P.D.E F2

övn

F

1988

sidor: 36

pris: ~~10:-~~ ~~15:-~~ 20:-

Övningsanteckningar PDE F2 1998

$\boxed{\text{Blandade övn. 2}}$

Visa att för lösn. $u(x,t)$ till $\begin{cases} \dot{u} - \Delta u = 0 & \text{i } \Omega \\ u = 0 & \text{på } \Gamma \ (= \partial\Omega) \\ u = u_0 & t = 0 \end{cases}$

gäller $\|\nabla u\| \leq \|\nabla u_0\|$ (dvs man kan inte få skarpare "veck" än man hade från början)

där $\|f\| = \left(\int_\Omega f^2 dx\right)^{1/2}$

Lösn. Mult. ekv. m $-\Delta u$ och integrera

$$0 = \int_\Omega \underbrace{\dot{u}(-\Delta u)}_{=\frac{1}{2}\frac{\partial}{\partial t}(\nabla u)^2} dx + \int_\Omega (\Delta u)^2 dx = \{\text{Part. int.}\} = \int_\Omega \nabla\dot{u}\cdot\nabla u \, dx -$$

$$-\underbrace{\int_\Gamma \dot{u}\cdot\frac{\partial u}{\partial n} ds}_{=0 \text{ ty } u = \text{konst på } \Gamma} + \int_\Omega (\Delta u)^2 dx =$$

$$= \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega (\nabla u)^2 dx + \int_\Omega (\Delta u)^2 dx = \frac{1}{2}\frac{\partial}{\partial t}\|\nabla u\|^2 + \|\Delta u\|^2$$

Integrera över tiden

$$0 = \int_0^t \frac{1}{2}\frac{\partial}{\partial s}\|\nabla u\|^2 ds + \int_0^t \|\Delta u\|^2 ds = \frac{1}{2}\left(\|\nabla u(t)\|^2 - \|\nabla \underbrace{u(t=0)}_{u_0}\|^2\right) +$$

$$+ \int_0^t \|\Delta u\|^2 ds$$

$$\therefore \quad \|\nabla u_0\|^2 = \|\nabla u\|^2 + \underbrace{2\int_0^t \|\Delta u\|^2 ds}_{\geq 0}$$

$\|\nabla u_0\|^2 \geq \|\nabla u\|^2 \qquad \sqrt{\cdots} \Rightarrow$ påst

$\boxed{\text{Bl. övn. 3}}$

Visa I) $\int_\varepsilon^t \|\dot{u}\| ds \le \frac{1}{\sqrt{2}} \ln(t/\varepsilon) \|u_0\|$

mha $\|\dot{u}\| \le \frac{1}{\sqrt{2}} \frac{1}{t} \|u_0\|$ (*)

II) $\int_\varepsilon^t \|\dot{u}\| ds \le \frac{1}{2} \sqrt{\ln \frac{t}{\varepsilon}} \|u_0\|$

mha $\int_0^t s\|\Delta u\|^2 ds \le \frac{1}{4} \|u_0\|^2$ (**)

Notera: $\dot{u} = \Delta u$ för homogena värmeledn. ekv.,

$u = 0$ på randen

<u>Lösn:</u> I) Integrera (*)

$$\int_\varepsilon^t \|\dot{u}\| ds \le \frac{1}{\sqrt{2}} \|u_0\| \int_\varepsilon^t \frac{1}{s} ds = \frac{1}{\sqrt{2}} \|u_0\| \ln \frac{t}{\varepsilon}$$

II) $$\int_\varepsilon^t \|\dot{u}\| ds = \int_\varepsilon^t \|\Delta u\| ds = \int_\varepsilon^t \underbrace{s^{-1/2}(s^{1/2}}_{=1} \|\Delta u\|) ds \le \{\text{Cauchy}\} \le$$

$$\left(\int fg\, dx \le \left(\int f^2 dx\right)^{1/2} \left(\int g^2 dx\right)^{1/2} \right)$$

$$\le \left(\int_\varepsilon^t \frac{1}{s} ds \right)^{1/2} \underbrace{\left(\int_\varepsilon^t s\|\Delta u\|^2 ds \right)^{1/2}}_{\le \int_0^t s\|\Delta u\|^2 ds} \overset{(**)}{\le} \left(\int_\varepsilon^t \frac{1}{s} ds \right)^{1/2} \frac{1}{2} \|u_0\| =$$

$$= \sqrt{\ln \frac{t}{\varepsilon}} \; \frac{\|u_0\|}{2}$$

Bl. övn. 8

Visa att $\|u_1 - u_2\| \le C_\Omega^2 \|f_1 - f_2\|$

där
$$(1)\begin{cases} -\Delta u_1 = f_1 & i\ \Omega \\ u_1 = 0 & \text{på } \Gamma \end{cases}$$

$$(2)\begin{cases} -\Delta u_2 = f_2 & i\ \Omega \\ u_2 = 0 & \text{på } \Gamma \end{cases}$$

Lösn. studera $\hat{u} = u_1 - u_2$

$(1) - (2) \Rightarrow$

$$(3)\begin{cases} -\Delta \hat{u} = \underbrace{f_1 - f_2}_{=\tilde{f}} & i\ \Omega \\ \hat{u} = 0 & \text{på } \Gamma \end{cases}$$

Mult. (3) m $\hat{u}$ och integrera

$$\int_\Omega -\Delta\hat{u}\,\hat{u}\,dx \overset{PI}{=} \int_\Omega (\nabla\hat{u})^2 dx - \underbrace{\int_\Gamma \hat{u}\frac{\partial\hat{u}}{\partial n}\,ds}_{=0} =$$

$$= \|\nabla\hat{u}\|^2 = \int_\Omega \tilde{f}\hat{u}\,dx \le \{\text{Cauchy}\} \le \|\hat{f}\|\,\|\hat{u}\|$$

$\therefore\ \|\nabla\hat{u}\|^2 \le \|\tilde{f}\|\,\|\hat{u}\|$

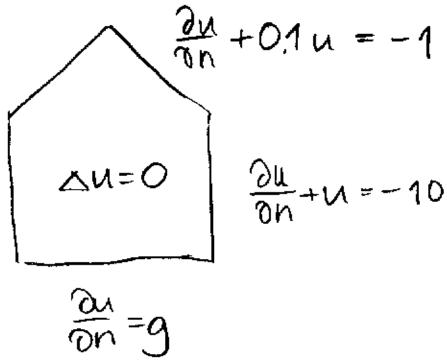
Finns konst. $C = C_\Omega$ sådan att

$\|u\| \le C_\Omega \|\nabla u\|$ (se övn 7)

$\|\hat{u}\|^{\cancel{2}} \le C_\Omega^2 \|\nabla\hat{u}\|^{\cancel{2}} \le C_\Omega^2 \|\tilde{f}\|\,\cancel{\|\hat{u}\|} \Rightarrow$ Påst.

Bl. övn. 12

Betrakta ett hus



Vad innebär randvillkoren?

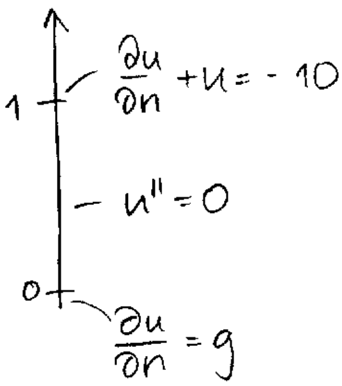
$\frac{\partial u}{\partial n} = \nabla u \cdot \hat{n}$ – värmeflödet in

$\frac{\partial u}{\partial n} + u = -10 \qquad \frac{\partial u}{\partial n} = k(T_{ute} - u)$

värmeflödet $\sim$ temp. skillnaden ute-inne

Taket bättre isolerat än väggarna.

