal and dual problems have when $\boldsymbol{Q}$ has a negative eigenvalue? Motivate your answers!

(3p) Question 5

(optimality conditions)

Farkas' Lemma can be stated as follows:

Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{b}$ an $m \times 1$ vector. Then exactly one of the systems

$$\begin{aligned} \boldsymbol{Ax} &= \boldsymbol{b}, \\ \boldsymbol{x} &\geq \boldsymbol{0}^n, \end{aligned} \tag{I}$$

and

$$\begin{aligned} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} &\leq \boldsymbol{0}^n, \\ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} &> 0, \end{aligned} \tag{II}$$

has a feasible solution, and the other system is inconsistent.

Prove Farkas' Lemma.

(3p) Question 6

(LP duality)

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{Ax} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned} \tag{P}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{c} \in \mathbb{R}^n$ are given matrices.

Assume that the program (P) has multiple optimal solutions. You are therefore interested in finding an optimal solution to (P) that has the minimum value with respect to another linear objective function, $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{x}$. Formulate a linear program which will yield such an optimal solution, without first solving the problem (P).

Hint: There is a means to describe the set of primal–dual optimal solutions to (P) as a system of linear inequalities.

(3p) Question 7

(sequential linear programming)

Consider the following nonlinear programming problem: find $\boldsymbol{x}^* \in \mathbb{R}^n$ that solves the problem to

$$\begin{aligned} &\text{minimize } f(\boldsymbol{x}), && && \text{(1a)}\\ &\text{subject to } g_i(\boldsymbol{x}) \leq 0, && i = 1, \dots, m, && \text{(1b)}\\ &\phantom{\text{subject to }} h_j(\boldsymbol{x}) = 0, && j = 1, \dots, \ell, && \text{(1c)} \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, g_i, and $h_j : \mathbb{R}^n \to \mathbb{R}$ are given functions in C^1 on $\mathbb{R}^n$.

We are interested in establishing that the classic Sequential Linear Programming (SLP) subproblem tells us whether an iterate $\boldsymbol{x}_k \in \mathbb{R}^n$ satisfies the KKT conditions or not, thereby establishing a natural termination criterion for the SLP algorithm.

Given the feasible iterate $\boldsymbol{x}_k$, the SLP subproblem is to

$$\begin{aligned} &\underset{p}{\text{minimize}} \ \nabla f(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p}, && && \text{(2a)}\\ &\text{subject to } g_i(\boldsymbol{x}_k) + \nabla g_i(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p} \leq 0, && i = 1, \dots, m, && \text{(2b)}\\ &\phantom{\text{subject to }} h_j(\boldsymbol{x}_k) + \nabla h_j(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p} = 0, && j = 1, \dots, \ell, && \text{(2c)}\\ &\phantom{\text{subject to }} -1 \leq p_s \leq 1, && s = 1, \dots, n. && \text{(2d)} \end{aligned}$$

This subproblem is natural: it is based on a linearization of both the objective function and the constraint functions, whereby it resembles the Frank–Wolfe method. The main difference, of course, is that the problem (1) has general and perhaps nonlinear constraints which in the subproblem (2) therefore are replaced by first-order Taylor approximations.

Establish the following statement: the vector $\boldsymbol{x}_k$ is a KKT point in the problem (1) if and only if $\boldsymbol{p} = \mathbf{0}^n$ is a globally optimal solution to the SLP subproblem (2). In other words, the SLP algorithm terminates if and only if $\boldsymbol{x}_k$ is a KKT point in the original problem (1).

Hint: Compare the KKT conditions of (1) and (2).

