Chalmers/GU
Mathematics

EXAM

TMA946/MAN280
APPLIED OPTIMIZATION

Date:	03–05–28
Time:	House V, morning
Aids:	Text memory-less calculator
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Richards Grzibovskis (0740-459022)
Result announced:	03–06–12 Short answers are also given at the end of the exam on the notice board for optimization in the MD building.

Exam instructions

When you answer the questions

State your methodology carefully.
Use generally valid methods and theory.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(The simplex method in linear programming)

Consider the following LP problem:

$$
\begin{array}{lrrrrrl}
\text{minimize} & z = & x_1 & -2x_2 & -4x_3 & +4x_4 & \\
\text{subject to} & & & -x_2 & +2x_3 & +x_4 & \leq 4, \\
& & -2x_1 & +x_2 & +x_3 & -4x_4 & \leq 5, \\
& & x_1 & -x_2 & & +2x_4 & \leq 3, \\
& & x_1, & x_2, & x_3, & x_4 & \geq 0.
\end{array}
$$

(2p) a) Find an optimal solution, or an extreme half-line along which the objective function diverges to $-\infty$, by using the Simplex method.

Some matrix inverses that might come in handy are

$$
\begin{pmatrix} 1 & -1 & 0 \\ -4 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} -1 & -1 & 0 \\ -4 & -1 & 0 \\ -2 & 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},
$$

$$
\begin{pmatrix} -1 & 2 & 1 \\ 1 & 1 & -4 \\ -1 & 0 & 2 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -4 & -9 \\ 2 & -1 & -3 \\ 1 & -2 & -3 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 0 \\ -1 & 2 & 3 \end{pmatrix}.
$$

(1p) b) Find a feasible solution which has the objective value $z = -418$.

Question 2

(Modelling)

Suppose that we are interested in describing a feasible region as the *union* of a finite number of convex sets $X_i \subset \Re^n$, that is, that $X := \cup_{i=1}^m X_i$ is the feasible set.

(1p) a) Establish through a counter-example that even though the sets X_i all are convex, their union X may not be.

(2p) b) Suppose that each set X_i is a polyhedron, and that the objective function $f : \Re^n \mapsto \Re$ is linear. Describe how we can formulate the problem to

$$
\underset{x \in X}{\text{minimize}} \ f(x)
$$

as a type of *mixed-integer linear program*, where the variables are of two types, binary variables and continuous variables, and where the problem reduces to an LP whenever the binary variables are fixed.

(3p) Question 3

(Modelling in linear programming)

Nilsson & Nilsson is a small company which manufactures furniture on their own, as well as acting as sub-contractors to other firms. Their main business is to manufacture simple bookshelves made from laminated particle-board (spånskiva). In order to manufacture bookshelves, the firm buys unlaminated particle-boards for b Skr per m^2, and laminates them using their own machines. Pre-laminated boards are available at c Skr per m^2. The boards are cut, and the resulting pieces are assembled into bookshelves. After packaging, the shelves are sold to local stores under the name KALLE for the price of p_1 Skr per item.

Due to a limited demand, the company may sell at most k shelves this way. Impressed by their capabilities, IKEA has asked them to become a supplier of their wildly famous BILLY bookshelf, paying them p_2 Skr per shelf. Each KALLE bookshelf requires a_1 m^2 of particle-board whereas BILLY requires a_2 m^2. Since the KALLE bookshelf is well adapted to the firm's equipment, it only takes e_1 minutes to cut the material into the required pieces for one shelf, whereas each BILLY requires e_2 minutes of cutting time.

Since the idea with IKEA furniture is that the customers themselves assemble the products, a BILLY shelf requires no assembly, whereas one KALLE requires f minutes of assembly-time. Finally, it takes g_1 minutes to package one KALLE shelf, whereas it takes g_2 minuteS to package one BILLY shelf. The company's machines may laminate at most L m^2 of particle-board per month. Two cutting-stations are available to the company, and together they provide 280h of cutting time each month. The company has 16 employees which may split their time between packaging and assembly, and each employee spends 120h per month working (the rest of the time is spent on sick-leave, vacations and so forth).

Formulate the problem to maximize the firm's profit under the given constraints.

Question 4

(The Frank–Wolfe algorithm)

(2p) a) Consider the following constrained nonlinear minimization problem:

$$\begin{aligned} &\underset{(x,y)}{\text{minimize}} \;\; f(x,y) := x^2 + y^2, \\ &\text{subject to} \begin{cases} -1 \le x \le 2, \\ -1 \le y \le 1. \end{cases} \end{aligned}$$

Starting at the point $(x_0, y_0) = (2, 1)$, perform one step of the Frank–Wolfe algorithm. Feel free to plot the problem and perform the algorithmic steps graphically, as long as you describe all the steps in detail with mathematical notation also.