Enkel modell i 1-D



Integrera ekv 2 ggr. $\quad u(y) = C_1 y + C_2$

RV $\quad y = 0 \quad \frac{\partial u}{\partial n} = u' \cdot \hat{n} = -u' = g \Rightarrow C_1 = -g$

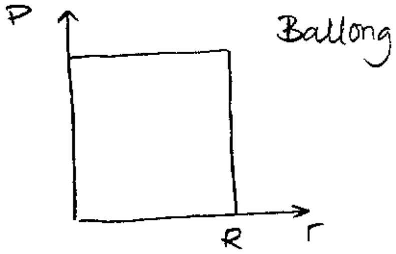
$y = 1 \quad u' + u = -10$

$-g - g + C_2 = -10 \qquad C_2 = -10 + 2g$

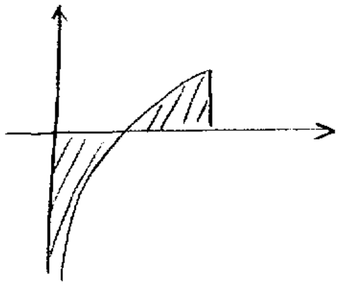
Medeltemp. $\bar{u} = \int_0^1 u \, dy = \int_0^1 -gy + 2g - 10 \, dy = \frac{3}{2}g - 10$

$\bar{u} = 20 \Rightarrow g = 20$

1, Sfäriska vågor



Spricker vid $t=0$



Kolla upp:
Totalt övertryck = konst?
Energin = konst?
$\|\dot{u}\|^2 - \|\Delta u\|^2 = \text{konst}$

2, Best. lösning till

$$\begin{cases} \dot{u} - \Delta u = \delta \\ u(x,0) = 0 \end{cases} \quad \text{i } \mathbb{R}^3$$

Fundamentallösn: $\dot{u} - \Delta u = 0$, $u(x,0) = \delta$

$$\text{Lösn: } u(x,t) = \frac{C}{t^{3/2}} \int_\Omega \underbrace{u_0(y)}_{=0} e^{-\frac{|x-y|^2}{4t}} dy + C \int_0^t \frac{1}{(t-s)^{3/2}} \underbrace{\int_\Omega \underbrace{f(y,s)}_{=\delta(y)} e^{-\frac{|x-y|^2}{4(t-s)}} dy}_{e^{-\frac{|x|^2}{4(t-s)}}} ds =$$

$$= C \int_0^t \frac{1}{(t-s)^{3/2}} e^{-|x|/4(t-s)} ds = \left\{ \begin{aligned} z &= \frac{|x|}{2(t-s)^{1/2}} \\ dz &= \frac{|x|}{4(t-s)^{3/2}} ds \end{aligned} \right\} =$$

$$= C \int_{\frac{|x|}{2t^{1/2}}}^{\infty} \frac{4}{|x|} e^{-z^2} dz$$

För att best. C utnyttjar vi att

$$* \; E(x,t) = \frac{C}{t^{3/2}} e^{-|x|/4t} \to \delta \text{ då } t \to 0^+$$

$$* \int_{\Omega} \delta \, dx = 1$$

$$\Rightarrow \int_{\mathbb{R}^3} \frac{C}{t^{3/2}} e^{-|x|/4t} dx = C 4\pi \int_0^{\infty} \frac{r^2}{t^{3/2}} e^{-r^2/4t} dr =$$

$$= \left\{ \begin{array}{l} s = \frac{r^2}{4t} \\ ds = \frac{r}{2t} dr \end{array} \right\} = C\, 16\pi \int_0^{\infty} s^{1/2} e^{-s} ds = C\, 16\pi \underbrace{\Gamma(3/2)}_{= \sqrt{\pi/2}} \Rightarrow$$

$$C = \frac{1}{8\pi\sqrt{\pi}}$$

$$\therefore \; u(x,t) = \frac{1}{2\pi^{3/2}|x|} \int_{\frac{|x|}{2t^{1/2}}}^{\infty} e^{-z^2} dz$$

då $t \to \infty \Rightarrow \dot{u} \Rightarrow 0 \Rightarrow -\Delta u = \delta$

Lösn. till denna är $u(x) = \frac{1}{4\pi|x|}$

$$\lim_{t \to \infty} u(x,t) = \frac{1}{2\pi^{3/2}|x|} \underbrace{\int_0^{\infty} e^{-z^2} dz}_{= \sqrt{\pi/2}} = \frac{1}{4\pi|x|}$$

Visa att

$$\|u\| \leq \int_0^T \|f\|\,dt + \|u_0\|$$

där u löser
$$\begin{cases} \dot{u} - \Delta u = f & i\ \Omega \times (0,T] \\ u(x,0) = u_0(x) & t = 0 \\ u(x,t) = 0 & \text{på } \Gamma \times (0,T) \end{cases}$$

Lösn: Mult. ekv. m. u & int.

$$\int_\Omega f u\,dx = \underbrace{\int_\Omega \dot{u} u\,dx}_{\frac{1}{2}\frac{d}{dt}u^2} - \int_\Omega \Delta u\, u\,dx =$$

$$= \frac{1}{2}\frac{d}{dt}\|u\|^2 + \int_\Omega \nabla u \cdot \nabla u\,dx - \int_\Gamma \frac{\partial u}{\partial n}\underbrace{u}_{=0}\,ds =$$

$$= \|u\|\frac{d}{dt}\|u\| + \|\nabla u\|^2 \geqslant \|u\|\frac{d}{dt}\|u\|$$

$$\|u\|\frac{d}{dt}\|u\| \leq \int_\Omega f u\,dx \underset{\text{CAUCHY}}{\leq} \|f\|\|u\|$$

om $\|u\| = 0$ så olikheten ok, så anta $\|u\| \neq 0$

$$\Rightarrow \frac{d}{dt}\|u\| \leq \|f\|$$

Integrera över $(0,T]$

$$\|u(T)\| - \|u_0\| \leq \int_0^T \|f\|\,dt$$

Lax-Milgrams sats:

Om 1) $|a(u,v)| \leq \beta \|u\|_V \|v\|_V < \infty$ (kontinuerliga)

2) $a(v,v) \geq \alpha \|v\|_V^2 \quad \alpha > 0$

3) $|\ell(v)| \leq \gamma \|v\|_V < \infty$ (kontinuerlig)

Så finns unikt u så att

$\underbrace{a(u,v) = \ell(v)}_{\text{variationsformulering}}$ för alla $v \in V$

$\int f \cdot g \, dx = \underbrace{(f,g)}_{\text{skalärprodukt}}$

Betrakta ekv.

$$-\Delta u + cu = 0 \quad \text{i } \Omega$$

$$\frac{\partial u}{\partial n} = g \quad \text{på } \Gamma$$

där $c = c(x) \geq c_0 > 0$

Mult. ekv. m. testfunktionen $v \in V$

$$\int_\Omega -\Delta u v \, dx + \int_\Omega cuv \, dx = \int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega \frac{\partial u}{\partial n} v \, ds + \int_\Omega cuv \, dx$$

$$\Rightarrow \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega cuv \, dx = \int_\Gamma gv \, ds \quad \forall v \in V$$

Sök $u \in V$ så att detta gäller

Sätt $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega cuv \, dx$

$\ell(v) = \int_\Gamma gv \, ds$

$$\|u\|_V = \left(\|\nabla u\|^2 + \|u\|^2\right)^{1/2}$$

Kolla förutsättningarna i L-M

1) $|a(u,v)| \leq \left|\int_\Omega \nabla u \cdot \nabla v\, dx\right| + \left|\int_\Omega c u v\, dx\right| \leq$

$$\leq \left|\int_\Omega \nabla u \cdot \nabla v\, dx\right| + \underbrace{\max(c)}_{C_1} \left|\int_\Omega u v\, dx\right| \leq \{\text{CAUCHY}\} \leq$$

$$\|\nabla u\| \cdot \|\nabla v\| + C_1 \|u\| \cdot \|v\| \leq \left\{ \begin{array}{l} \|v\| \leq (\|v\|^2 + \|\nabla v\|^2)^{1/2} = \|v\|_V \\ \|\nabla v\| \leq \cdots \end{array} \right\} \leq$$

$$\leq (\|\nabla u\| + C_1\|u\|)\|v\|_V \leq (1 + C_1)\|u\|_V \|v\|_V$$

2) $a(v,v) \geq \int_\Omega \nabla v \cdot \nabla v\, dx + c_0 \int_\Omega v v\, dx =$

$$= \|\nabla v\|^2 + c_0 \|v\|^2 \geq \min(1, c_0)\|v\|_V^2$$

3) $|l(v)| = \left|\int_\Gamma g v\, ds\right| \underset{\text{cauchy}}{\leq} \|g\|_\Gamma \|v\|_\Gamma \underset{\text{Thm 21.5}}{\leq} C\|g\|_\Gamma \|v\|_V$

ok om $\|g\|_\Gamma < \infty$

$\therefore$ Lax-Milgram $\Rightarrow$ finns lösning till diffekv.

$$\begin{cases} -\Delta u = f & \text{i } \Omega \\ u = g & \text{på } \Gamma \end{cases}$$

$g = 0$

$$\int -\Delta u \cdot v = \int_\Omega \nabla u \cdot \nabla v\, dx - \int_\Gamma \frac{\partial u}{\partial n} v\, ds = \int_\Omega \nabla u \cdot \nabla v\, dx \quad \Rightarrow$$

tag $v = 0$ på Γ

Best. $u \in V_0$ s.a.