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EXAM SOLUTION

TMA947/MAN280
APPLIED OPTIMIZATION

Date: 11–12–12
Examiner: Michael Patriksson

Question 1

(the simplex method)

(2p) a) We first rewrite the problem on standard form. We multiply the objective by (-1) to obtain a minimization problem, multibly the second constraint by (-1) to obtain a positive r.h.s., and introduce slack variables s_1 and s_2.

$$\begin{array}{lrrrrrl} \text{minimize} & z = & -x_1 & -2x_2 & & & \\ \text{subject to} & & x_1 & +x_2 & -s_1 & & = 1 \\ & & -x_1 & +x_2 & & +s_2 & = 2 \\ & & x_1, & x_2, & s_1, & s_2 & \geq 0. \end{array}$$

In phase I the artificial variable a is added in the first constraint, s_2 is used as the second basic variable in order to obtain a unit matrix as the first basis. We obtain the phase I problem

$$\begin{array}{lrrrrrrl} \text{minimize} & w = & & & & & a & \\ \text{subject to} & & x_1 & +x_2 & -s_1 & & +a & = 1 \\ & & -x_1 & +x_2 & & +s_2 & & = 2 \\ & & x_1, & x_2, & s_1, & s_2, & a & \geq 0. \end{array}$$

The starting BFS is thus $(a, s_2)^{\mathrm{T}}$. Calculating the vector of reduced costs for the non-basic variables x_1, x_2, s_1 yields $(-1, -1, 1)^{\mathrm{T}}$. We can choose between x_1 and x_2 as entering variable. We let x_2 enter the basis. The minimum ratio test shows that a should leave the basis. We thus have a BFS without artificial variables, and may proceed with phase II.

We have the basic variables (x_2, s_2). The vector of reduced costs for the non-basic variables x_1 and s_1 is $(1, -2)$. We let s_1 enter the basis. The minimum ratio test implies that s_2 leaves the basis. We now have x_2, s_1 as basic variables. The vector of reduced costs for the non-basic variables x_1 and s_1 is $(-1, 2)^{\mathrm{T}}$. Thus we let x_1 enter the basis. We have that the column corresponding to x_1 is $\boldsymbol{B}^{-1}\boldsymbol{N}_1 = (-1, -2)^{\mathrm{T}}$. Hence the problem is unbounded.

(1p) b) The non-basic variable $s_2 = 0$, as we let $x_1 = \mu$ we have that

$$(x_2, s_1)^{\mathrm{T}} = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}_1\mu = (2, 1)^{\mathrm{T}} + (1, 2)^{\mathrm{T}}\mu.$$

Returning to the original variables we have that

$$(x_1, x_2)^{\mathrm{T}} = (0, 2)^{\mathrm{T}} + (1, 1)^{\mathrm{T}}\mu$$

is the direction of unboundedness. To see that this is correct draw the problem!

(3p) Question 2

(modeling) Let x_i be the amount of fuel purchased at city i, $i = 1, \ldots, n$. We also introduce a variable y_i to denote the amount of fuel in the plane when leaving city i. Then we can formulate the problem as

$$\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{n} c_i x_i, && (1)\\
\text{subject to} \quad & x_i \leq K_i, \quad i = 1, \ldots, n && (2)\\
& z_i - w_i = y_i, \quad i = 1, \ldots, n, && (3)\\
& y_i \leq M, \quad i = 1, \ldots, n, && (4)\\
& x_i \leq K_i, \quad i = 1, \ldots, n, && (5)\\
& y_i \geq \alpha_i z_i, \quad i = 1, \ldots, n, && (6)\\
& x_{i+1} + y_i - \alpha_i z_i = y_{i+1}, \quad i = 1, \ldots, n-1, && (7)\\
& x_i, y_i, z_i \geq 0, \quad i = 1, \ldots, n. && (8)
\end{aligned}$$

Question 3

(interior penalty methods)

(1p) a) All functions involved are in C^1. The conditions on the penalty function are fulfilled, since $\phi'(s) = 1/s^2 \geq 0$ for all $s < 0$. Further, LICQ holds everywhere. The answer is yes.