Is the point obtained an optimal solution? Why/why not?

(1p) b) Consider the nonlinear optimization problem with linear constraints:

$$(P) \begin{cases} f^* := \underset{x \in \Re^n}{\text{minimum}} \;\; f(x), \\ \qquad\qquad \text{subject to } Ax \le b, \end{cases}$$

where $f : \Re^n \to \Re$ is convex and differentiable, $A \in \Re^{n \times m}$, $b \in \Re^m$.

The value f^* is the optimal value of f over the feasible set in the problem (P). Let further L_k^* denote the optimal value of the Frank–Wolfe subproblem at the feasible point x_k (where $Ax_k \le b$):

$$(P_L) \begin{cases} L_k^* := \underset{z \in \Re^n}{\text{minimum}} \;\; \nabla f(x_k)^{\mathrm{T}}(z - x_k), \\ \qquad\qquad \text{subject to } Az \le b. \end{cases}$$

Show that $f(x_k) - f^* \le -L_k^*$ always holds. (That is, the optimal value of the Frank–Wolfe subproblem allows us to estimate the distance to the optimum in terms of the objective function by providing a lower bound on f^*.)

Question 5

(Interior penalty methods in nonlinear programming)

Consider the problem to

$$\begin{aligned} \text{minimize } & f(x) := x_1 + x_2, \\ \text{subject to } & g_1(x) := -x_1^2 + x_2 \geq 0, \\ & g_2(x) := x_1 \geq 0. \end{aligned}$$

(1p) a) What is the globally optimal solution to this problem?

Feel free to plot the problem and determine the solution graphically, as long as you can motivate your solution.

(2p) b) Suppose that we attack this problem with the use of an interior penalty (or, barrier) method, where the barrier function is chosen as $\phi(g(x)) := -\log(g(x))$. Describe the steps of the algorithm. Further, suppose that we are able to find the globally optimal solution to each unconstrained penalty problem. Describe these optimal solutions, and prove that their sequence converges to the unique optimal solution to the problem.

Question 6

(Optimality conditions in linear programming)

The *semi-assignment* problem arises in some relaxation methods for problems in integer programming. Its statement is as follows:

$$\begin{aligned} \text{minimize } & \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}, \\ \text{subject to } & \sum_{i=1}^{n} x_{ij} = 1, \qquad j = 1, \dots, n, \\ & x_{ij} \geq 0, \qquad i = 1, \dots, n, \quad j = 1, \dots, n. \end{aligned}$$

Its interpretation as describing a semi-assignment becomes clear if one adds $x_{ij} \in \{0, 1\}$ for all i and j to the constraints: then it is clear that we can interpret each pair (i, j) as a job to be assigned to a particular machine (or member of staff), where i denotes machine and j denotes job. A semi-assignment is one which assigns precisely one machine to each job. (An *assignment* is one in which we also request that each machine is assigned exactly one job; a semi-assignment can assign many jobs to the same machine, and no jobs at all to some others.)

(1p) a) Prove that the constraints $x_{ij} \in \{0,1\}$ are redundant. That is, establish that by solving the above LP problem, we can *automatically* make sure that the resulting optimal solution is binary.

(2p) b) The semi-assignment problem is a very simple LP problem, which can be solved by the following, very simple, greedy algorithm:

For each $j = 1, \dots, n$, do the following:

(1) Find the smallest value of c_{ij} over all $i \in \{1, \dots, n\}$. Let i_j^* be the index corresponding to that least value. (This corresponds to finding the cheapest machine for the job.)

(2) Assign machine i_j^* to job j: let $x_{ij}^* = 1$ for $i = i_j^*$, and $x_{ij}^* = 0$ for $i \neq i_j^*$.

The resulting solution is clearly feasible, since it assigns each job to exactly one machine, and the solution x^* is also non-negative. Prove, by using linear programming duality, in particular the *primal–dual optimality conditions*, that this algorithm is correct, that is, that it does solve the semi-assignment problem.

Hint: Write down the LP dual problem, state the primal–dual optimality conditions, and show that the solution given by the above algorithm satisfies these conditions.