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall\, v \in V = \{v : \|\nabla v\|^2 + \|v\|^2 < \infty, v = 0 \text{ på } \Gamma\}$$

Ekvivalent minimeringsproblem

$$\min F(v) = \min\left(\frac{1}{2}\|v\|^2 - \int_\Omega f v \, dx\right)$$

Skriv $v = \underset{\substack{\uparrow \\ \min}}{u} + tw$

$\underline{g \neq 0}$ finn $u \in V_g$ så att

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall\, v \in V_0$$

V_g : funktionen är g på randen

V_0 : — " — 0 — " —

1. Variationsformulera

$$-(xu')' = 7 \quad , \quad x \in [1,2]$$

$u'(1) = u(1)$ Robin
$u(2) = 0$ (eller $u(2) = 3$) Dirichlet

Testfkn $v(x)$

$$-\int_1^2 (xu')'v(x)\,dx = 7\int_1^2 v(x)\,dx$$

$$\int_1^2 xu'(x)v'(x)\,dx - 2u'(2)v(2) + u'(1)v(1) = 7\int_1^2 v(x)\,dx$$

Dirichlet $\Rightarrow$ inkluderas i def. av fknsrum där lösn sökes.

Neumann el. Robin $\Rightarrow$ satisfieras genom variationsformulering.

Hitta $u(x) \in H^1[1,2]$ $u(2) = 0$
för alla $v \in H^1[1,2]$ $v(2) = 0$

$$\int_1^2 xu'(x)v'(x)\,dx + u(1)v(1) = 7\int_1^2 v(x)\,dx$$

Om u. s.a. var.f. gäller <u>OCH</u> $u \in C^2[1,2]$

$$-\int_1^2 (xu'(x))'v(x)\,dx + \underbrace{xu'v\Big|_1^2}_{=2u'(2)v(2)-u'(1)v(1)} + u(1)v(1) = 7\int_1^2 v(x)\,dx$$

$$\int_1^2 (-(xu'(x))' - 7)v(x)\,dx + [u(1) - u'(1)]v(1) = 0$$

$v(x)$: $v(1) = 0$

$$\int_1^2 (-(xu'(x))' - 7)v(x)\,dx \Rightarrow -(xu'(x))' - 7 = 0 \text{ punktvis}$$

2. $\begin{cases} u_t - u_{xx} = f, & x \in [0,1], \quad t \in [0,+\infty) \\ u(0,t) = 0 & \text{Dirichlet} \\ u'(1,t) = g(t) \\ u(x,0) = u_0(x) \end{cases}$

cG1/cG1
tid rum

Variationsform: Hitta $u \in H^1([0,1] \times I_n)$ i $u(0,t) = 0$, $u(x,0) = u_0(x)$

$$\int_{I_n} \int_0^1 (u_t v + u_x v_x)\, dx\,dt - \int_{I_n} \underset{\overset{\|}{g(t)}}{u_x(1,t)} v(1,t) - \underbrace{u_x(0,t) v(0,t)}_{v(0,t)=0} dt =$$

$$= \int_{I_n} \int_0^1 f v \, dx\,dt$$

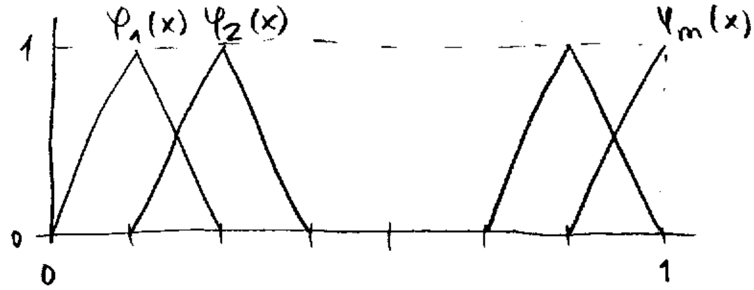
för alla $v \in H^1(\ )$, $v(0,t) = 0$

Ansätter: $\begin{cases} U(x,t) = U_{n-1}(x) \dfrac{t_n - t}{k} + U_n(x) \dfrac{t - t_{n-1}}{k} \\ U_n(x) = U_{n,1} \varphi_1(x) + \dots + U_{n,m} \varphi_m(x) \end{cases}$

för varje I_n, testfkn v - styckvis konstanta i tiden,
styckvis linjära i rummet
$v(0,t) = 0$

$$\int_{I_n} \int_0^1 (U_t v + U_x v_x)\, dx\,dt - \int_{I_n} g(t) v(1,t)\, dt = \int_{I_n} \int_0^1 f v \, dx\,dt$$

$v(x,t) = c_1 \varphi_1(x) + \dots + c_m \varphi_m(x)$, $t \in I_n$



$$\int_{I_n}\int_0^1 (\underbrace{U_t}_{\frac{U_n(x)-U_{n-1}(x)}{k}}\varphi_j + \underbrace{U_x}_{=U_n'(x)\cdot\frac{t_n-t}{k}+U_{n-1}'(x)\frac{t-t_{n-1}}{k}}\varphi_{jx})\,dx\,dt - \int_{I_n} g(t)\begin{Bmatrix} 0, & j\neq m \\ 1, & j=m\end{Bmatrix} dt = \int_{I_n}\int_0^1 fv\,dx\,dt$$

$$\int_0^1 (U_n(x)-U_{n-1}(x))\varphi_j(x)\,dx + \underbrace{\int_{I_n}\frac{t_n-t}{k}\,dt}_{=k/2}\int_0^1 U_n'(x)\varphi'(x)\,dx +$$

$$+\underbrace{\int_{I_n}\frac{t-t_{n-1}}{k}}_{=k/2}\int_0^1 U_{n-1}'(x)\,\varphi'(x)\,dx - \int_{I_n} g(t)\begin{Bmatrix} 0, & j\neq m \\ 1, & j=m\end{Bmatrix} dt = \int_{I_n}\int_0^1 fv\,dx\,dt$$

$$\int_0^1 (U_n(x)-U_{n-1}(x))\varphi_j(x)\,dx + \frac{k}{2}\int_0^1 \left[U_n'(x)+U_{n-1}'(x)\right]\varphi'(x)\,dx -$$

$$\int_{I_n} g(t)\{\ldots\} = \int_{I_n}\int_0^1 fv\,dx\,dt$$

$$U(x,t) = U_n(x)\frac{t_n-t}{k} + U_{n-1}(x)\frac{t-t_{n-1}}{k}$$

$$U_n(x) = U_{n,1}\varphi_1(x) + \cdots + U_{n,m}\varphi_m(x)$$

$$M = [m_{jl}], \quad m_{jl} = \int_0^1 \varphi_j(x)\varphi_l(x)\,dx$$

$$S = [s_{jl}], \quad s_{jl} = \int_0^1 \varphi_j'(x)\varphi_l'(x)\,dx$$

$$M = h\cdot\begin{bmatrix} 2/3 & 1/6 & & & 0 \\ 1/6 & 2/3 & 1/6 & & \\ & 1/6 & 2/3 & \ddots & \\ & & \ddots & \ddots & \\ 0 & & & 2/3 & 1/6 \\ & & & 1/6 & 1/3 \end{bmatrix}, \quad S = \frac{1}{h}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & \\ & & & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}$$

$$M(U_n - U_{n-1}) + \frac{k}{2} S (U_n + U_{n-1}) = F + G$$

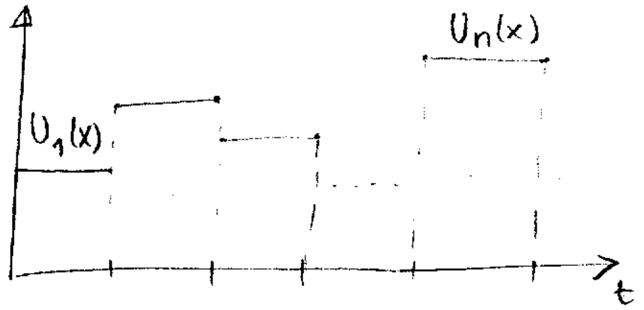
$$U_n = \begin{bmatrix} U_{n,1} \\ \vdots \\ U_{n,m} \end{bmatrix}, \quad F = \begin{bmatrix} \int_{I_n}\int_0^1 f\varphi_1(x)\,dx \\ \vdots \\ \int_{I_n}\int_0^1 f\varphi_m(x)\,dx \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \int_{I_n} g(t)\,dt \end{bmatrix}$$

