Matematisk fysik 26/10

Olika metoder att räkna "fysikintegraler"

1) Komplexifiering

$$I(a) = \int_0^\infty e^{-ax}\cos(bx)\,dx = \mathrm{Re}\int_0^\infty e^{-ax}e^{ibx}\,dx =$$

$$= \mathrm{Re}\,\frac{1}{a-ib} = \frac{a}{a^2+b^2}$$

2) Derivering med avseende på parameter

$$I = \int_0^\infty e^{-ax}\cos(bx)\cdot x\,dx \underset{\text{Leibniz regel}}{=} -\frac{d}{da}I(a)$$

(I(a) definierad ovan)

OBS! Vid parameterberoende integrationsgränser

$$\frac{d}{d\alpha}\left[\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\,dx\right] = \int_{a(\alpha)}^{b(\alpha)} \frac{d f(x,\alpha)}{d\alpha}\,dx + \int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\frac{dx}{d\alpha}\,dx =$$

$$= \int_{a(\alpha)}^{b(\alpha)} \frac{d f(x,\alpha)}{d\alpha}\,dx + \frac{d\,b(\alpha)}{d\alpha}\int_{a(\alpha)}^{b(\alpha)} f'(x,\alpha)\,dx - \frac{d\,a(\alpha)}{d\alpha}\int_{a(\alpha)}^{b(\alpha)} f'(x,\alpha)\,dx$$

$$+ \frac{d\,b(\alpha)}{d\alpha}\left[f(b(\alpha),\alpha)\right] - \frac{d\,a(\alpha)}{dx}\left[f(a(\alpha),\alpha)\right]$$

3) Kombinerade derivator $\rightarrow$ diff. ekv

$$I(\alpha) = \int_0^\infty \frac{e^{-\alpha x}}{1+x^2}\,dx$$

$$I'(\alpha) = \int_0^\infty \frac{-x e^{-\alpha x}}{1+x^2}\,dx$$

$$I''(\alpha) = \int_0^\infty \frac{x^2 e^{-\alpha x}}{1+x^2}\,dx = \int_0^\infty e^{-\alpha x}\frac{(x^2+1)-1}{x^2+1}\,dx$$

$$= \int_0^\infty e^{-\alpha x}\,dx - \int_0^\infty \frac{e^{-\alpha x}}{x^2+1} = \left[\frac{e^{-\alpha x}}{-\alpha}\right]_0^\infty - I(\alpha)$$

$$= \frac{1}{\alpha} - I(\alpha) \qquad /\!/E$$

$$I''(\alpha) + I(\alpha) = \frac{1}{\alpha}$$

(DE) $\quad y''(x) + y(x) = \frac{1}{x}$

homogen lösning $\quad y_h(x) = C_1 y_1(x) + C_2 y_2(x)$

y_h: $r^2+1=0$

$r = \pm i$

$y_1 = e^0 \cos x$

$y_2 = e^0 \sin x$

Använd parametervariation

Ansätt $\quad y(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \qquad (1)$

$$y' = u_1 y_1' + u_2 y_2' + \boxed{u_1' y_1 + u_2' y_2} \qquad (2)$$

välj u_1, u_2 så att boxen blir $=0$

$$y'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'' \qquad (3)$$

stoppa in (1), (2) & (3) i problemet (DE)

$$u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2' + u_1 y_1 + u_2 y_2 = \frac{1}{x}$$

$$\Rightarrow u_1\underbrace{(y_1''+y_1)}_{=0} + u_2\underbrace{(y_2''+y_2)}_{=0} + u_1' y_1' + u_2' y_2' = \frac{1}{x} \qquad (4)$$

ty lösningar till homogena (DE)

$$\Rightarrow (2)\,\&\,(4) \Rightarrow \begin{pmatrix} y_1' & y_2' \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} \frac{1}{x} \\ 0 \end{pmatrix}$$

linjär algebra-problem

$$u_1' = -\frac{\sin x}{x}$$

$$u_2' = \frac{\cos x}{x}$$

Resultat

$$I(\alpha) = \underset{y_1}{-\cos\alpha} \underset{u_1}{\int_\infty^\alpha \frac{\sin(t)}{t}\,dt} + \underset{y_2}{\sin\alpha} \underset{u_2}{\int_\infty^\alpha \frac{\cos t}{t}\,dt}$$

$$= \sin\alpha\, Ci(\alpha) + \cos\alpha\left(\frac{\pi}{2} - Si(\alpha)\right)$$

obs!