(2p) b) With the given data, it is clear that the only constraint is (almost) fulfilled with equality: $(\boldsymbol{x}_6)_1^2 - (\boldsymbol{x}_6)_2 \approx -0.005422 \approx 0$. We set up the KKT conditions to see whether it is fulfilled approximately. Indeed, we have the following corresponding to the system $\nabla f(\boldsymbol{x}_6) + \hat{\mu}_6 \nabla g(\boldsymbol{x}_6) = \mathbf{0}^2$:

$$\begin{pmatrix} -6.4094265 \\ 3.39524 \end{pmatrix} + 3.385 \begin{pmatrix} 1.88778 \\ -1 \end{pmatrix} \approx \begin{pmatrix} -0.01929 \\ 0.01024 \end{pmatrix},$$

and the right-hand side can be considered near-zero. Since $\hat{\mu}_6 \geq 0$ we approximately fulfill the KKT conditions.

For the last part, we establish that the problem is convex. The feasible set clearly is convex, since g is a convex function and the constraint is on the "$\leq$"-form. The Hessian matrix of f is

$$\begin{pmatrix} 12(x_1 - 2)^2 + 2 & -4 \\ -4 & 8 \end{pmatrix},$$

which is positive semidefinite everywhere (in fact, positive definite outside of the region defined by $x_1 = 2$); hence, f is convex on $\mathbb{R}^2$. We conclude that our problem is convex, and hence the KKT conditions imply global optimality. The vector $\boldsymbol{x}_6$ therefore is an approximate global optimal solution to our problem.

Question 4

(Lagrangian duality)

(1p) a) We begin by constructing the Lagrangian function

$$L(\boldsymbol{x}, \boldsymbol{\mu}) = \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{\mu}^{\mathrm{T}}(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}).$$

The dual function is defined as

$$q(\boldsymbol{\mu}) = \min_{x \in \mathbb{R}^n} L(\boldsymbol{x}, \boldsymbol{\mu}).$$

We have that $\nabla_x^2 L(\boldsymbol{x}, \boldsymbol{\mu}) = \boldsymbol{Q}$ which is positive definite, thus the unconstrained problem defining q is convex. We solve the sufficient optimality condition $\nabla_x L(\boldsymbol{x}, \boldsymbol{\mu}) = \boldsymbol{0}$ and obtain

$$\begin{aligned} \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\mu} = \boldsymbol{0}&, \\ \boldsymbol{x} = \boldsymbol{Q}^{-1}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\mu} - \boldsymbol{c})&. \end{aligned}$$

Inserting this into the definition of the Lagrangian function we obtain

$$\begin{aligned} q(\boldsymbol{\mu}) &= \frac{1}{2}(\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{A} - \boldsymbol{c}^{\mathrm{T}})\boldsymbol{Q}^{-1}\boldsymbol{Q}\boldsymbol{Q}^{-1}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\mu} - \boldsymbol{c}) + (\boldsymbol{c}^{\mathrm{T}} - \boldsymbol{\mu}\boldsymbol{A})\boldsymbol{Q}^{-1}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\mu} - \boldsymbol{c}) + \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{b} \\ &= -\frac{1}{2}(\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{A} - \boldsymbol{c})\boldsymbol{Q}^{-1}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\mu} - \boldsymbol{c}) + \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{b}. \end{aligned}$$

The dual problem is $\min_{\boldsymbol{\mu} \geq 0} q(\boldsymbol{\mu})$ which is in the same form as the original quadratic program after appropriate restructure of terms.

(1p) b) The Hessian of the dual is

$$\nabla^2 q(\boldsymbol{\mu}) = -\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^{\mathrm{T}}.$$

The dual function is always concave, so we know that all eigenvalues are non-negative. The question is if $\boldsymbol{Q}$ has strictly positive eigenvalues, does it imply that the Hessian to q has strictly positive eigenvalues? The answer is no. Consider $\boldsymbol{Q} = \boldsymbol{I}$ and

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

We have that

$$-\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

Adding the first rows to the third shows that the rows are linearly dependent, hence $-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ has zero as an eigenvalue. In fact, if $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $m > n$ then we always obtain 0 as an eigenvalue. A simpler counter-example is possible with one variable and two constraints, but one of the constraints will then be redundant.