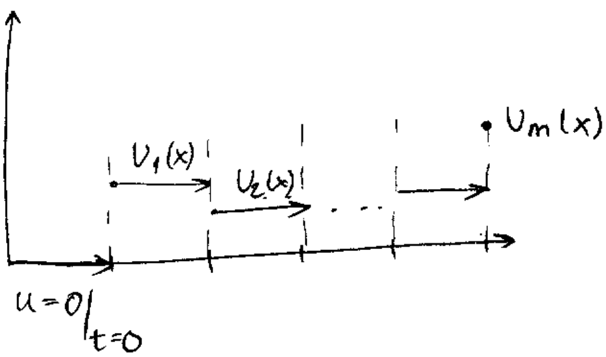
$$G \approx \begin{bmatrix} 0 \\ \vdots \\ 0 \\ kg(t_n) \end{bmatrix}$$

$$\left(M + \frac{k}{2} S\right) U_n = \left(M - \frac{k}{2} S\right) U_{n-1} + F + G$$

cG1 / dG0 → implicit Euler
rum — tid → explicit Euler

styckvis konstant i t





$$\int_0^1 (U_m - U_{m-1})\varphi_j\,dx + k\int_0^1 U_n'(x)\varphi_j'(x)\,dx - \int_I g(t) \begin{Bmatrix} 0 & j \neq m \\ 1, & j = m \end{Bmatrix} =$$

$$= \int_{I_n} \int_0^1 fv$$

$$M(U_n - U_{n-1}) + kSU_n = F_n + G_n$$ <u>implicit Euler</u>

$$(M + kS)U_n = MU_{n-1} + F_n + G_n$$

$$\int_0^1 (U_{n+1} - U_n)\varphi_j \, dx + k\int_0^1 U_n'(x)\varphi_j'(x)\, dx - \int_{I_n} \cdots = \cdots$$

$$M(U_{n+1} - U_n) + kSU_n = F_n + G_n$$ <u>Explicit Euler</u>

$$MU_{n+1} = (M - kS)U_n + F_n + G_n$$

Dagens program: 1. * Elliptisk regularitet (Fö 6)

2. * Ickehomogena Dirichlet RV (Fö 8)

3. * Energikonservering hos cG1 för vågekv (Fö 8)

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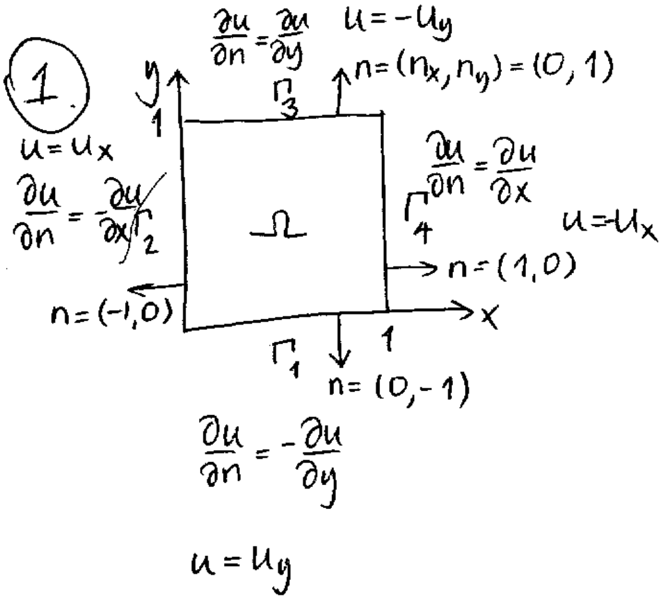
(1)



$$\frac{\partial u}{\partial n} + u = 0 \quad \text{på} \quad \Gamma = \bigcup_{i=1}^{4} \Gamma_i$$

Påst: $\|D^2 u\|_{L_2(\Omega)} \leq \|\Delta u\|_{L_2(\Omega)}$ Elliptisk regularitet

$$D^2 u(x,y) = (u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2)^{1/2}$$

Bevis:

$$\|\Delta u\|^2_{L_2(\Omega)} = \int_\Omega (\Delta u)^2 \, dxdy = \int_\Omega (u_{xx} + u_{yy})^2 \, dxdy =$$

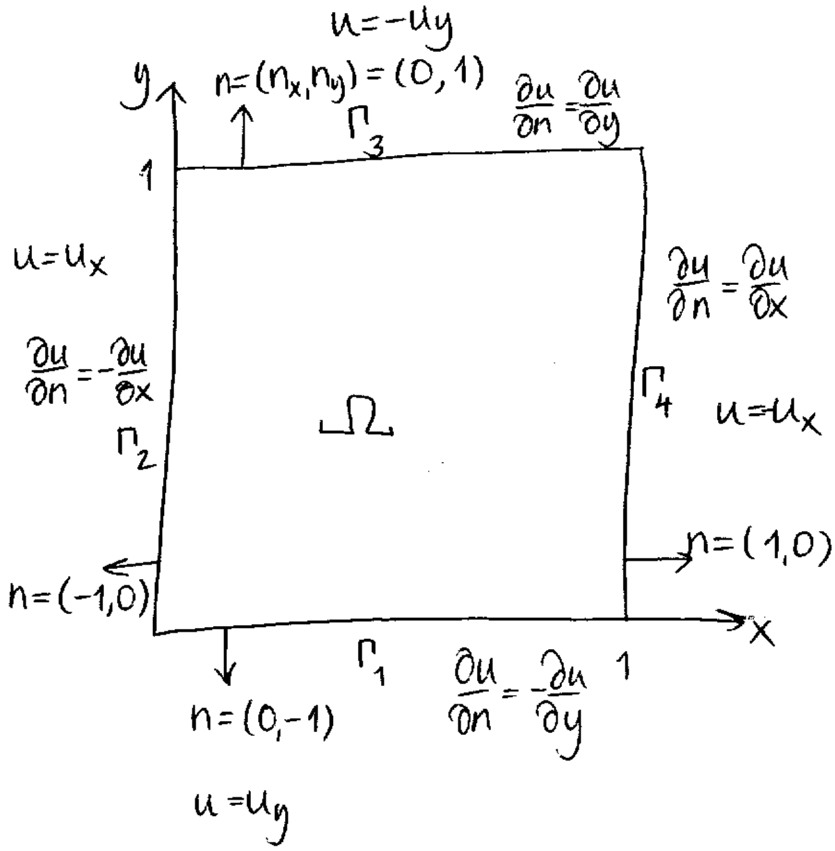
$$= \int_\Omega (u_{xx}^2 + u_{yy}^2 + 2u_{xx}u_{yy}) \, dxdy$$

Men

$$\int_\Omega u_{xx}u_{yy} \, dxdy \overset{\text{Green}}{=} \int_\Gamma u_x u_{yy} \underbrace{n_x}_{n_x = 0 \text{ på } \Gamma_1 \,\&\, \Gamma_3} ds - \int_\Omega u_x \underbrace{u_{yyx}}_{= u_{xyy}} \, dx\,dy =$$

$$= -\int_{\Gamma_2} u_x u_{yy} \, ds + \int_{\Gamma_4} u_x u_{yy} \, ds - \int_\Gamma u_x u_{xy} \underbrace{n_y}_{= 0 \text{ på } \Gamma_2 \cup \Gamma_4} ds + \int_\Omega u_{xy} u_{xy} \, dxdy =$$