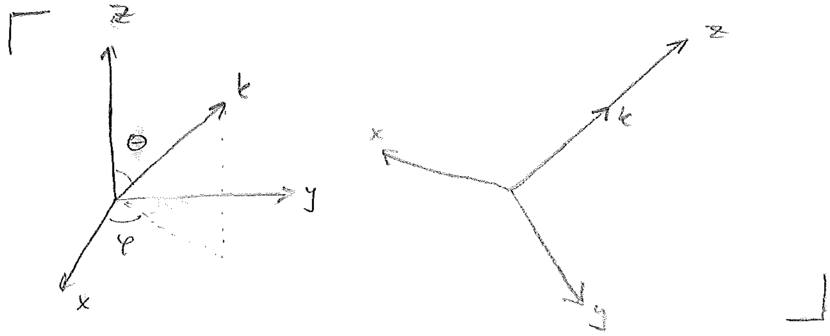
$$Ci(\alpha) = \int_\infty^\alpha \frac{\cos t}{t}\,dt$$

$$Si(\alpha) = \int_0^\alpha \frac{\sin t}{t}\,dt$$

4) Symmetriargument

$$I_1(\mathbf{k}) = \int \frac{d\Omega}{1+\mathbf{k}\cdot\hat{r}} = \int_0^{2\pi}\left(\int_{-1}^{1} \frac{d(\cos\theta)}{1+\mathbf{k}\cdot\hat{r}}\right) d\varphi = \int_0^{2\pi}\left(\int_{-1}^{1} \frac{d(\cos\theta)}{1+k\cos\theta}\right) d\varphi =$$

$$= \{ y = k\cos\theta \} = \frac{2\pi}{k} \ln\left(\frac{1+k}{1-k}\right)$$



$$I_2(\mathbf{k}) = \int \frac{d\Omega}{(1+\mathbf{k}\cdot\hat{r})^2} = -\frac{d}{d\alpha}\int \frac{d\Omega}{\alpha+\mathbf{k}\cdot\hat{r}}\Bigg|_{\alpha=1} = \frac{4\pi}{1-k^2}$$

$$\vdots$$

$$I_m(\mathbf{k}) = (-1)^{m-1} \frac{d^{m-1}}{d\alpha^{m-1}} \int \frac{d\Omega}{\alpha+\mathbf{k}\cdot\hat{r}}\Bigg|_{\alpha=1}$$

Typisk spridningsteori $\quad I_m(\mathbf{k}) = \int \frac{d\Omega}{(1+\mathbf{k}\cdot\hat{r})^m}$

Ett lite svårare problem

$$I_1(\mathbf{k},\mathbf{a}) = \int \frac{\mathbf{a}\cdot\hat{r}}{1+\mathbf{k}\cdot\hat{r}} d\Omega = \mathbf{a}\cdot\underbrace{\int \frac{\hat{r}}{1+\mathbf{k}\cdot\hat{r}} d\Omega}_{\mathbb{J}(\mathbf{k})} = \mathbf{a}\cdot\mathbb{J}(\mathbf{k}) = \mathbf{a}\cdot A\mathbf{k}$$

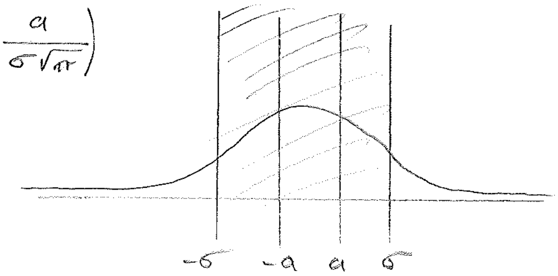
Ty $\mathbb{J}$ kan bara bero på $\mathbf{k}$

$$\Rightarrow \mathbf{k}\cdot\mathbb{J}(\mathbf{k}) = A\,\mathbf{k}\cdot\mathbf{k} = Ak^2 \rightarrow A = \frac{1}{k^2}\,\mathbf{k}\cdot\mathbb{J}(\mathbf{k}) =$$

$$= \frac{1}{k^2}\int \frac{\mathbf{k}\cdot\hat{r}}{1+\mathbf{k}\cdot\hat{r}} d\Omega = \frac{1}{k^2}\int\left(1 - \frac{1}{1+\mathbf{k}\cdot\hat{r}}\right) d\Omega = \frac{4\pi}{k^2}\left(1 - \frac{1}{2k}\ln\left(\frac{1+k}{1-k}\right)\right)$$

Asymptotisk serie

$\mathrm{erf}\left(\frac{a}{\sigma\sqrt{\pi}}\right)$



Gauss felfunktion

$\mathrm{erf}(x)$

Serieutveckling

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \overset{\text{Taylor}}{=} \frac{2}{\sqrt{\pi}}\int_0^x \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \dots\right)dt =$$

$$= \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \dots\right) \quad \text{konvergent}$$

Leibniz: En serie är konvergent om $|a_n|$ är monotont avtagande och $a_n \to 0$ då $n \to \infty$

$$1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\left(\underbrace{\int_0^\infty e^{-t^2}dt}_{\text{kommer av ettan}} - \int_0^x e^{-t^2}dt\right) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt = \left[\begin{matrix} t^2 = y \\ dt = \frac{dy}{2\sqrt{y}}\end{matrix}\right] =$$

(upprepade partiella integrationer)

$$= \underbrace{\frac{2}{\sqrt{\pi}}e^{-x^2}\left(\frac{1}{2x} - \frac{1}{2^2x^3} + \frac{1\cdot 3}{2^3x^5} - \frac{1\cdot 3\cdot 5}{2^4x^7} + \dots\right)}_{S_n(x)} + \underbrace{(-1)^n\left(\frac{1\cdot 3\cdot 5\cdot 7\dots(2n-1)}{2^{n-1}}\right)\frac{2}{\sqrt{\pi}}\int_x^\infty \frac{e^{-t^2}}{t^{2n}}dt}_{R_n(x)}$$