(3p) Question 7

(Convexity and optimality)

Consider the convex optimization problem to

$$(P) \left\{ \underset{x \in X}{\text{minimize}} \; f(x), \right.$$

where $f : \Re^n \mapsto \Re$ is a continuously differentiable function which is convex on the set X; the set $X \subset \Re^n$ is convex and non-empty. We suppose that the set X is compact (that is, closed and bounded), so that the problem is guaranteed to have a non-empty set of optimal solutions, denoted by X^*.

Establish the following interesting result: for all optimal solutions x^*, the value of the gradient of f is the same! In other words, if x^* and $\hat{x}$ both are optimal solutions (that is, are in the set X^*), then $\nabla f(x^*) = \nabla f(\hat{x})$ holds.

Hint: Utilize the convexity of the problem. In particular, utilize that a convex function f is such that

$$f(y) \geq f(x) + \nabla f(x)^{\mathrm{T}}(y - x), \qquad x, y \in \Re^n,$$

holds; this inequality can in fact be used as a definition of the gradient of the differentiable, convex function f at x. Also utilize the following characterization of an optimal solution to the problem (P): a feasible solution x^* to the convex problem (P) is globally optimal in (P) if and only if

$$\nabla f(x^*)^{\mathrm{T}}(y - x^*) \geq 0, \qquad y \in X.$$

Solutions to exam May 28, 2003

TMA 946 / MAN 280 Applied optimization

1. a) By introducing the slack variables x_5, x_6 and x_7 we get

$$\begin{array}{llrrrrrrrcl}
\text{minimize} & z = & x_1 & -2x_2 & -4x_3 & +4x_4 & & & & & \\
\text{subject to} & & & -x_2 & +2x_3 & +x_4 & +x_5 & & & = & 4, \\
 & & -2x_1 & +x_2 & +x_3 & -4x_4 & & +x_6 & & = & 5, \\
 & & x_1 & -x_2 & & +2x_4 & & & +x_7 & = & 3, \\
 & & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7 & \geq & 0.
\end{array}$$

Let $x_B = [x_5, x_6, x_7]$ and $x_N = [x_1, x_2, x_3, x_4]$ be the initial basic and nonbasic vector respectively. The reduced costs of the nonbasic variables then become

$$c_N^T - c_B^T B^{-1} N = [1, -2, -4, 4],$$

and thus x_3 is the entering variable. Further, we have

$$B^{-1}b = [4, 5, 3]^T,$$
$$B^{-1}N_3 = [2, 1, 0]^T,$$

which gives

$$\underset{j,\,(B^{-1}N_3)_j > 0}{\arg\min} \frac{(B^{-1}b)_j}{(B^{-1}N_3)_j} = 1,$$

so x_5 is the leaving variable. The new basic and nonbasic vectors are $x_B = [x_3, x_6, x_7]$ and $x_N = [x_1, x_2, x_5, x_4]$, and the reduced costs of the nonbasic variables become

$$c_N^T - c_B^T B^{-1} N = [1, -4, 2, 6],$$

so x_2 is the entering variable, and

$$B^{-1}b = [2, 3, 3]^T,$$
$$B^{-1}N_2 = [-1/2, 3/2, -1]^T,$$

which gives

$$\underset{j,\,(B^{-1}N_2)_j > 0}{\arg\min} \frac{(B^{-1}b)_j}{(B^{-1}N_2)_j} = 2,$$

and thus x_6 is the leaving variable. The new basic and nonbasic vectors become $x_B = [x_3, x_2, x_7]$ and $x_N = [x_1, x_6, x_5, x_4]$, and the reduced costs of the nonbasic variables are

$$c_N^T - c_B^T B^{-1} N = [-13/3, 8/3, 2/3, -6],$$

so x_4 is the entering variable and

$$B^{-1}b = [3, 2, 5]^T,$$
$$B^{-1}N_4 = [-1, -3, -1]^T,$$

which means that the problem is unbounded and the objective function diverges to $-\infty$ along the extreme half-line given by

$$x(t) = [x_1, x_2, x_3, x_4] = [0, 2, 3, 0] + t[0, 3, 1, 1], \quad t \geq 0. \tag{1}$$

b) The desired feasible solution can be found along the half-line given by (1). More precisely,

$$c^T x(t) = 0 - 2 \cdot (2 + 3t) - 4 \cdot (3 + t) + 4t = -6t - 16,$$

so by choosing $t = 67$ we get the feasible solution

$$x(67) = [x_1, x_2, x_3, x_4] = [0, 203, 70, 67],$$

and since $c^T x(67) = -418$ we are done.