Förtydligande till figur på föreg. sida



$$= \int_{\Gamma_1} u_x u_{xy}\, ds - \int_{\Gamma_2} u_x u_{yy}\, ds - \int_{\Gamma_3} u_x u_{xy}\, ds + \int_{\Gamma_4} u_x u_{yy}\, ds +$$

$$+ \int_{\Omega} u_{xy}^2\, dxdy \overset{(*)}{\geq} \int_{\Omega} u_{xy}^2\, dxdy \quad \Rightarrow$$

$$\|\Delta u\|^2_{L_2(\Omega)} \geq \int_{\Omega} (u_{xx}^2 + u_{yy}^2 + 2(u_{xy})^2)\, dxdy = \int_{\Omega} (D^2 u)^2\, dxdy =$$

$$= \|D^2 u\|^2_{L_2(\Omega)}$$

Visar slutligen (*)

$$\int_{\Gamma_1} u_x u_{xy}\, ds = \int_0^1 u_x u_{xy}\, dx = \int_0^1 u_x \cdot \frac{\partial}{\partial x}(u_y)\, dx =$$

$$= \int_0^1 u_x u_x\, dx \geq 0$$

$$-\int_{\Gamma_2} u_x u_{yy}\, ds = -\int_0^1 u_x u_{yy}\, dy = -\int_0^1 u\, u_{yy}\, dy = \{P.1\} =$$

$$= \int_0^1 u_y^2\, dy - u(0,1)u_y(0,1) + u(0,0)u_y(0,0) =$$


$$= u(0,1)u(0,1) + u(0,0)u(0,0) + \int_0^1 u_y^2\, dy \geq 0$$

Γ_3 & Γ_4 kan behandlas helt analogt.

i hörnen: $u_x(0,1) = u_y(0,1)$ eftersom u är kontinuerlig

(2) $-u'' = 1 \quad , \quad 0 < x < 1 \, , \quad \underbrace{u(0) = 7}_{\text{väsentligt RV}} \, , \quad \underbrace{u'(1) = 0}_{\text{naturligt RV}}$

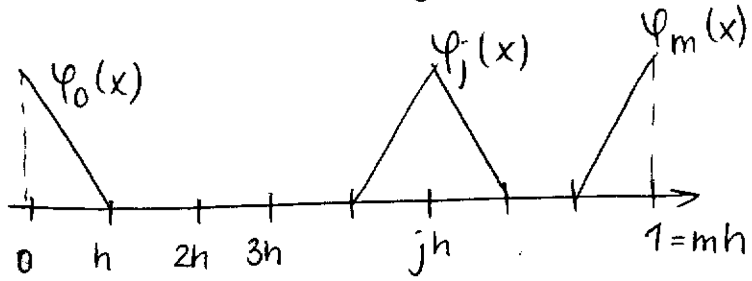
Variationsformulering:

Finn $u \in V$ s.a. $\int_0^1 u'v' \, dx = \int_0^1 v \, dx \quad \forall v \in V_0$

$V = \left\{ v : \int_0^1 (v'^2 + v^2) \, dx < \infty \, , \, v(0) = 7 \right\}$ Egentligen inget linjärt rum, snarare en mängd.

$V_0 = \left\{ v : \int_0^1 (v'^2 + v^2) \, dx < \infty \, , \, v(0) = 0 \right\}$ Detta är dock linj. rum.

FEM: (med steglängd h)



Ansätt: $U(x) = 7\varphi_0(x) + U_1\varphi_1(x) + \cdots + U_m\varphi_m(x)$

$\int_0^1 U'\varphi_j' \, dx = \int_0^1 \varphi_j \, dx \quad , \quad j = 1, \ldots, m$

$\underbrace{7\int_0^1 \varphi_0' \varphi_j' \, dx}_{a_{j0}} + \sum_{k=1}^{m} U_k \underbrace{\int_0^1 \varphi_k' \varphi_j' \, dx}_{a_{jk}} = \underbrace{\int_0^1 \varphi_j \, dx}_{b_j}$

$\underset{(m+1)\times 1}{\tilde{U} = \begin{bmatrix} 7 \\ U_1 \\ \vdots \\ U_m \end{bmatrix}} \qquad \underset{m \times 1}{\tilde{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}} = \begin{bmatrix} h \\ h \\ \vdots \\ \frac{h}{2} \end{bmatrix}$

$$a_{10} = \int_0^1 \varphi_0' \varphi_1' \, dx = \int_0^h \varphi_0' \varphi_1' \, dx = \int_0^h \left(-\frac{1}{h}\right)\frac{1}{h} \, dx = -\frac{1}{h}$$

$a_{j0} = 0 \quad 2 \le j \le m$, a_{jk} enl. tidigare

$1 \le j, k \le m$

$$\tilde{A} = (a_{jk}) = \begin{bmatrix} -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & 0 \\ 0 & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ 0 & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & \\ \vdots & & & \ddots & \ddots & \ddots \\ \vdots & & 0 & & & \frac{1}{h} \end{bmatrix}$$

$m \times (m+1)$

$\tilde{A}\hat{U} = \hat{b}$

Alt: $\sum_{k=1}^{m} a_{jk} U_k = b_j - 7a_{j0} \quad , \; j = 1, \dots, m$

$AU = b$

$A = (a_{jk})_{j,k=1,\dots,m}$

$$U = \begin{bmatrix} U_1 \\ \vdots \\ U_m \end{bmatrix} \qquad b = \begin{bmatrix} b_1 - 7a_{10} \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} h + \frac{7}{h} \\ h \\ \vdots \\ h \\ \frac{h}{2} \end{bmatrix}$$

③ $$\begin{cases} \ddot{u} - u'' = 0, & x \in (0,1), \quad t > 0 \\ u(0,t) = 0, \quad u'(1,t) = 0, & t > 0 \\ u(x,0) = u_0(x), \quad \dot{u}(x,0) = \dot{u}_0(x), & 0 < x < 1 \end{cases}$$

$$\begin{cases} \dot{u} - v = 0 \\ \dot{v} - u'' = 0 \end{cases}$$

Vet från Fö 8 att cG(1) ger:

$$\begin{cases} MU_n - \frac{k}{2} MV_n = MU_{n-1} + \frac{k}{2} MV_{n-1} \\ \frac{k}{2} SU_n + MV_{m} = -\frac{k}{2} SU_{n-1} + MV_{n-1} \end{cases}$$

<u>Påst:</u> $\| U_n' \|^2 + \| V_n \|^2 = \| U_{n-1}' \|^2 + \| V_{n-1} \|^2$

(jfr avsnitt 17.2.4 i CDE)

<u>Bevis:</u> Enl. hint:

$$(U_{n-1} + U_n)^t S \cancel{M^{-1}} \cancel{M} (U_n - \tfrac{k}{2} V_n) = (U_{n-1} + U_n)^t S \cancel{M^{-1}} \cdot$$
$$\cdot (\cancel{M} (U_{n-1} + \tfrac{k}{2} V_{n-1})) \qquad (1)$$

$$(V_{n-1} + V_n)^t (\tfrac{k}{2} SU_n + MV_n) = (V_{n-1} + V_n)^t (-\tfrac{k}{2} SU_{n-1} + MV_{n-1}) \quad (2)$$

Notera: $\| U_n' \|^2 = (U_n', U_n') = \sum_{k=1}^{m} U_{n,k} \varphi_k' \sum_{j=1}^{m} U_{n,j} \varphi_j' =$

$$\sum_{j,k=1}^{m} U_{n,k} \underbrace{(\varphi_k', \varphi_j')}_{S_{jk}} U_{n,j} = U_n^t S U_n$$