Serien ej konvergent för något x, ty $\frac{|a_{n+1}|}{|a_n|} \to \infty$ då $n \to \infty$

$$(1 - \mathrm{erf}(x)) - S_n(x) = R_n(x) \xrightarrow[x\to\infty]{} 0 \quad \text{för godtyckligt } n!$$

Definition $\lim_{x\to\infty} x^n\left(f(x) - S_n(x)\right) = 0$ (Asymptotisk serie)

Jämför: konvergent serie $S_n(x) \xrightarrow[n\to\infty]{} f(x)$ ($\forall x$)

asymptotisk serie $S_n(x) \xrightarrow[x\to\infty]{} f(x)$ ($\forall n$)

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Asymptotiska serier

$$\lim_{x\to\infty} x^N\left(f(x)-S_N(x)\right)=0 \implies S_N(x)\xrightarrow[x\to\infty]{} f(x) \quad \forall N$$

Watsons Lemma

(Whitaker-Watson, "A course in modern analysis")

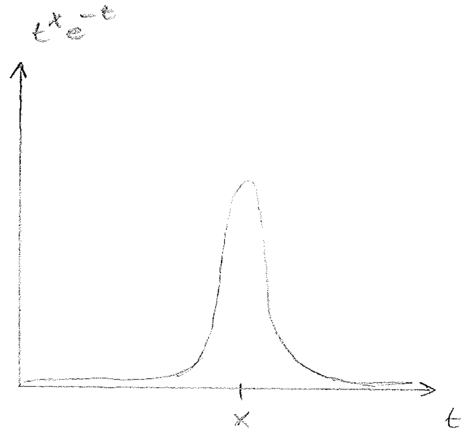
Ex. $$J(x)=\int F(t)e^{-xt}\,dt \underset{x\to\infty}{=} ?$$

$$F(t)\sim t^b\sum_{n=0}^{\infty} c_n t^n, \quad b>-1$$

$$x\to\infty \quad J(x)\sim\sum_{n=0}^{\infty} c_n \frac{\Gamma(n+b+1)}{x^{n+b+1}}, \text{ där } \Gamma(\xi+1)=\int_0^\infty t^\xi e^{-t}\,dt$$

Sadelpunktsmetoden

$$\Gamma(x+1)=\int_0^\infty t^x e^{-t}\,dt, \quad (\Gamma(n+1)=n!)$$



skriv integranden på exponentiell form

$$e^{f(t)}=t^x e^{-t}=e^{x\ln t-t}$$

$$f(t)=x\ln t-t$$

$$f'(t)=\frac{x}{t}-1 \quad (=0 \text{ då } x=t)$$

$$f''(t)=-\frac{x}{t^2} \quad (<0 \text{ då } x=t)$$

Serieutveckla kring maximat $t=x$ $\quad f(t)=f(x)+f'(t)\big|_{t=x}(t-x)\ldots$

$$\Gamma(x+1)=\int_0^\infty \exp\Big\{\underbrace{(x\ln x-x)}_{f(t)}+\underbrace{0}_{f'(t)}+\underbrace{\tfrac{1}{2}\left(-\tfrac{1}{x}\right)(t-x)^2}_{f''(t)}+\cancel{\ldots}\Big\}\,dt$$

$$\approx e^{x\ln x-x}\underbrace{\int_0^\infty e^{-\frac{1}{2x}(t-x)^2}\,dt}=x^x e^{-x}\sqrt{2\pi x}$$

första termen i Stirlings formel

$$\begin{bmatrix} t-x=t' \\ 2x=a \end{bmatrix} \quad \int_0^\infty e^{-\frac{t'^2}{a}}\,dt=\sqrt{\pi a}$$

Man kan visa att

$$\Gamma(n+1) = n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} + \dots\right) \qquad \text{STIRLING}$$

$$\ln n! \approx \frac{1}{2}\ln 2\pi + \left(n+\frac{1}{2}\right)\ln n - n + \sum_{j=1}^{N} \frac{B_{2n}}{2j(2j-1)} n^{1-2j} = S_N(n)$$

$$B_j = \frac{(-1)^{j-1}\, 2\,(2j!)}{(2\pi)^{2j}} \sum_{p=1}^{\infty} p^{-2j}$$

$S_N(n)$ Asymptotisk serieutveckling av $\ln n!$

dvs. $\ln n! \xrightarrow[n \to \infty]{} S_N(n) \quad \forall N$

Viktig användning i Statistisk fysik

$S = k_B \ln W$ N - gitterpunkter

$\uparrow$ n - partiklar

entropi

$$S = k_B \ln W = k_B \ln \binom{N}{n} = k_B \ln\left(\frac{N!}{n!(N-n)!}\right) = k_B\left\{\ln N! - \ln n! - \ln(N-n)!\right\}$$

Stirlings formel fungerar ty storleksordningen är Avogadros tal

Residykalkyl

$f = u + iv$ analytisk

Cauchy-Riemann

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Cauchys integralformel

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0} dz$$

Laurentutveckling

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n$$

där

$$a_n = \frac{1}{2\pi i} \oint \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$$

