Chalmers/Gothenburg University
Mathematical Sciences

EXAM

TMA947/MAN280
OPTIMIZATION, BASIC COURSE

Date:	08–08–28
Time:	House M, morning
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3.
	Questions are *not* numbered by difficulty.
	To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Christoffer Cromvik (0762-721860)
Result announced:	08–04–02
	Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$\begin{array}{llrl}
\text{minimize} & z = & x_1 + x_2 + 3x_3, & \\
\text{subject to} & & -x_2 + 3x_3 & \leq -1, \\
 & & -2x_1 + x_2 - x_3 & \leq 1, \\
 & & x_1, \quad x_2, \quad x_3 & \geq 0.
\end{array}$$

(2p) a) Solve this problem by using phase I and phase II of the simplex method. [*Aid:* Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

for producing basis inverses.]

(1p) b) Is the solution obtained unique? Motivate!

Question 2

(modelling)

Consider the following *portfolio selection problem*. An investor must choose a portfolio $\boldsymbol{x} = (x_1, x_2, \dots, x_n)^{\mathrm{T}}$, where x_j is the proportion of the assets allocated to the j:th security. The return on the portfolio has the mean value $\bar{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{x}$ and the variance $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{V}\boldsymbol{x}$, where $\bar{\boldsymbol{c}}$ is the vector denoting mean returns and $\boldsymbol{V}$ is the matrix of covariances of the returns. The investor would (essentially) like to maximize his/her expected return, while at the same time minimize the variance and hence the risk.

A portfolio is called *efficient* (or *(weakly) Pareto optimal*) if there is no other portfolio having both a larger expected return and a smaller variance.

(2p) a) Formulate an optimization problem whose optimal solution corresponds to an efficient portfolio. (Different models are possible.) Motivate why your model leads to an efficient portfolio.

(1p) b) Suggest a systematic way of generating multiple efficient solutions.

Question 3

(interior penalty method)

Consider the following problem:

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := \tfrac{1}{2}(x_1+1)^2 + \tfrac{1}{2}(x_2+1)^2, \\ \text{subject to} \quad & x_1 \geq 0, \\ & x_2 \geq 0. \end{aligned}$$

(1p) a) Apply the logarithmic interior penalty method for this problem, and show that it converges to a limit point.

(1p) b) Is the limit point a KKT-point?

(1p) c) Under what conditions for a general nonlinear program can we be certain that the interior penalty method converges to a KKT-point?

Question 4

(necessary local and sufficient global optimality conditions)

Consider an optimization problem of the following general form:

$$\text{minimize } f(\boldsymbol{x}), \tag{1a}$$

$$\text{subject to } \boldsymbol{x} \in S, \tag{1b}$$

where $S \subseteq \mathbb{R}^n$ is nonempty, closed and convex, and $f : \mathbb{R}^n \to \mathbb{R}$ is in C^1 on S.

(1p) a) Establish the following result on the local optimality of a vector $\boldsymbol{x}^* \in S$ in this problem.

PROPOSITION 1 (necessary optimality conditions, C^1 case) *If $\boldsymbol{x}^* \in S$ is a local minimum of f over S then*

$$\nabla f(\boldsymbol{x}^*)^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}^*) \geq 0, \qquad \boldsymbol{x} \in S \tag{2}$$

holds.

(2p) b) Establish the following result on the global optimality of a vector $\boldsymbol{x}^* \in S$ in this problem.

THEOREM 2 (necessary and sufficient global optimality conditions, C^1 case) *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is convex on S. Then,*

$$\boldsymbol{x}^* \text{ is a global minimum of } f \text{ over } S \quad \Longleftrightarrow \quad (2) \text{ holds.}$$

Question 5

(Lagrangian duality)

Consider the following quadratic programming problem:

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := x_1^2 + 2x_2^2 - 4x_1 - 8x_2, \\ \text{subject to} \quad & x_1 + x_2 \leq 2, \qquad (1) \\ & x_1, \ x_2 \geq 0. \end{aligned}$$

We will attack this problem by using Lagrangian duality.