Analogt: $\| V_n \|^2 = V_n^t M V_n$

Addera (1) & (2) ledvis

$$U_{n-1}^t S U_n + \underline{\|U_n'\|^2} - \frac{k}{2} U_{n-1}^t S V_n - \frac{k}{2} U_n^t S V_n +$$

$$+ V_{n-1}^t \frac{k}{2} S U_n + \frac{k}{2} V_n^t S U_n + V_{n-1}^t M V_n + \underline{\|V_n\|^2} =$$

$$= \underline{\|U_{n-1}'\|^2} + U_n^t S U_{n-1} + \frac{k}{2} U_{n-1}^t S V_{n-1} + \frac{k}{2} U_n^t S V_{n-1} - \frac{k}{2} V_{n-1}^t S U_{n-1} -$$

$$- \frac{k}{2} V_n^t S U_{n-1} + \underbrace{V_{n-1}^t M V_{n-1}}_{= \underline{\|V_{n-1}\|^2}} + V_n^t M V_{n-1} \quad \ldots \quad \Rightarrow$$

$$\|U_n'\|^2 + \|V_n\|^2 = \|U_{n-1}'\|^2 + \|V_{n-1}\|^2$$

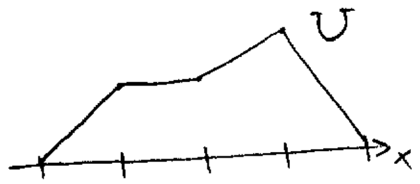
ty $\underset{\text{tal}}{V^t S W} = (V^t S W)^t = W^t \underset{\substack{\uparrow \\ \text{symmetrisk}}}{S^t} V = W^t S V$

Fög, sista sidan

Visa $\|e'\| \leq C_i \|hr\|$

$e = u - U$, $r = f - U''$

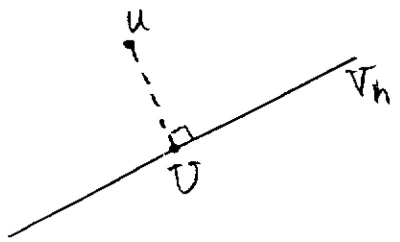
$-u'' = f$, $0 < x < 1$, $u(0) = u(1) = 0$



$\int_0^1 U'v' = \int_0^1 fv$ för alla $v \in V_h = \{v(x) : v$ st. linj, kont, $v(0) = v(1) = 0\}$

Obs $\int_0^1 \underbrace{(u - U)'}_{e} v' = 0$ för alla $v \in V_h$ ← någon slags ortogonalitet

Felekvationen



Har visat att $\|u' - U'\| \leq \|u' - v'\|$ för alla $v \in V_h$

Kommer ihåg:

$$\|e'\|^2 = \int_0^1 e'e' = \int_0^1 e'e' - \underbrace{\int_0^1 e'v'}_{=0} = \int_0^1 e'(e - v)' =$$

$$= \int_0^1 \underbrace{(-e'')}_{f + U'' = r}(e - v) = \int_0^1 r(e - v) = \int_0^1 hr \cdot h^{-1}(e - v) \leq \{\text{Cauchy}\}$$

$\leq \|hr\| \cdot \|h^{-1}(e - v)\|$ $h = h(x)$ lokala intervallängden

Påst: Finns $v \in V_h$ s.a.

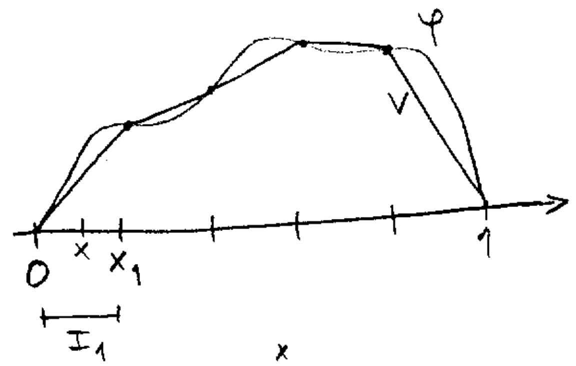
$$\|h^{-1}(e-v)\| \le C\|e'\|$$

Detta ger $\|e'\|^2 \le C\|hr\|\cdot\|e'\| \Leftrightarrow$

$$\|e'\| \le C\|hr\|$$

Visa att om v interpolerar φ så blir

$$\|h^{-1}(\varphi - v)\| \le C\|\varphi'\|$$



Cauchy

$$(\varphi - v)(x) = \int_0^x (\varphi - v)' \implies |(\varphi - v)(x)| \le \int_{I_1} |\varphi' - v'| \le$$

$$\le \underbrace{\left(\int_{I_1} 1^2\right)^{1/2}}_{h_1^{1/2}} \underbrace{\left(\int_{I_1} (\varphi' - v')^2\right)^{1/2}}_{\|\varphi' - v'\|_{I_1}}$$

$$\|\varphi - v\|_{I_1}^2 = \int_{I_1} |\varphi - v|^2 \le h_1^2 \|\varphi' - v'\|_{I_1}^2$$

$$\|\varphi - v\|_{I_1} \le h_1 \|\varphi' - v'\|_{I_1} \implies \|h^{-1}(\varphi - v)\|_{I_1} \le \|\varphi' - v'\|_{I_1}$$

$$\implies \|h^{-1}(\varphi - v)\|^2 = \sum_i \|h^{-1}(\varphi - v)\|_{I_i}^2 \le \sum_i \|(\varphi - v)'\|_{I_i}^2 = \|\varphi' - v'\|^2$$

Har visat

$$\|h^{-1}(\varphi - v)\| \le \|\varphi' - v'\|$$

Komb. med $\|\varphi' - v'\|_{I_i} \le \|\varphi'\|_{I_i} + \|v'\|_{I_i}$

$$v' = \frac{v(x_i) - v(x_{i-1})}{h_i} = \frac{\varphi(x_i) - \varphi(x_{i-1})}{h_i} = \int_{x_{i-1}}^{x_i} \varphi'/h_i$$

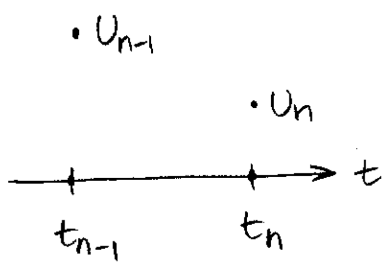
ger $$\|v'\|_{I_i}^2 = \int_{I_i} |v'|^2 \le h_i \left(\frac{\int_{x_{i-1}}^{x_i} |\varphi'|}{h_i} \right)^2 \le h_i^{-1} h_i \int_{I_i} |\varphi'|^2 = \|\varphi'\|_{I_i}^2$$

Cauchy

Sammantaget

$$\|h^{-1}(\varphi - v)\| \le 2\|\varphi'\| \qquad C_i \le 2$$

Övn (S7:3) $\dot{u} + au = 0$, $t > 0$, $u = u(t)$, $a = \text{konst}$.



Expl. Euler

$$\frac{U_n - U_{n-1}}{k} + a U_{n-1} = 0 \qquad \text{dvs}$$

$$\boxed{U_n = (1 - ka) U_{n-1}}$$

cG1 $\quad U_n - U_{n-1} + ak\,\dfrac{U_{n-1}+U_n}{2} = 0$

$$\left(1+\frac{k}{2}a\right)U_n = \left(1-\frac{k}{2}a\right)U_{n-1}$$

$$\boxed{U_n = \frac{\left(1-\frac{k}{2}a\right)}{\left(1+\frac{k}{2}a\right)}\,U_{n-1}}$$

dG0 $\quad \dfrac{U_n - U_{n-1}}{k} + a\,U_n = 0 \quad$ eller

$$U_n - U_{n-1} + ka\,U_n = 0$$

dvs $\quad (1+ka)\,U_n = U_{n-1}$

$$\boxed{U_n = \frac{U_{n-1}}{1+ka}}$$

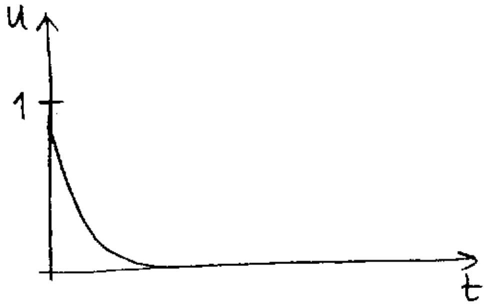
$a = 30$, $k = 0{,}1$ ger

Expl. Euler: $\quad U_n = -2\,U_{n-1} = (-2)^2\,U_{n-2} = \cdots$

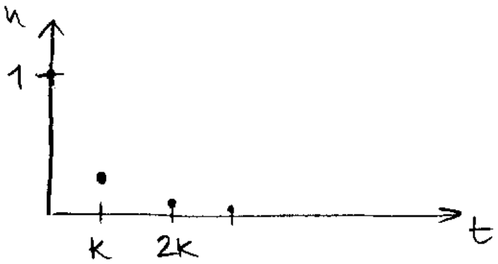
cG1: $\quad U_n = -\frac{1}{5}\,U_{n-1} = \left(-\frac{1}{5}\right)^2 U_{n-2} = \cdots$