(1p) c) If $\boldsymbol{Q}$ is p.d. then the following holds: Since $\boldsymbol{Q}$ is the Hessian of the primal objective, if $\boldsymbol{Q}$ is p.d. then the primal problem is convex. The dual problem is always a convex problem. The dual function is differentialble since it is a second degree polynomial. For a convex problem, the dual gap is zero.

If $\boldsymbol{Q}$ has a negative eigenvalue then the primal problem is no longer convex. Let $\boldsymbol{v}$ be an eigenvector of $\boldsymbol{Q}$ with negative eigenvalue $\lambda < 0$. We have that

$$L(\alpha\boldsymbol{v}, \boldsymbol{\mu}) = \frac{1}{2}\boldsymbol{\lambda}\alpha^2\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v} + \alpha\boldsymbol{c}^{\mathrm{T}}\boldsymbol{v} + \boldsymbol{\mu}^{\mathrm{T}}(\boldsymbol{b} - \alpha\boldsymbol{A}\boldsymbol{v}) \to -\infty,$$

as $\alpha \to \infty$. This implies that $q(\boldsymbol{\mu}) := -\infty$ for all $\boldsymbol{\mu}$. Hence the dual gap is no longer zero unless the primal problem is unbounded.

(3p) Question 5

(optimality conditions)

Farkas' Lemma is established in Theorem 11.10.

(3p) Question 6

(LP duality)

We can write the dual problem as

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}, \\ \text{subject to} \quad & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}, \\ & \boldsymbol{y} \geq \boldsymbol{0}^m. \end{aligned}$$

From weak duality, we know that for any primal feasible $\boldsymbol{x}$ and dual feasible $\boldsymbol{y}$, we have $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \geq \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$. If $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \leq \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$ for a primal feasible $\boldsymbol{x}$ and a dual feasible $\boldsymbol{y}$, we obtain from strong duality that $\boldsymbol{x}$ is optimal in the primal problem, and $\boldsymbol{y}$ is optimal in the dual problem. Hence, all solutions $\boldsymbol{x}$ (respectively, $\boldsymbol{y}$) to the linear inequality system

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} \qquad\quad & \geq \boldsymbol{b}, \\ \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} & \leq \boldsymbol{c}, \\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} & \leq 0, \\ \boldsymbol{x} \qquad\quad & \geq \boldsymbol{0}^n, \\ \boldsymbol{y} & \geq \boldsymbol{0}^m, \end{aligned}$$

will be optimal solutions to the primal (respectively, dual) problem. To find the best optimal solution to the primal problem with respect to the linear function $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{x}$, we can therefore solve the linear program to

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \qquad\quad \geq \boldsymbol{b}, \\ & \qquad\quad \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}, \\ & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \leq 0, \\ & \boldsymbol{x} \qquad\quad \geq \boldsymbol{0}^n, \\ & \qquad\quad \boldsymbol{y} \geq \boldsymbol{0}^m. \end{aligned}$$

(3p) Question 7

(sequential linear programming)

Suppose that $\boldsymbol{p} = \boldsymbol{0}^n$ solves the SLP subproblem (2). When representing the optimality conditions for this problem, we then note that the bound constraints

(2d) on $\boldsymbol{p}$ are redundant. Writing down the KKT conditions for $\boldsymbol{p}$ in the problem (2), we therefore obtain the conditions that

$$\nabla f(\boldsymbol{x}_k) + \sum_{i=1}^{m} \mu_i \nabla g_i(\boldsymbol{x}_k) + \sum_{j=1}^{\ell} \lambda_i \nabla h_i(\boldsymbol{x}_k) = \mathbf{0}^n, \tag{1a}$$

$$\mu_i g_i(\boldsymbol{x}^*) = 0, \qquad i = 1, \ldots, m, \tag{1b}$$

$$\boldsymbol{\mu} \geq \mathbf{0}^m. \tag{1c}$$

But this is a statement that $\boldsymbol{x}^*$ is a KKT point in the original problem.