(1p) a) Consider Lagrangian relaxing the complicating constraint (1). Write down explicitly the resulting Lagrangian subproblem of minimizing the Lagrange function over the remaining constraints. Construct an explicit formula for the Lagrangian dual function. Establish that it is a concave function.

(1p) b) Solve the Lagrangian dual problem. State in particular the optimal solution, and the optimal value of the dual objective function.

(1p) c) Utilize the result in b) to generate an optimal solution to the original, primal, problem. Verify that strong duality holds. Is the primal optimal solution unique?

(3p) Question 6

(convexity)

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \|\boldsymbol{A}\boldsymbol{x}\|_2^2 \leq 1, \end{aligned}$$

where $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{A}$ is invertible and $\boldsymbol{c} \neq \boldsymbol{0}$. Show that the problem is convex and derive an explicit expression for the optimal solution.

Question 7

(linear programming duality and matrix games)

Let $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^m$, and $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, and consider the canonical LP problem

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned}$$

and its associated dual LP problem. In the following, we denote the respective problem by (P) and (D).

(1p) a) If $m = n$ and $\boldsymbol{A}^{\mathrm{T}} = -\boldsymbol{A}$, we then say that the matrix $\boldsymbol{A}$ is *skew-symmetric.*

Suppose that in the problem (P), the matrix $\boldsymbol{A}$ is skew-symmetric and that $\boldsymbol{b} = -\boldsymbol{c}$ also holds. Establish that if an optimal solution to (P) exists, then $z^* = 0$ holds.

(2p) b) The problem studied in a) is known as a *self-dual* LP problem.

Consider again the canonical primal–dual pair (P), (D) of LP problems. Construct a self-dual LP problem in $n + m$ variables and $n + m$ linear constraints which is equivalent to (P), (D). By "equivalent" we refer to the property that any primal–dual optimal solutions $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ to the pair (P), (D) are obtained immediately as an optimal solution to the problem constructed. (In other words, we can solve any primal–dual pair of canonical LP problems as a self-dual LP problem in a higher dimension.)

Good luck!

Chalmers/Gothenburg University
Mathematical Sciences

EXAM SOLUTION

TMA947/MAN280
APPLIED OPTIMIZATION

Date: 08-08-28
Examiner: Michael Patriksson

Question 1

(the simplex method)

(2p) a) To transform the problem to standard form, the sign on the first constraint must be changed. Then subtract a non-negative slack variable s_1 in the first constraint and add one non-negative slack variable s_2 in the second. A BFS cannot be found directly, hence begin with phase 1 with an artificial variable $a \geq 0$ added in the first constraint - the slack varible s_2 can be used as the other basic variable. The objective is to minimize $w = a$. Start with the BFS given by (a, s_2). In the first iteration of the simplex algorithm, x_2 is the only variable with a negative reduced cost (-1), and is therefore the only eligable incoming variable. The minimum ratio test shows that either a or s_2 can be removed from the basis. By choosing a as the outgoing variable, we can proceed to phase 2.

The reduced costs in the first iteration of the phase 2 problem are

$$\tilde{\boldsymbol{c}}^{\mathrm{T}}_{(x_1,x_3,s_1)} = (1, 6, 1) \geq \mathbf{0},$$

and thus the optimality condition is fulfilled for the current basis. We have $\boldsymbol{x}^*_B = (1, 0)^{\mathrm{T}}$, or in the original variables, $\boldsymbol{x}^* = (x_1, x_2, x_3)^* = (0, 1, 0)^{\mathrm{T}}$, with the optimal value $z^* = 1$.

(1p) b) At the obtained optimal solution all reduced cost are strictly greater than zero, hence the obtained optimal solution must be unique.