2a) Take $X_1 = [1,2]$ and $X_2 = [3,4]$. The union of the two intervals clearly is not convex.

b) For each X_i, let $x^i \in X_i$ be one type of constraint. In order to have x belonging to the union of the sets X_i we define an indicator $\delta_i \in \{0,1\}$, for each set. The resulting constraints are as follows:

$$x = \sum_{i=1}^{m} \delta_i x^i$$

$$x^i \in X_i \, , \quad i = 1, \ldots, m$$

$$\sum_{i=1}^{m} \delta_i = 1 \, ,$$

$$\delta_i \in \{0,1\}, \quad i = 1, \ldots, m$$

When δ_i, $i = 1, \ldots, m$ is fixed to a feasible value, we have selected one of the sets X_i for which $x = x^i$ holds. Clearly, then, the problem to minimize $f(x)$ subject to the above constraints is a mixed-integer linear program according to the definition.

3.

Variables:

x_1: Bought unlaminated particle-board
x_2: Bought laminated particle-board
Y_1: No of Kalle made and sold
Y_2: No of Billy made and sold

maximize: $x_1 P_1 + x_2 P_2 - b x_1 - C x_2$ (income - expenses)

subject to: $Y_1 \leq k$ (limit on demand of Kalle)

$x_1 + x_2 \geqslant a_1 Y_1 + a_2 Y_2$: (We must have enough particle-board)

$e_1 Y_1 + e_2 Y_2 \leqslant 280 \cdot 60$: (We have 280·60 minutes of cutting time)

$f Y_2 + g_1 Y_1 + g_2 Y_2 \leqslant 120 \cdot 16 \cdot 60$: (We have 120·16·60 minutes of packing and assembly time available)

$X_1 \leq L$ (limit on how much we may laminate ourselves)

$0 \leq x_1, x_2, Y_1, Y_2$ No negative production

Question 4.

a) We start from $(x_0, y_0) = (2, 1)$

$$\nabla f(x_0, y_0) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

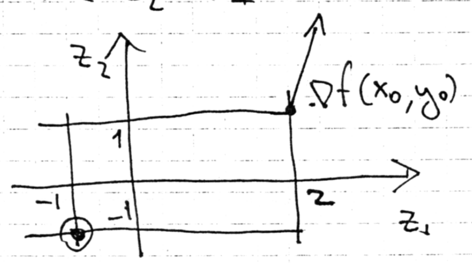
We can solve the two-dim. subproblem

$$\min \begin{pmatrix} 4 \\ 2 \end{pmatrix}^t \left(z - \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right)$$

$$\text{s.t.} \quad -1 \le z_1 \le 2$$

$$-1 \le z_2 \le 1$$

graphically:



$$z^* = (-1, -1)$$

Let's solve the line search problem:

$$\min_{0 \le \alpha \le 1} f(x(\alpha), y(\alpha))$$

where
$$x(\alpha) = (1-\alpha) x_0 + \alpha z_1^* = 2(1-\alpha) - \alpha = 2 - 3\alpha$$
$$y(\alpha) = (1-\alpha) y_0 + \alpha z_2^* = 1 \cdot (1-\alpha) - \alpha = 1 - 2\alpha$$

$$f(x(\alpha), y(\alpha)) = (2-3\alpha)^2 + (1-2\alpha)^2 = 4 - 12\alpha + 9\alpha^2 + 1 - 4\alpha + 4\alpha^2 =$$
$$= 13\alpha^2 - 16\alpha + 5$$

The min is achieved at $\alpha^* = \frac{16}{26} = \frac{8}{13}$ (e.g. $f'_\alpha(x(\alpha), y(\alpha)) =$

$\Rightarrow$ can calculate $(x(\alpha^*), y(\alpha^*)) = \left(\frac{2}{13}, -\frac{3}{13}\right)$.

This is, of course, not the optimal solution $(x^*, y^*) = (0, 0)$
(objective f-n is positive $\forall (x, y) \neq (x^*, y^*)$, while
$f^* = f(x^*, y^*) = 0$).

Question 4.

b)

The function f is convex, as well as the set $X = \{ x \mid Ax \leq b \}$. Therefore, $\forall x \in X$ we have:

$$f(x_k) + \nabla f^t(x_k)(x - x_k) \leq f(x).$$

Taking the min over $x \in X$ of the both sides yelds:

$$f(x_k) + \min_{x \in X} \nabla f^t(x_k)(x - x_k) \leq \min_{x \in X} f(x),$$

which is the same as

$$f(x_k) + L_k^* \leq f^*, \text{or} \quad f(x_k) - f^* \leq -L_k^*$$

■

5\. a) $x^* = (0,0)^T$.

