Chalmers/Gothenburg University
Mathematical Sciences

EXAM

TMA947/MAN280
OPTIMIZATION, BASIC COURSE

Date:	11–04–26
Time:	House V, 8^{30}–12^{30}
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Adam Wojciechowski (0703-088304)
Result announced:	11–05–10 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$\begin{aligned} \text{maximize} \quad & c_1x_1 + c_2x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 4, \\ & x_1 - 2x_2 \geq -2, \\ & x_1 \geq 0, \\ & x_2 \geq 0. \end{aligned}$$

(2p) a) Solve this problem for $c_1 = 1$ and $c_2 = 4$ using phase I (if necessary) and phase II of the simplex method.

Aid: Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(1p) b) For which costs $\boldsymbol{c} = (c_1, c_2)^{\mathrm{T}}$ is the solution optimal?

[*Hint:* Use a graphical representation of the problem and illustrate the answer to the question therein.]

Motivate your answer!

(3p) Question 2

(the Separation Theorem)

The Separation Theorem can be stated as follows.

Suppose that the set $S \subseteq \mathbb{R}^n$ is closed and convex, and that the point $\boldsymbol{y}$ does not lie in S. Then, there exist a vector $\boldsymbol{\pi} \neq \mathbf{0}^n$ and $\alpha \in \mathbb{R}$ such that $\boldsymbol{\pi}^{\mathrm{T}}\boldsymbol{y} > \alpha$ and $\boldsymbol{\pi}^{\mathrm{T}}\boldsymbol{x} \leq \alpha$ for all $\boldsymbol{x} \in S$.

Establish the theorem using basic results from the course. If you rely on other results when performing your proof of the above theorem, then those results must be stated; they may however be utilized without proof.

Question 3

(descent and optimality in optimization)

(1p) a) Consider the function

$$f(\boldsymbol{x}) := x_1^2 - x_1x_2 + 5x_2^3 - 12.$$

At $\boldsymbol{x} = (1,1)^{\mathrm{T}}$, is the vector $\boldsymbol{p} = (1,-2)^{\mathrm{T}}$ a direction is descent?

(2p) b) A continuously differentiable function f has a local minimum at the origin, that is, at $\boldsymbol{x}^* = \mathbf{0}^2$, subject to the constraints that $x_1 - x_2 \leq 0$ and $2x_1 + x_2 \leq 0$. Determine the possible values of $\nabla f(\boldsymbol{x})$ at $\boldsymbol{x}^* = \mathbf{0}^2$.

(3p) Question 4

(optimality conditions)

Consider the problem to

$$\begin{aligned} \text{maximize} \quad & f(\boldsymbol{x}) := \boldsymbol{b}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \sum_{j=1}^{n} x_j^2 \leq 1, \end{aligned}$$

where $\boldsymbol{b} \neq \mathbf{0}^n$ is a given vector.

Show that the unique globally optimal solution is $\boldsymbol{x}^* = \boldsymbol{b}/\|\boldsymbol{b}\|$.

Question 5

(modeling)

Consider a company that sell a product which is imported from a set of producers, $\{1, \ldots, n\}$. Your assignment is to plan the import of the company over a planning period from time 0 to time T. Let c_{jt} be the price per unit product from producer j at time t and let k_{jt} be the maximum amount that can be imported from producer j at time t. Note that if products are imported from a producer at time

t, they will arrive at the company at time $t+1$. The demand for the product at time t is d_t units ($d_0 = 0$). The company has a warehouse where it can store at most M units between each time step, at a cost of f per unit.

(2p) a) Formulate a linear optimization model for the minimization of the cost of importing the product, while fulfilling the demand at each time step.

(1p) b) You are told that there is a possibility of not fulfilling the demand at each time. If the company does not fulfill the demand at some time step, a cost γ per unit shortage has to be paid. Reformulate the model to include this additional fact.