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1) Smarta kurvor

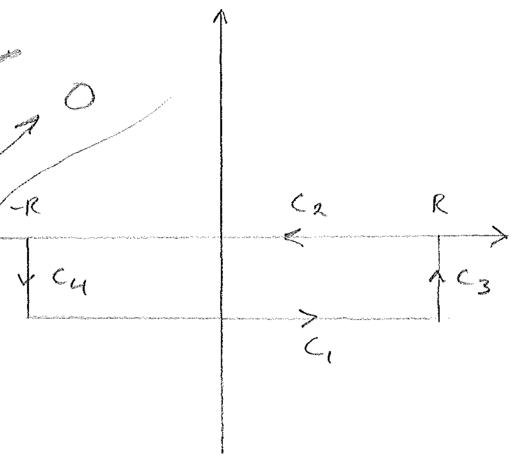
$$I = \int_{-\infty}^{\infty} e^{iax - bx^2} dx \qquad b > 0 \qquad a, b \in \mathbb{R}$$

kvadratkomplettera

$$e^{-a^2/4b} \lim_{R\to\infty} \int_{-R}^{R} e^{-b\underbrace{\left(x - \frac{ia}{2b}\right)}_{=z}{}^2} dx = e^{-a^2/4b} \underbrace{\lim_{R\to\infty} \int_{-R-\frac{ia}{2b}}^{R-\frac{ia}{2b}} e^{bz^2} dz}_{I_R}$$

$$I_R + \cancel{\int_{C_3}} + \int_{C_2} + \cancel{\int_{C_4}} = 0$$

$$\int_{C_3} e^{-bz^2} dz = \int_{-\frac{ia}{2b}}^{0} e^{-b(R+iy)^2} i\,dy = ie^{-bR^2} \int_{-\frac{ia}{2b}}^{0} e^{by^2 - 2ibRy} dy \xrightarrow{R\to\infty} 0$$



$$I_R = -\int_{C_2} = -\int_{R}^{-R} e^{-bx^2} dx = \int_{-R}^{R} e^{-bx^2} dx$$

$$\Rightarrow I_R = \lim_{R\to\infty} e^{-\frac{a^2}{4b}} \sqrt{\pi/b} = \sqrt{\pi/b}\, e^{-a^2/4b}$$

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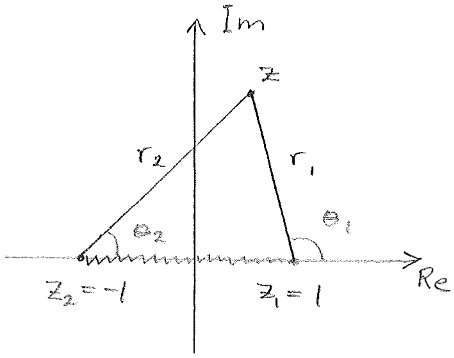
Grenpunkter

$\log z$: $z=0,\infty$ (jfr $\log z = -\log\frac{1}{z}$)

$z^{1/n}$: $z=0,\infty$ (jfr $z^{1/n} = e^{1/n \log z}$)

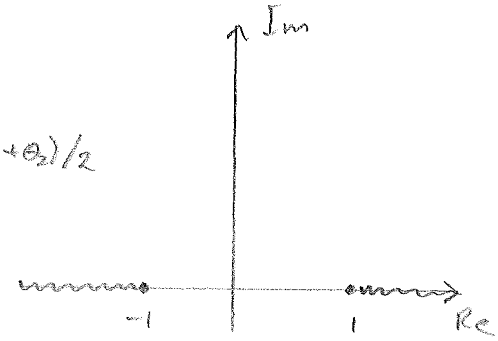
$(z^2-1)^{1/2}$: $z=\pm 1,\infty$ ($(z^2-1)^{1/2} = \sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2}$)



$z-1 = r_1 e^{i\theta_1}$

$z+1 = r_2 e^{i\theta_2}$

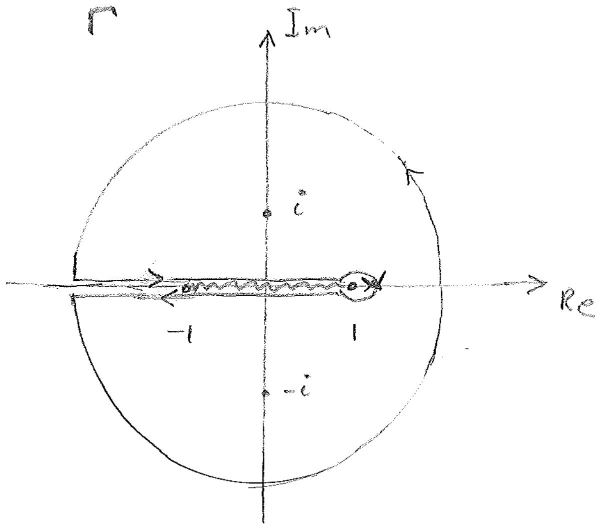
$(z^2-1)^{1/2} = \sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2}$



alternativt snitt

Integralberäkning

Ex $$I = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}\,(1+x^2)}$$ komplexifiera $$\oint_\Gamma \frac{dz}{\sqrt{1-z^2}\,(1+z^2)}$$