b) The interior penalty function is

$$x_1 + x_2 - \sigma \ln(-x_1^2 + x_2) - \sigma \ln(x_1).$$

In $\mathbb{R}^2_{++}$, this function is in C^2. Differentiating yields that

$$1 + \frac{\sigma \cdot 2x_1}{-x_1^2 + x_2} - \frac{\sigma}{x_1} = 0 \quad \text{and}$$

$$1 - \frac{\sigma}{-x_1^2 + x_2} = 0,$$

that is, $$x_1(\sigma) = (-1 + \sqrt{1+8\sigma})/4,$$

$$x_2(\sigma) = \frac{(-1 + \sqrt{1+8\sigma})^2}{16} + \sigma.$$

As $\sigma \downarrow 0$, $x_1(\sigma) \downarrow 0$ and $x_2(\sigma) \downarrow 0$ so

$(x_1, x_2) \to (0,0)$.

6a) The semi-assignment problem is possible to solve as n simple problems of the form

$$\text{minimize } \sum_i c_{ij} x_{ij}$$

$$\text{subject to } \sum_i x_{ij} = 1, \qquad (j = 1, \dots, n)$$

$$x_{ij} \geq 0, \quad i = 1, \dots, n.$$

The reason is that the problem is linear and separable over j. The extreme points of the feasible set of the above problem are all integer; they are of the form $\mathbb{R}^n \ni \bar{x} = (0, 0, \dots, 1, 0, \dots, 0)^T$, where there is exactly one element with value 1, and the remaining elements are zero. Solving the problem using the simplex method, for example, yields an extreme point automatically, and hence an optimal problem to the binary problem where "$x_{ij} \geq 0$" is replaced by "$x_{ij} \in \{0, 1\}$".

b) (P_j) minimize $\sum_{i=1}^{n} c_{ij} x_{ij}$

subject to $\begin{cases} \sum_{i=1}^{n} x_{ij} = 1, & \mid \pi \\ x_{ij} \geq 0, \quad i=1,\dots,n \end{cases}$

(D_j) maximize π

subject to $\pi \leq c_{ij}, \quad i=1,\dots,n$

To solve (D_j), it is clear that the optimal solution is to set

$$\pi^* = \min_i \{c_{ij}\}.$$

The optimality conditions state, by complementarity, that

$$x_{ij}^* \cdot (c_{ij} - \pi^*) = 0, \quad i=1,\dots,n.$$

Taking any i for which $c_{ij} = \pi^*$ (at least one must exist by the definition of π^*), we then set $x_{ij}^* = 1$ for this index, say, i^*, and $x_{ij} = 0$ for all $i \neq i^*$.

This is a primal-dual solution that satisfies the primal constraints ($\sum x_{ij}^* = 1$, $x_{ij}^* \geq 0 \; \forall i$), dual constraints ($\pi^* \leq c_{ij}, \forall i$) and complementarity. It is therefore optimal. The procedure described is exactly the one described here. We are done.

Question 7.

Let f^* be ~~the~~ optimal value of the problem (P); let further x_1^*, x_2^* be two optimal solutions to (P).

Since f is convex

$$f^* = f(x_1^*) \geq f(x_2^*) + \nabla f^t(x_2^*)(x_1^* - x_2^*) =$$
$$= f^* + \nabla f^t(x_2^*)(x_1^* - x_2^*)$$

$$\Rightarrow \nabla f^t(x_2^*)(x_1^* - x_2^*) \leq 0$$

On the other hand, since x_2^* is an optimal solution, and $x_1^* \in X \Rightarrow$

$$\nabla f^t(x_2^*)(x_1^* - x_2^*) \geq 0 \quad \text{[optimality conditions]}.$$

Therefore, $\nabla f^t(x_2^*)(x_1^* - x_2^*) = 0.$ $\qquad (*)$

Now, $\forall y \in \mathbb{R}^n$:

$$f(y) \geq f(x_2^*) + \nabla f^t(x_2^*)(y - x_2^*) \qquad \text{[by convexity]}$$
$$= f^* + \nabla f^t(x_2^*)(y - x_1^*) + \nabla f^t(x_2^*)(x_1^* - x_2^*)$$
$$= f(x_1^*) + \nabla f^t(x_2^*)(y - x_1^*).$$

because $f(x_1^*) = f^*$ and $(*)$

Therefore, $\nabla f(x_2^*) \in \partial f(x_1^*)$!

But $\partial f(x_1^*) = \{\nabla f(x_1^*)\} \Rightarrow$

$\nabla f(x_1^*) = \nabla f(x_2^*)$ for arbitrary optimal solutions x_1^*, x_2^* ∎