(3p) **Question 6**

(the gradient projection algorithm)

The gradient projection algorithm is a generalization of the steepest descent method to problems over convex sets. Given a feasible point $\boldsymbol{x}^k$, the next point is obtained according to $\boldsymbol{x}^{k+1} = \text{Proj}_X\left(\boldsymbol{x}^k - \alpha_k \nabla f(\boldsymbol{x}^k)\right)$, where X is the convex set over which we minimize, $\alpha_k > 0$ is the step length, and $\text{Proj}_X(y) = \arg\min_{\boldsymbol{x}\in X} \|\boldsymbol{x} - \boldsymbol{y}\|$ denotes the closest point to y in X.

Consider the problem to

$$
\begin{aligned}
\underset{\boldsymbol{x}}{\text{minimize}} \quad & f(\boldsymbol{x}) := x_1^2 + 2x_2^2 - 2x_1x_2 - 2x_1 - 3x_2, \\
\text{subject to} \quad & 0 \leq x_1 \leq 3, \\
& 0 \leq x_2 \leq 2.
\end{aligned}
$$

Start at the point $\boldsymbol{x}^0 = (0, 0)^{\mathrm{T}}$ and perform two iterations of the gradient projection algorithm using step length $\alpha_k = 1$ for all k. You may solve the projection problem in the algorithm graphically. Is the point obtained a global/local minimum? Motivate why/why not.

Question 7

(short questions)

Answer these short questions. You must motivate your answers; an answer without motivation results in zero points.

(1p) a) Consider the following problem:

$$\begin{aligned} \text{maximize} \quad & f(\boldsymbol{x}), \\ \text{subject to} \quad & g_i(\boldsymbol{x}) \leq 0, \quad i \in \{1, \dots, m\}, \\ & \boldsymbol{x} \in \mathbb{R}^n. \end{aligned}$$

Assume that f and g_i, $i \in \{1, \dots, m\}$, are convex, differentiable functions. If $\hat{\boldsymbol{x}}$ is a KKT point for the above problem, can we conclude that $\hat{\boldsymbol{x}}$ is optimal? If not, give a counter-example!

(2p) b) Consider the primal problem to

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}), && \text{(1a)} \\ \text{subject to} \quad & x_1^2 + x_2^2 \leq 4, && \text{(1b)} \\ & x_1 + x_2 \geq 1, && \text{(1c)} \\ & (x_1, x_2) \in X, && \text{(1d)} \end{aligned}$$

where $X \subset \mathbb{R}^2$ is a closed set.

We formulate the Lagrangian dual problem by performing a Lagrangian relaxation of constraints (1b) and (1c), and denote the dual objective function by q. We solve the Lagrngian dual problem with an algorithm that produces a sequence of points converging to a dual point $\hat{\boldsymbol{\mu}}$, which we assume is a KKT point in the Lagrangian dual problem. Can we conclude anything about the optimal objective value of the dual problem? Can we conclude anything about the optimal objective value of the primal problem? Can we conclude something more if X is a convex set and f is a convex function?

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EXAM SOLUTION

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Date: 11–04–26

Examiner: Michael Patriksson

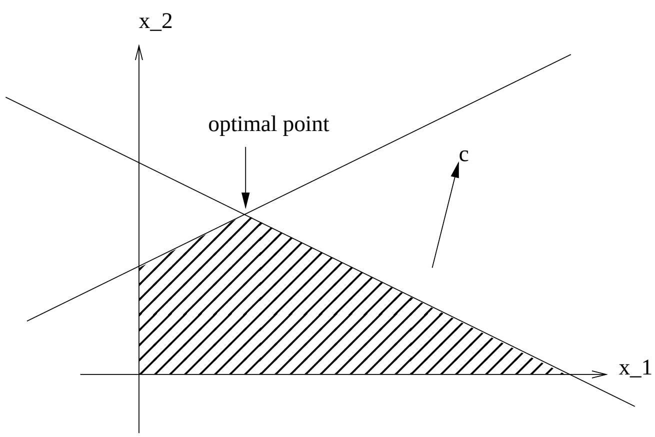
(3p) Question 1

(the simplex method)