Question 2

(modelling)

(2p) a) One possibility is the following formulation:

$$\begin{aligned}
\text{minimize} \quad & f(\boldsymbol{x}) := \mu(-\bar{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{x}) + (1-\mu)\boldsymbol{x}^{\mathrm{T}}\boldsymbol{V}\boldsymbol{x}, \\
\text{subject to} \quad & \sum_{i=1}^{n} x_i = 1, \\
& x_i \geq 0, \qquad\qquad i = 1, \ldots, n,
\end{aligned}$$

where $\mu \in [0,1]$ is a parameter balancing the two objectives. To show that the obtained solution $\boldsymbol{x}^*$ is efficient assume that it is not. Then there is another solution $\boldsymbol{y}^*$ such that $\bar{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{y}^* < \bar{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{x}^*$ and $\boldsymbol{y}^{*\mathrm{T}}\boldsymbol{V}\boldsymbol{y}^* < \boldsymbol{x}^{*\mathrm{T}}\boldsymbol{V}\boldsymbol{x}^*$. But then obviously $f(\boldsymbol{y}^*) < f(\boldsymbol{x}^*)$ which is a contradiction to $\boldsymbol{x}^*$ being optimal.

Another possible formulation is:

$$\begin{aligned}
\text{minimize} \quad & f(\boldsymbol{x}) := -\bar{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{x}, \\
\text{subject to} \quad & \boldsymbol{x}^{\mathrm{T}}\boldsymbol{V}\boldsymbol{x} \leq b, \\
& \sum_{i=1}^{n} x_i = 1, \\
& x_i \geq 0, \qquad\qquad i = 1, \ldots, n,
\end{aligned}$$

where b is a parameter setting the maximum value of the variance objective. Also here, the efficiency can be shown through contradiction.

(1p) b) By varying the parameter values μ and b, respectively, different efficient solutions can be found.

(3p) Question 3

(interior penalty method)

The logarithmic penalty function is

$$P(\boldsymbol{x};\nu) = \tfrac{1}{2}(x_1+1)^2 + \tfrac{1}{2}(x_2+1)^2 - \nu\log x_1 - \nu\log x_2.$$

It is convex, and the first order optimality conditions is

$$\nabla P(\boldsymbol{x};\nu) = \begin{pmatrix} x_1 + 1 - \frac{\nu}{x_1} \\ x_2 + 1 - \frac{\nu}{x_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives a unique optimal solution

$$\boldsymbol{x}^*(\nu) = \frac{-1+\sqrt{1+4\nu}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

due to the requirement $x_1 > 0, x_2 > 0$. As $\nu \to \infty$, we get $\boldsymbol{x}^* = (0,0)^{\mathrm{T}}$. It converges to a KKT-point. According to Theorem 13.6, we can guarantee convergence to a KKT-point if LICQ holds.

Question 4

(necessary local and sufficient global optimality conditions)

(1p) a) See Proposition 4.23.

(2p) b) See Theorem 4.24.

Question 5

(Lagrangian duality)

(1p) a) From the stationarity conditions for the Lagrangian we get that

$$x_1(\mu) = \begin{cases} (4-\mu)/2, & 0 \le \mu \le 4, \\ 0, & 4 \le \mu, \end{cases}$$

respectively,

$$x_2(\mu) = \begin{cases} (8-\mu)/4, & 0 \le \mu \le 8, \\ 0, & 8 \le \mu. \end{cases}$$

We then get the following expression for the Lagrangian dual function, to be maximized over $\mu \ge 0$:

$$q(\mu) = \begin{cases} -(3/8)\mu^2 + 2\mu - 12, & 0 \le \mu \le 4, \\ -(1/8)\mu^2 - 8, & 4 \le \mu \le 8, \\ -2\mu, & \mu \ge 8. \end{cases}$$

The corresponding derivatives then are:

$$q'(\mu) = \begin{cases} -(3/4)\mu + 2, & 0 \leq \mu \leq 4, \\ -(1/4)\mu, & 4 \leq \mu \leq 8, \\ -2, & \mu \geq 8. \end{cases}$$

It is clear that q is concave and differentiable for every $\mu \geq 0$. It is in fact strictly concave.

(1p) b) Setting $q'(\mu) = 0$ as a first trial, we obtain that $q'(\mu) = 0$ for $\mu = 8/3$. Since the dual problem is convex this is the optimal dual solution: $\mu^* = 8/3$. The corresponding objective value is $q(\mu^*) = -9\frac{1}{3}$.