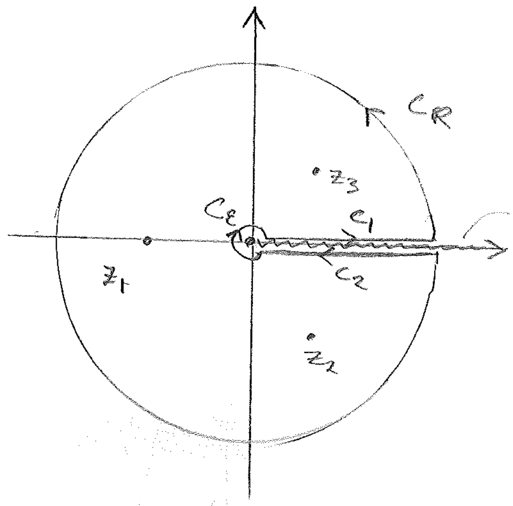


$$\Rightarrow I = \pi i \sum_{j=1}^{2} \mathrm{Res}\left[\frac{1}{\sqrt{1-z_j^2}\,(1+z_j^2)}\right]$$

Ex

$$I = \int_0^\infty \frac{dx}{x^3+1} \xrightarrow{\text{komplexifiera}} \oint \frac{\ln z}{z^2+1}\,dz$$

inför en flervärd hjälpfunktion



poler: $z_1 = -1 = e^{i\pi}$

$z_2 = e^{5i\pi/3}$

$z_3 = e^{i\pi/3}$

(1) $$\int_\Gamma \frac{\ln z}{z^3+1}\,dz = 2\pi i \sum_{j=1}^{3} \operatorname{Res}\left(\frac{\ln z_j}{z_j^3+1}\right)$$

cancelleras

(2) $$\int_\Gamma \frac{\ln z}{z^3+1}\,dz = \int_{c_1} \frac{\ln x}{x^3+1}\,dx + \underbrace{\int_{C_R} \frac{\ln z}{z^3+1}\,dz}_{\substack{\to 0 \\ R\to\infty}} + \int_{c_2=-c_1} \frac{\ln x + 2\pi i}{1+x^3}\,dx + \int_{C_\varepsilon} \frac{\ln z}{1+z^3}\,dz$$

(1) & (2) $\Rightarrow$ $$-2\pi i I + \int_{C_\varepsilon} \frac{\ln z}{1+z^3}\,dz = 2\pi i \sum \operatorname{Res}\left(\frac{\ln z}{z^3+1}\right)$$

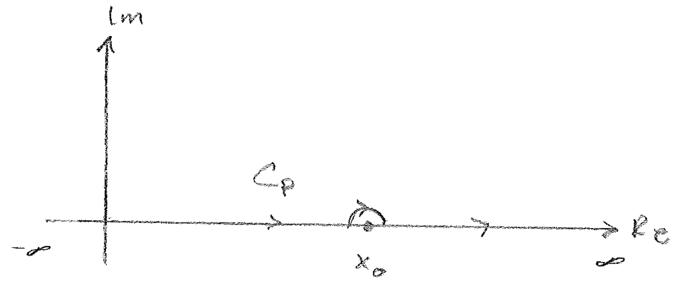
$$\int_0^{2\pi} \frac{\ln|r| + i\theta}{1+(re^{i\theta})^3}\, ire^{i\theta}\,d\theta \xrightarrow[r\to 0]{} 0$$

$$\Rightarrow I = -\sum_{j=1}^{3} \operatorname{Res}\left(\frac{\ln z_j}{z_j^3+1}\right) = \frac{2\pi}{3\sqrt{3}}$$

Räkna själv direkt m.h.a. smart val av kurva!

Principalvärden

$$\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}\,dx$$



Definiera integralen via sitt s.k. principalvärde

$$\int_{C_p} \frac{f(x)}{x-x_0}\,dx = \underbrace{\int_{-\infty}^{x_0-\varepsilon} \frac{f(x)}{x-x_0}\,dx + \int_{x_0+\varepsilon}^{\infty} \frac{f(x)}{x-x_0}\,dx}_{= P\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}\,dx} + \int_{C_0} \frac{f(z)}{z-x_0}\,dz \qquad \text{låt } \varepsilon \to 0$$

$$= P\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}\,dx \qquad \text{Principalvärdet}$$

$$\lim_{\varepsilon\to 0} \int_{C_0} \frac{f(z)}{z-x_0}\,dz = \left[z-x_0 = \varepsilon e^{i\theta} \right] = \lim_{\varepsilon\to 0} \int_{\pi}^{0} \frac{f(\varepsilon e^{i\theta}+x_0)}{\cancel{\varepsilon e^{i\theta}}}\, i\cancel{\varepsilon e^{i\theta}}\,d\theta \longrightarrow$$

ty f antas vara kontinuerlig

$$\longrightarrow \int_{\pi}^{0} f(x_0)\,i\,d\theta = i f(x_0) \int_{\pi}^{0} d\theta = -i\pi f(x_0)$$

$$\int_{C_p} \frac{f(z)}{z-x_0}\,dz = P\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}\,dx - i\pi f(x_0)$$