(2p) a) We first rewrite the problem on standard form. We multiply the objective by −1 to obtain a minimization problem, multiply one of the constraints by −1 to obtain a non-negative r.h.s. and introduce the slack variables s_1 and s_2. We obtain

$$\begin{array}{llrrrrl} \text{minimize} & z = & -x_1 & -4x_2 & & & \\ \text{subject to} & & x_1 & +2x_2 & +s_1 & & = 4 \\ & & -x_1 & +2x_2 & & +s_2 & = 2 \\ & & x_1, & x_2, & s_1, & s_2 & \geq 0. \end{array}$$

An obvious starting BFS is (s_1, s_2) and we can thus begin with phase II. The vector of reduced costs for x_1 and x_2 yields $(-1, -4)$. Thus x_2 enters the basis. The minimum ratio tests shows that s_2 leaves the basis. The new BFS is thus (s_1, x_2). The reduced costs for x_1 and s_2 are $(-3, 2)$. Thus x_1 enters the basis and minimum ratio test yields that s_1 leaves the basis. The new BFS is (x_1, x_2) and the reduced costs corresponding to s_1 and s_2 are $(3/2, 1/2)$. Hence the solution $(x_1, x_2) = B^{-1}b = (1, 3/2)$ is optimal. We can see that the calculations are correct by drawing a picture.



(1p) b) For the BFS to be optimal, the reduced costs must fulfill

$$c_N^T - c_B^T B^{-1} N \geq 0.$$

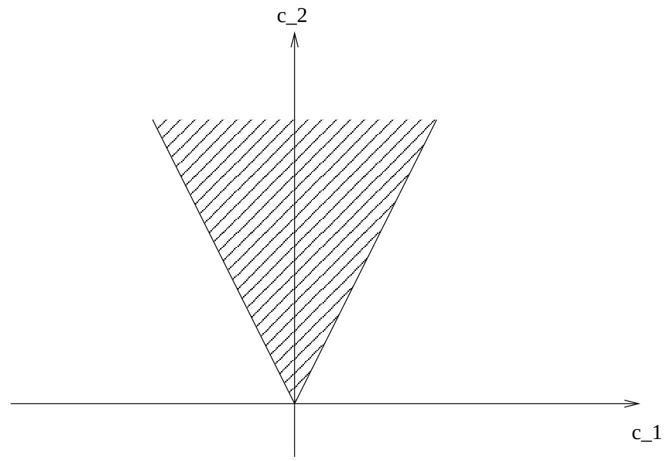
Since $N = I$,

$$B^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ 1/4 & 1/4 \end{pmatrix},$$

$c_N = (0, 0)$ and $c_B = (-c_1, -c_2)$, we obtain

$$\begin{aligned} 1/2c_1 + 1/4c_2 &\geq 0, \\ -1/2c_2 + 1/4c_2 &\geq 0. \end{aligned}$$

The same conclusion can be drawn from the KKT conditions. Drawing the region we obtain:



(3p) **Question 2**

(the Separation Theorem)

See Theorem 4.28 in The Book.

Question 3

(descent and optimality in optimization)

(1p) a) At $\boldsymbol{x} = (1, 1)^{\mathrm{T}}$, $\nabla f(\boldsymbol{x}) = (1, 14)^{\mathrm{T}}$, so $\nabla f(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{p} = (1, 14)(1, -2)^{\mathrm{T}} = -27 < 0$. Yes, the vector $\boldsymbol{p}$ is a vector of descent.