dG0: $\quad U_n = \frac{1}{4}\,U_{n-1} = \frac{1}{16}\,U_{n-2} = \cdots$

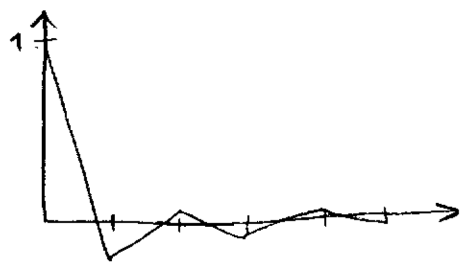
$u(t) = e^{-at}$ om $u(0) = 1$



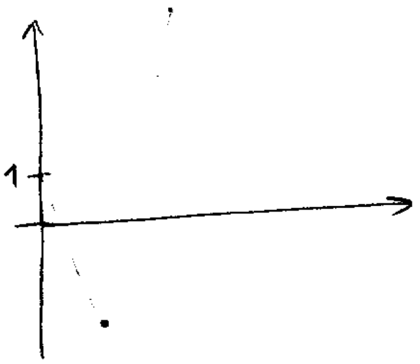
Exakt lösn



dG0



cG1



dG0

instabilt pga för stort tidssteg!

För värmeledningsekv. ersätts a med $-\Delta$

$-\Delta$ stor om u oscillerar i rumsled

Måste ta $k \lesssim h^2$

tex rumssteg $1/100 \Rightarrow$ tidssteg $1/10000$!

Probl. kvarstår för Expl. Euler även med godt. litet tidssteg om $a = i$.

Kontenta

Dämpat problem: diskontinuerlig Galerkin,
kanske dG1

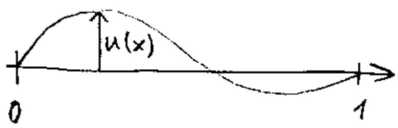
Odämpat: Metod utan dämpning
kanske cG1

Tenta 970826

1, $u = u(x,t)$

$$\ddot{u} + b\dot{u} - u'' = 0 \qquad 0 < x < 1 \qquad \text{i)}$$

$$u(0,t) = u(1,t) = 0 \qquad b \geq 0$$

a) 

"vibrerande sträng"

$\ddot{u}$: acceleration

u'': kraft från snörspänning

$b\dot{u}$: (dissipativ) kraft från tex luftmotstånd.

b) $$E(t) = \underbrace{\int_0^1 \dot{u}^2\,dx}_{\text{"kinetisk energi"}} + \underbrace{\int_0^1 u'^2\,dx}_{\text{"inre potentiell energi"}}$$

Multiplicera i) med $\dot{u}$ och integrera över $(0,1)$

$$\int_0^1 \ddot{u}\dot{u}\,dx + \int_0^1 b\dot{u}^2\,dx - \int_0^1 u''\dot{u}\,dx = 0$$

$$\frac{1}{2}\int_0^t \frac{\partial}{\partial t}(\dot{u}^2)\,dx + \int_0^1 b\dot{u}^2\,dx + \underbrace{\int_0^1 u'\dot{u}'\,dx}_{\frac{1}{2}\int_0^1 \frac{\partial}{\partial t}(u'^2)\,dx} = 0$$

$$\frac{d}{dt}\left[\int_0^1 \dot{u}^2\,dx + \int_0^1 u'^2\,dx\right] = -2\int_0^1 \underbrace{b\dot{u}^2}_{\geq 0}\,dx \leq 0$$

$\frac{dE}{dt} \leq 0 \implies E$ avtar med t

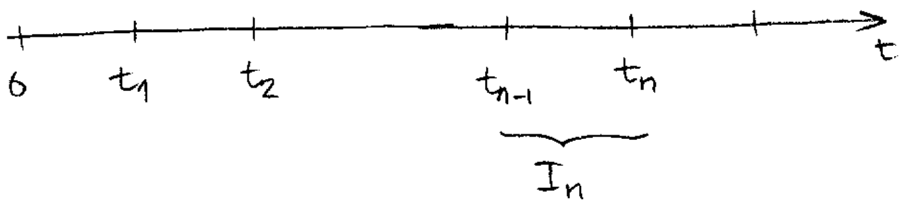
Om $b=0$ fås $\frac{dE}{dt}=0$, dvs E bevaras.

---

2) Inför ny funktion $v = \dot{u}$

Systemform $\begin{cases} \dot{u} - v = 0 \\ \dot{v} + bv - u'' = 0 \end{cases}$

$k_n = t_n - t_{n-1}$



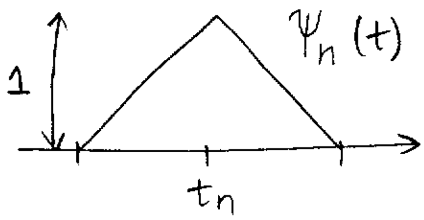
För $n = 1, 2, 3, \ldots$:

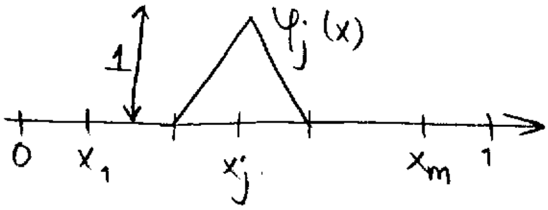
$$\int_{I_n} \int_0^1 \dot{u} w_1 \, dx dt - \int_{I_n} \int_0^1 v w_1 \, dx dt = 0$$

$$\int_{I_n} \int_0^1 \dot{v} w_2 \, dx dt + \int_{I_n} \int_0^1 b v w_2 \, dx dt + \int_{I_n} \int_0^1 u' w_2' \, dx dt = 0$$

w_1 & w_2 uppfyller samma randvillkor som ekvationen vi försöker lösa, dvs $w_i(0,t) = w_i(1,t) = 0$
$i = 1, 2$

FEM (cG1 i rum & tid)





$$\begin{cases} U(x,t) = U_{n-1}(x)\,\psi_{n-1}(t) + U_n(x)\,\psi_n(t) & t \in I_n \\ U_n(x) = \sum_{k=1}^{m} U_n^k \varphi_k(x) \quad (*) \end{cases}$$

och p.s.s.

$$\begin{cases} V(x,t) = V_{n-1}(x)\,\psi_{n-1}(t) + V_n(x)\,\psi_n(t) & t \in I_n \\ V_n(x) = \sum_{k=1}^{m} V_n^k \varphi_k(x) \quad (*) \end{cases}$$

$$\int_{I_n}\int_0^1 \dot{U}(x,t)\varphi_{j1}(x)\,dx\,dt - \int_{I_n}\int_0^1 V(x,t)\,\varphi_j(x)\,dx\,dt = 0$$

$$\int_{I_n}\int_0^1 \dot{V}(x,t)\varphi_{j2}(x)\,dx\,dt + \int_{I_n}\int_0^1 bV(x,t)\varphi_{j2}(x)\,dx\,dt +$$

$b = b(x,t)$

$$+\int_{I_n}\int_0^1 U'(x,t)\varphi_{j2}(x)\,dx\,dt = 0$$

$$\dot{U}(x,t) = U_{n-1}(x)\left(-\frac{1}{k_n}\right) + U_n(x)\left(\frac{1}{k_n}\right) \qquad t \in I_n$$

$$\dot{V}(x,t) = V_{n-1}(x)\left(-\frac{1}{k_n}\right) + V_n(x)\cdot\left(\frac{1}{k_n}\right) \qquad t \in I_n$$

$$\int_{I_n} \psi_{n-1}(t)\,dt = \int_{I_n} \psi_n(t)\,dt = \frac{k_n}{2}$$

$$\int_0^1 U_n(x)\varphi_{j1}(x)\,dx - \frac{k_n}{2}\int_0^1 V_n(x)\varphi_{j1}(x)\,dx = \int_0^1 U_{n-1}(x)\varphi_{j1}(x)\,dx +$$

$$+ \frac{k_n}{2}\int_0^1 V_{n-1}(x)\varphi_{j1}(x)\,dx$$

$$\int_0^1 V_n(x)\varphi_{j2}(x)\,dx + \int_{I_n}\int_0^1 b(x,t)\,V_n(x)\cdot\Psi_n(t)\cdot\varphi_{j2}(x)\,dx\,dt +$$

$$+ \frac{k_n}{2}\int_0^1 U_n'(x)\varphi_{j2}'(x)\,dx = \int_0^1 V_{n-1}(x)\varphi_{j2}(x)\,dx -$$

$$- \int_{I_n}\int_0^1 b(x,t)\,V_n(x)\,\Psi_{n-1}(t)\,\varphi_{j2}(x)\,dx\,dt - \frac{k_n}{2}\int_0^1 U_{n-1}'(x)\varphi_{j2}'(x)\,dx$$