Analogt för C_u i det undre halvplanet

$$\int_{C_u} \frac{f(z)}{z-x_0}\,dz = P\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}\,dx + i\pi f(x_0)$$

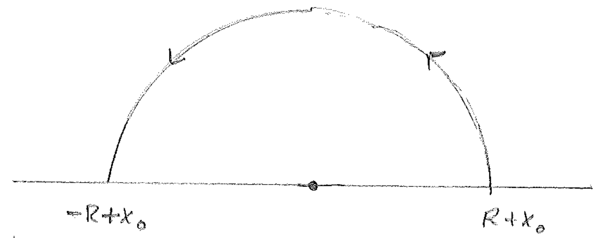
Hur beräknas principalvärdet?

Använd residysatsen

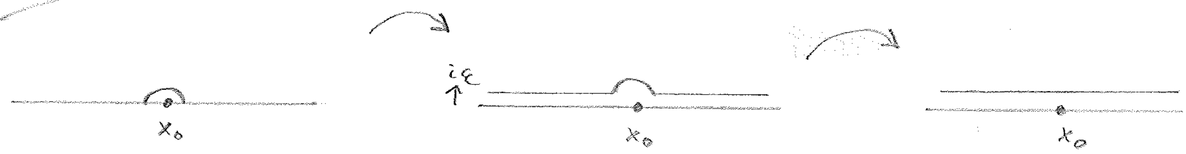
$$P\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}\,dx = \overset{+}{(-)}\, i\pi f(x_0) + 2\pi i \sum \mathrm{Res}$$

+ för övrehalvplanet
− för nedre halvplanet

i "rätt" halvplan



Feynmans trick



$$\int_{C_P}\frac{f(z)}{z-x_0}\,dz = \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon}\frac{f(\zeta+i\varepsilon)}{\zeta+i\varepsilon-x_0}\,d\zeta = \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon}\frac{f(x)}{x-x_0+i\varepsilon}\,dx$$

$[\zeta = z+i\varepsilon]$

[ombenämning $\zeta \to x$]

där vi låtit $\varepsilon \to 0$ i täljaren ty f kontinuerlig (av notationstekniska skäl)

Låt nu $\varepsilon \to 0$

$$P\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}\,dx = i\pi f(x_0) + \int_{-\infty}^{\infty}\frac{f(x)}{x-x_0+i\varepsilon}\,dx$$

På samma sätt för undre halvplanet

$$P\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}\,dx = -i\pi f(x_0) + \int_{-\infty}^{\infty}\frac{f(x)}{x-x_0+i\varepsilon}$$

"$\dfrac{1}{x-x_0\pm i\varepsilon} = P\dfrac{1}{x-x_0} \mp i\pi\,\delta(x-x_0)$" Fysikjargong

$$\Rightarrow \int_{-\infty}^{\infty}\frac{f(x)}{x-x_0\pm i\varepsilon} = P\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}\,dx \mp i\pi\int_{-\infty}^{\infty}\delta(x-x_0)f(x)\,dx$$

Kramers-Kronigrelationerna

$F(z)$ analytisk i övre halvplanet

och F avtar fortare än $\frac{1}{|z|}$ då $|z| \to \infty$

Beräkna $\frac{1}{2\pi i}\oint \frac{F(\xi)}{\xi - z}\,d\xi = \begin{cases} F(z), & \text{om } \operatorname{Im} z > 0 \\ 0, & \text{om } \operatorname{Im} z < 0 \end{cases}$

cauchys integralformel

$z = x + i\varepsilon$, $(\varepsilon \to 0)$

$$\frac{1}{2\pi i}\int_C \frac{F(\xi)}{\xi - (x+i\varepsilon)}\,d\xi = F(x+i\varepsilon) \to F(x) \quad \text{då } \varepsilon \to 0 \qquad (1)$$

$$\int_{-\infty}^{\infty} \frac{F(\xi)}{\xi - x - i\varepsilon}\,d\xi = P\int_{-\infty}^{\infty} \frac{F(\xi)}{\xi - x}\,d\xi + i\pi F(x) \qquad (2)$$

Kombination av (1) & (2)

$$2\pi i F(x) = P\int_{-\infty}^{\infty} \frac{F(\xi)}{\xi - x}\,d\xi + i\pi F(x)$$

$$F(x) = \frac{1}{i\pi} P\int_{-\infty}^{\infty} \frac{F(\xi)}{\xi - x}\,d\xi$$

$$\begin{cases} \operatorname{Re} F(x) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\operatorname{Im} F(\xi)}{\xi - x}\,d\xi \\ \operatorname{Im} F(x) = -\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\operatorname{Re} F(\xi)}{\xi - x}\,d\xi \end{cases}$$

Kramers-Kronig

(Hilberttransform)

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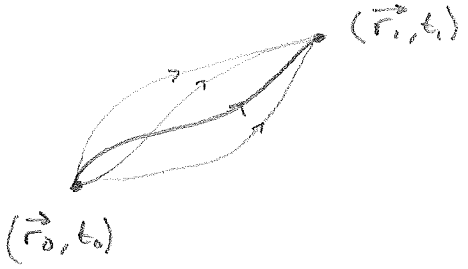
Sadelpunktsmetoden redux