(2p) b) As the only two constraints are affine (in fact linear) Abadie's CQ is fulfilled at every feasible point. Hence, at the local minimum $\boldsymbol{x}^* = \mathbf{0}^2$, the KKT conditions must be fulfilled. The two constraints have non-negative multipliers, μ_1 and μ_2; as the two inequalities are fulfilled with equality at $\boldsymbol{x}^*$, complementarity slackness is fulfilled even if $\mu_i > 0$ for any of $i = 1, 2$, so the only requirements sofar are that $\mu_i \geq 0$, $i = 1, 2$. What remains is to study the requirements from dual feasibility—the first row of the KKT conditions. From the requirement that $\nabla f(\boldsymbol{x}^*) + \boldsymbol{\mu}^{\mathrm{T}} \nabla g(\boldsymbol{x}^*) = \mathbf{0}^2$ we get that

$$\nabla f(\boldsymbol{x}^*) + \mu_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \mu_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \mathbf{0}^2,$$

so we conclude that the value of $\nabla f(\boldsymbol{x}^*)$ is of the form

$$\nabla f(\boldsymbol{x}^*) = \mu_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

for any non-negative values of $\boldsymbol{\mu} \geq \mathbf{0}^2$. (This is the cone spanned by the active constraints at $\boldsymbol{x}^*$, or, in other words, the normal cone to the faesible set at $\boldsymbol{x}^*$.)

(3p) Question 4

(optimality conditions) We rewrite the problem as that to minimize $f(\boldsymbol{x}) := -\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x}$, in order to fit with the standard formulation of the optimality conditions. The problem is now convex, and its only constraint has an interior point, so Slater's CQ is fulfilled. This means that the KKT conditions are necessary as well as sufficient for a global optimal solution.

The solution suggested, $\boldsymbol{x}^* = \boldsymbol{b}/\|\boldsymbol{b}\|$, fulfills the only constraint with equality, whence the Lagrange multiplier may be positive. The first row of the KKT conditions then states that

$$-\boldsymbol{b} + 2\mu^* \boldsymbol{x}^* = \mathbf{0}^n,$$

that is,

$$-\boldsymbol{b} + 2\mu^* \boldsymbol{b}/\|\boldsymbol{b}\| = \mathbf{0}^n.$$

With the identification $\mu^* = \|\boldsymbol{b}\|/2$, we verify that $\boldsymbol{x}^* = \boldsymbol{b}/\|\boldsymbol{b}\|$ fulfills the KKT conditions. As noted above, the problem is a convex one, so $\boldsymbol{x}^* = \boldsymbol{b}/\|\boldsymbol{b}\|$ is indeed the unique globally optimal solution to the problem.

Question 5

(modeling)

(2p) a) Introduce the variables x_{jt} for the amount imported from producer j between time t and $t+1$. Let y_t be the amount stored between time t and $t+1$ and let $y_0 = 0$. The model is to

$$
\begin{aligned}
\text{minimize} \quad & \sum_{t=0}^{T-1} \left(\sum_{j=1}^{n} c_{jt} x_{jt} + f y_t \right), \\
\text{subject to} \quad & \sum_{j=1}^{n} x_{jt} + y_t - y_{t+1} = d_{t+1}, \quad t = 0, \dots, T-1, \\
& y_0 = 0, \\
& y_t \leq M, \quad t = 1, \dots, T, \\
& x_{jt} \leq k_{jt}, \quad t = 0, \dots, T-1, \\
& y_t \geq 0, \quad t = 1, \dots, T, \\
& x_{jt} \geq 0, \quad j = 1, \dots, n, \; t = 0, \dots, T.
\end{aligned}
$$

(1p) b) Introduce z_t as the shortage at time step t. The model is to

$$
\begin{aligned}
\text{minimize} \quad & \sum_{t=0}^{T-1} \left(\sum_{j=1}^{n} c_{jt} x_{jt} + f y_t + \gamma z_t \right), \\
\text{subject to} \quad & \sum_{j=1}^{n} x_{jt} + y_t - y_{t+1} + z_{t+1} = d_{t+1}, \quad t = 0, \dots, T-1, \\
& y_0 = 0, \\
& y_t \leq M, \quad t = 0, \dots, T, \\
& x_{jt} \leq k_{jt}, \quad t = 0, \dots, T-1, \\
& y_t \geq 0, \quad t = 1, \dots, T, \\
& z_t \geq 0, \quad t = 1, \dots, T, \\
& x_{jt} \geq 0, \quad j = 1, \dots, n, \; t = 0, \dots, T.
\end{aligned}
$$