Insättning av (*) ger

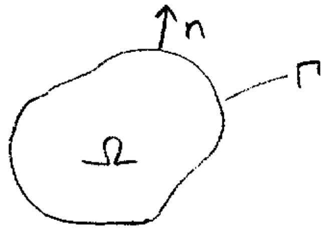
$$\begin{cases} MU_n - \frac{k_n}{2}MV_n = MU_{n-1} + \frac{k}{2}MV_{n-1} \\ MV_n + B_nV_n + \frac{k_n}{2}SU_n = MV_{n-1} - B_{n-1}V_{n-1} - \frac{k_n}{2}SU_{n-1} \end{cases}$$

$$U_n = \begin{bmatrix} U_n^1 \\ \vdots \\ U_n^m \end{bmatrix} \qquad V_n = \begin{bmatrix} V_n^1 \\ \vdots \\ V_n^m \end{bmatrix} \qquad M = (m_{jk}),$$

$$m_{jk} = \int_0^1 \varphi_k\,\varphi_j\,dx$$

$$S = (s_{jk}), \qquad s_{jk} = \int_0^1 \varphi_k'\,\varphi_j'\,dx$$

$$B_n = (b_{jk}^n), \qquad b_{jk}^n = \int_{I_n}\int_0^1 b(x,t)\cdot\Psi_n(t)\,\varphi_k(x)\,\varphi_j(x)\,dx\,dt$$

4) 

$\Delta u = 0$ i Ω

$\frac{\partial u}{\partial n} + u = g$ på Γ

(a) har ej betonat så mycket i kursen

b, Påst. $\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Gamma u^2 ds \leq \frac{1}{2}\int_\Gamma g^2 ds$

<u>bevis</u>: $\int_\Omega \Delta u \cdot u \, dx = 0$ {Greens formel}

$$\int_\Gamma \underbrace{\frac{\partial u}{\partial n}}_{g-u} \cdot u \, ds - \int_\Omega \nabla u \cdot \nabla u \, dx = 0$$

$$\int_\Omega |\nabla u|^2 dx + \int_\Gamma u^2 ds = \int_\Gamma gu \, ds \leq \int_\Gamma \frac{1}{2}(g^2+u^2) ds$$

$\left\{ a \cdot b \leq \frac{1}{2}(a^2+b^2) \right\}$

... VSB

c) Sök variationsformulering

$$\int_\Omega (\Delta u) v \, dx = 0$$

$$\int_\Gamma \underbrace{\frac{\partial u}{\partial n}}_{=g-u} \cdot v \, ds - \int_\Omega \nabla u \cdot \nabla v \, dx = 0 \quad \Longrightarrow$$

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Gamma uv \, ds = \int_\Gamma gv \, ds$$

VL = $a(u,v)$

HL = $L(v)$

Variationsformulering:

Finn $u \in V$ s.a. $a(u,v) = L(v) \quad \forall v \subset V$

där $V = H^1(\Omega) = \{ v : \underbrace{\int_\Omega (|\nabla v|^2 + v)\, dx}_{\|v\|_V^2} < \infty \}$

Lax-Milgram (Th. 21.1 i CDE) ger existens & entydighet <u>om</u>

* $a(v,w)$ bilinjär
* $L(v,w)$ linjär

} KLART!

* $a(v,w)$ är "V-elliptisk", dvs $a(v,v) \geq \mathcal{K}_1 \|v\|_V^2$ $\forall v \in V$ (kan visas med Th. 21.4 Poincaré-Friedrichs olikhet:

$$\|v\|^2_{L_2(\Omega)} \leq C\left(\|\nabla v\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(\Omega)} \right)$$

* $a(v,w)$ är kontinuerlig, dvs $|a(v,w)| \leq \mathcal{K}_2 \|v\|_V \|w\|_V$ $\forall v,w \in V$

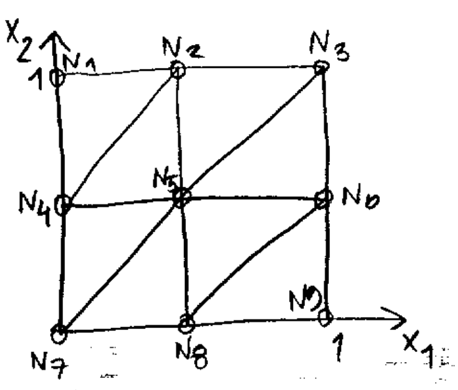
* $L(v)$ är kontinuerlig dvs $|L(v)| \leq \mathcal{K}_3 \|v\|_V$

kan visas med Th. 21.5: "Spårolikhet"

$\|v\|_{L_2(\Gamma)} \leq C \|v\|_V$

---

6) (fortsättning på 4)



$\Omega = [0,1] \times [0,1]$

FEM (cG1): Finn $U \in V_h$ s.a.

$$\int_\Omega \nabla U \cdot \nabla v \, dx + \int_\Gamma U \cdot v \, ds = \int_\Gamma g v \, ds \quad \forall v \in V_h \quad (*)$$

$V_h = \{ v :$ v kont & styckvis linjär på ovanst. triangulering av $\Omega \}$

Bas för $V_h : \{\varphi_i\}_{i=1}^{9}$ där $\begin{cases} \varphi_i \in V_h \\ \varphi_i(N_j) = \begin{cases} 1 & \text{om } i=j \\ 0 & \text{om } i \neq j \end{cases} \end{cases}$

Ansätt $U(x_1, x_2) = \sum_{j=1}^{9} U_j \varphi_j(x_1, x_2)$

$$(*) \iff \sum_{j=1}^{9} U_j \underbrace{\int_\Omega \nabla \varphi_j \cdot \nabla \varphi_i \, dx}_{a_{ij}^1} + \sum_{j=1}^{9} U_j \underbrace{\int_\Gamma \varphi_j \varphi_i \, ds}_{a_{ij}^2} = \underbrace{\int_\Gamma g \varphi_i \, ds}_{g_i}$$

$i = 1, \dots, 9$

$A^1 = (a_{ij}^1)$

$A^2 = (a_{ij}^2)$

$A = A^1 + A^2 \qquad U = \begin{bmatrix} U_1 \\ \vdots \\ U_9 \end{bmatrix} \qquad g = \begin{bmatrix} g_1 \\ \vdots \\ g_9 \end{bmatrix}$

$$\boxed{AU = g}$$

Betrakta $i = 6$ och $g = 1$:

$a_{66}^1 = 2$, $a_{63}^1 = -\frac{1}{2}$, $a_{65}^1 = -1$, $a_{69}^1 = -\frac{1}{2}$ övr. $a_{6j}^1 = 0$

$a_{66}^2 = \frac{1}{3}$, $a_{69}^2 = \frac{1}{12}$, $a_{63}^2 = \frac{1}{12}$ övr. $a_{6j}^2 = 0$

$g_6 = \frac{1}{2}$ ger ekv. nr 6 i systemet.

7) Kommentar:

Energinormsfeluppskattningar ($\|\nabla e\| \leq \dots$)

A priori: som i 1D

A posteriori: måste vara försiktig när man integrerar partiellt på termen $(\nabla U, \nabla(e - \pi_h e))$ ty ∇U är diskontinuerlig över triangelsidor där $(e - \pi_h e) \neq 0$

Dela upp integralen i en summa över samtliga deltrianglar. Se avsnitt 15.2.2.