1) Kvantfysikalisk tillämpning



$|A|$ - sannolikhet att en partikel i $(\vec{r}_0, t_0)$ rört sig till $(\vec{r}_1, t_1)$

Feynmans recept

$$A = \langle \vec{r}_1, t_1 | \, \underbrace{e^{-i H(t_1 - t_0)/\hbar}}_{\text{tidsutvecklingsoperator}} \, | \vec{r}_0, t_0 \rangle =$$

$$= \int \mathcal{D}[\vec{r}(t_1 - t_0)] \, e^{-i S(r(t))/\hbar}$$

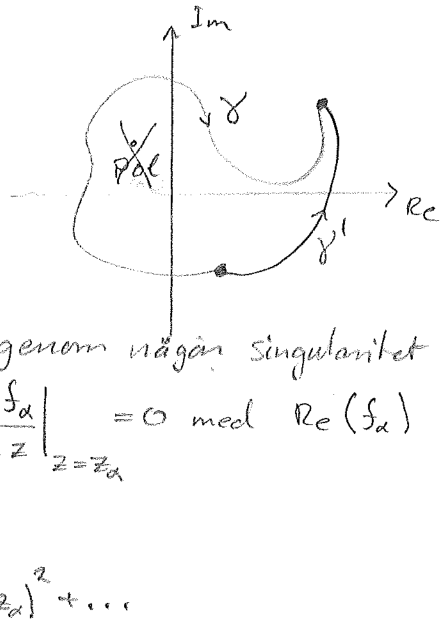
summerar över alla vägar

2) Statistisk fysik

$$Z = \sum_{\{i\}} e^{-\beta E_i} \longrightarrow \int \mathcal{D}[\gamma] \, e^{-S[\gamma]} \quad , \quad \beta = \frac{1}{k_B T}$$

$\{i\}$ summa över m kvanttillstånd

$$I_\alpha \equiv \int_\gamma e^{-f_\alpha(z)} dz \quad , \quad \alpha \in \mathbb{C}$$



Debyes recept

Deformera kurvan γ till γ' utan att passera genom någon singularitet så att på γ' $\mathrm{Im}(f_\alpha) =$ konstant och $\left.\frac{df_\alpha}{dz}\right|_{z=z_\alpha} = 0$ med $\mathrm{Re}(f_\alpha)$ minimal. Utveckla f_α kring $z = z_\alpha$

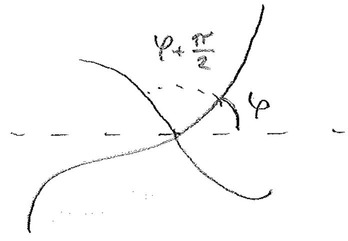
$$f_\alpha(z) = f_\alpha(z_\alpha) + \left.\frac{\partial f_\alpha}{\partial z}\right|_{z=z_\alpha} (z - z_\alpha) + \frac{1}{2} \left.\frac{\partial^2 f_\alpha}{\partial z^2}\right|_{z=z_\alpha} (z - z_\alpha)^2 + \dots$$

sätt $\left.\frac{\partial^2 f_\alpha}{\partial z^2}\right|_{z=z_\alpha} = g e^{i\theta}$, $z - z_\alpha = r e^{i\varphi}$

$$\begin{cases} \mathrm{Re}\left[f_\alpha(z) - f_\alpha(z_\alpha) \right] = \frac{1}{2} r^2 g \cos(2\varphi + \theta) + \cancel{\mathcal{O}(r^3)} \\ \mathrm{Im}\left[f_\alpha(z) - f_\alpha(z_\alpha) \right] = \frac{1}{2} r^2 g \sin(2\varphi + \theta) + \cancel{\mathcal{O}(r^3)} \end{cases}$$

1) $2\varphi + \theta = 0$, $z - z_\alpha = r e^{i\varphi}$

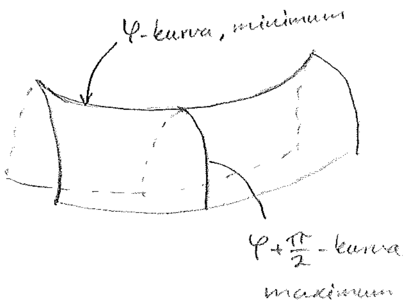
2) $2\varphi + \theta = \pi$, $z - z_\alpha = r e^{i(\varphi + \pi/2)}$



sätt in i relationena ovan

$$f_\alpha(z) = f_\alpha(z_\alpha) + \frac{t^2}{2} + \mathcal{O}(t^3)$$

$$t = (z - z_\alpha)\sqrt{\left.\frac{\partial^2 f_\alpha(z)}{\partial z^2}\right|_{z=z_\alpha}}$$



$$I_\alpha \approx e^{-f_\alpha(z_\alpha)} \int e^{-t^2/2} \frac{dz(t)}{dt} dt =$$

$$\left[\frac{dz}{dt} = \left(\left.\frac{\partial^2 f_\alpha}{\partial z^2}\right|_{z=z_\alpha} \right)^{-1/2} \right.$$

Utvidga intervallet på γ till $(-\infty, \infty)$
för varje minimum z_α.... $\Big]$

$$= e^{-f_\alpha(z_\alpha)} \left\{ \frac{2\pi}{\left.\frac{\partial^2 f_\alpha}{\partial z^2}\right|_{z=z_\alpha}} \right\}^{1/2}$$

Gaussisk integral

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

Samma teknik för rent imaginära exponenter
kallas stationära fasapproximationen.