(3p) Question 6

(the gradient projection algorithm)

Iteration 1: We have $\nabla f(\boldsymbol{x}^0) = (-2, -3)^{\mathrm{T}}$. We need to project the point $(0,0)^{\mathrm{T}} - (-2,-3)^{\mathrm{T}} = (2,3)^{\mathrm{T}}$ on the feasible region X. From the figure, we see that $\text{Proj}_{\boldsymbol{x} \in X}\left((2,3)^{\mathrm{T}}\right) = (2,2)^{\mathrm{T}}$. Hence, $\boldsymbol{x}^1 = (2,2)^{\mathrm{T}}$.

Iteration 2: We have $\nabla f(\boldsymbol{x}^1) = (-2, 1)^{\mathrm{T}}$. We need to project the point $(2,2)^{\mathrm{T}} -$

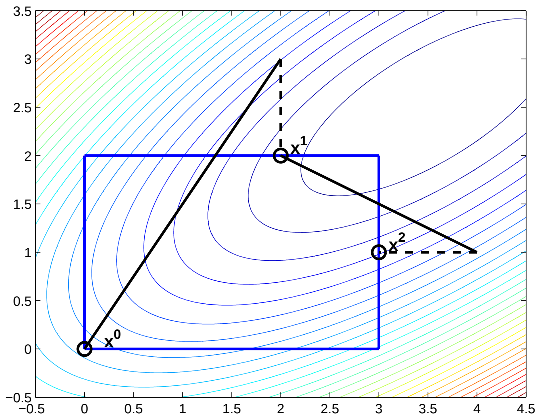


Figure 1: Path taken by the gradient projection algorithm.

$(-2,1)^{\mathrm{T}} = (4,1)^{\mathrm{T}}$ on the feasible region X. From the figure, we see that $\mathrm{Proj}_{\boldsymbol{x}\in X}\left((4,1)^{\mathrm{T}}\right) = (3,1)^{\mathrm{T}}$. Hence, $\boldsymbol{x}^2 = (3,1)^{\mathrm{T}}$.

To check if $\boldsymbol{x}^2$ is a global/local minima, we consider the KKT conditions. All constraints are convex and there exists an inner point, which implies that Slater's constraint qualification holds. Hence, the KKT conditions are necessary for optimality. We can also note that the objective function is convex, which implies that the KKT conditions are also sufficient for optimality. At $\boldsymbol{x}^2$, the only active constraint is $g(\boldsymbol{x}) := x_1 - 3$, and we have $\nabla g(\boldsymbol{x}) = (1,0)^{\mathrm{T}}$. Since $\nabla f(\boldsymbol{x}^2) = (2,-5)$, we note that $\nabla f(\boldsymbol{x}) \neq \mu \nabla g(\boldsymbol{x})$ for any positive μ, and hence $\boldsymbol{x}^2$ is not a KKT point, and is therefore neither a global nor local minima.

Question 7

(short questions)

(1p) a) No! Consider the problem to

$$\underset{x\in[-1,2]}{\text{maximize}}\, x^2;$$

the point $x = -1$ is a KKT point, but it is only a local maximum (not a global). Further, $x = 0$ is also a KKT point, but it is not even a local maximum.

(2p) b) According to Theorem 6.4 (p. 144 in the course book) the dual function is always concave. Hence the dual problem (which we maximize) is a convex problem, which in turn implies that KKT is sufficient for optimality. Thus we can conclude that $\hat{\mu}$ is an optimal solution to the dual problem. Weak duality implies that we have a lower bound on the primal problem, that is, we know that $f(x^*) \geq q(\hat{\mu})$. If the set X is convex and f is convex, we have a convex problem (as the two additional constraints are also convex). Further, an interior point exists with respect to the constraints. We can therefore conclude (from Theorem 6.9 on page 149) that no duality gap exists. Hence $f(x^*) = q(\hat{\mu})$.