(1p) c) The Lagrangian optimal solution in $\boldsymbol{x}$ for $\mu = \mu^*$ is, from a), $\boldsymbol{x} = (2/3, 4/3)^{\mathrm{T}}$. This is feasible in the primal problem, and $f(\boldsymbol{x}) = q(\mu^*)$ so it is also optimal, by the weak duality theorem. According to duality theory for convex problems over polyhedral sets, all primal optimal solutions are generated from Lagrangian optimal solutions given an optimal dual vector. Since $\boldsymbol{x}(\mu^*)$ here is the unique vector $\boldsymbol{x}^* = (2/3, 4/3)^{\mathrm{T}}$ this must also be the unique optimal solution to the primal problem.

(3p) Question 6

(convexity)

The objective function is convex and the constraint

$$\|\boldsymbol{A}\boldsymbol{x}\|_2^2 = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} \leq 1$$

is convex since it is quadratic and the Hessian $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ is positive definite, since $\boldsymbol{A}$ is invertible. (It is positive semidefinite even if it is not invertible, and hence convex.) This means that the problem is convex. The optimal solution can be computed from the KKT-conditions since e.g. Slater CQ holds.

$$\begin{aligned} c + 2\lambda \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} &= \boldsymbol{0} \\ \|\boldsymbol{A}\boldsymbol{x}\|_2^2 &\leq 1 \\ \lambda\left(\|\boldsymbol{A}\boldsymbol{x}\|_2^2 - 1\right) &= 0 \end{aligned}$$

Since the objective function is linear with nonzero gradient, the optimal solution must be at the boundary of the constraint. The first condition gives

$$\boldsymbol{x} = -\frac{1}{2\lambda}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\right)^{-1}\boldsymbol{c},$$

and hence

$$1 = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} \Rightarrow \lambda = \sqrt{\tfrac{1}{4}\boldsymbol{c}^{\mathrm{T}}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\right)^{-1}\boldsymbol{c}} = \tfrac{1}{2}\|\boldsymbol{A}^{-\mathrm{T}}\boldsymbol{c}\|_2.$$

The optimal solution is

$$\boldsymbol{x}^* = -\frac{1}{\|\boldsymbol{A}^{-\mathrm{T}}\boldsymbol{c}\|_2}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\right)^{-1}\boldsymbol{c}.$$

Question 7

(linear programming duality and matrix games)

(1p) a) Under the given conditions we have that

$$\begin{aligned} z^* &= \text{minimum } \{\, \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}^n \,\} \\ &= \text{maximum } \{\, \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \mid \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}, \quad \boldsymbol{y} \geq \boldsymbol{0}^m \,\} \\ &= \text{maximum } \{\, (-\boldsymbol{c})^{\mathrm{T}}\boldsymbol{y} \mid -\boldsymbol{A}\boldsymbol{y} \leq -\boldsymbol{b}, \quad \boldsymbol{y} \geq \boldsymbol{0}^n \,\} \\ &= \text{maximum } \{\, (-\boldsymbol{c})^{\mathrm{T}}\boldsymbol{y} \mid \boldsymbol{A}\boldsymbol{y} \geq \boldsymbol{b}, \quad \boldsymbol{y} \geq \boldsymbol{0}^n \,\} \\ &= -z^*, \end{aligned}$$

which implies that $z^* = 0$.

(2p) b) The self-dual skew symmetric LP problem sought is

$$\begin{aligned} &\text{minimize } \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}, \\ &\text{subject to } \begin{pmatrix} \boldsymbol{0}^{m\times n} & -\boldsymbol{A}^{\mathrm{T}} \\ \boldsymbol{A} & \boldsymbol{0}^{n\times m} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \geq \begin{pmatrix} -\boldsymbol{c} \\ \boldsymbol{b} \end{pmatrix}, \\ &\qquad\qquad\qquad\qquad (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0}^n \times \boldsymbol{0}^m. \end{aligned}$$