{Matematisk Fysik 9/11}

Lebesgues Konvergensteorem

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \left\{ f_n(x) \to f(x) \right\} = \int_a^b f(x)\,dx$$

om $\exists g : |f_n(x)| \leq g(x)$ nästan överallt.

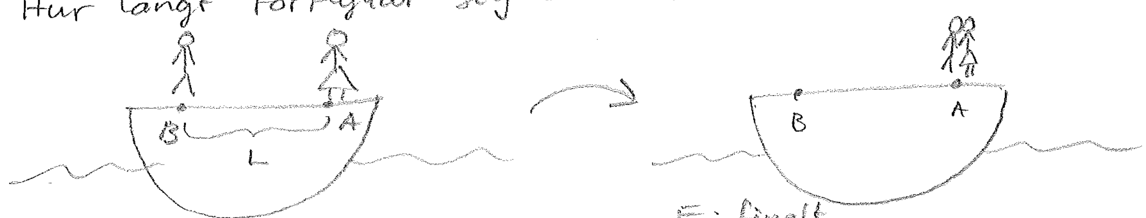
Fubini - Tonelli

$$\int_y \left(\int_x f(x,y)\,dx \right) dy = \int_x \left(\int_y f(x,y)\,dy \right) dx$$

om $f(x,y)$ ej växlar tecken.

Ordning av gränsvärden - Två sedelärande exempel

1) Hur långt förflyttar sig båten?



I: initialt F: finalt

a) perfekt fluid $\eta = 0$ $(\eta_{H_2O} \approx 10^{-3}\,\mathrm{Pa\,s})$

inga yttre krafter

Masscentrum R_{CM}

$$\vec{R}_{C.M.I}\Big(\underbrace{\text{Båten} + \text{Julia}}_{M} + \underbrace{\text{Romeo}}_{m}\Big) = \vec{R}_{C.M.F}\,(M+m)$$

Välj koordinatsystem $x_{BJ,i} = 0$

$$\Rightarrow \frac{M x_{BJ,i} + m x_{R,i}}{\cancel{M+m}} = \frac{M x_{BJ,f} + m x_{R,f}}{\cancel{M+m}}$$

$$\Rightarrow \underbrace{(x_{R,f} - x_{R,i})m}_{(L + x_{BJ,f})m} = -M x_{BJ,f}$$

$$\Rightarrow x_{BJ,f} = -\left(\frac{m}{m+M}\right)L$$

b) reellfluid $\eta \neq 0$

x − masscentrum Båten + Julia

y − masscentrum Romeo

$\longrightarrow x, y$

$$M\ddot{x} + m\ddot{y} = -\eta \dot{x}$$

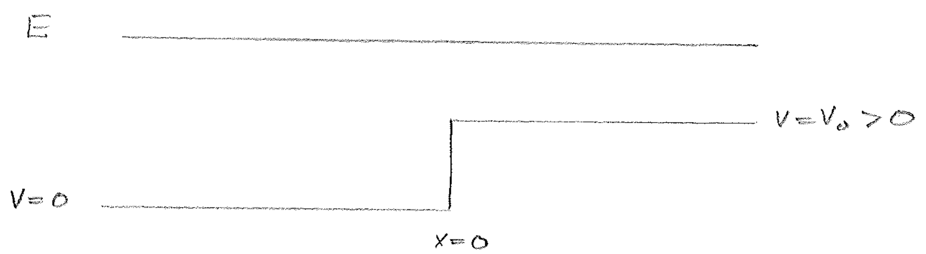
$$\int_0^\infty M\ddot{x} + m\ddot{y} \, dt = -\eta \int_0^\infty \dot{x} \, dt$$

$$M\dot{x}(t) + m\dot{y}(t) \Big|_0^\infty = -\eta \, x(t) \Big|_0^\infty$$

haetigheten är 0 både då $t=0$ och $t=\infty$

$$\Rightarrow 0 - 0 = -\eta \left(x(\infty) - x(0) \right) = -\eta \, \ell(\eta) = 0$$

$$\Rightarrow \lim_{\eta \to 0} \ell(\eta) = 0 \quad !$$



energi hos inkommande partikel $E = \frac{(\hbar k)^2}{2m} \Rightarrow k = \frac{\sqrt{2mE}}{\hbar}, \; x<0 \; ; \; k' = \frac{\sqrt{2m(E-V_0)}}{\hbar}, \; x>0$

Klassiskt: total transmission

Kvantmekaniskt: Viss reflektion R

$$\psi(x) = \begin{cases} e^{ikx} + B e^{-ikx} & , x<0 \\ A e^{ik'x} & , x>0 \end{cases} \quad \left| \quad \begin{array}{ll} \psi(0_-) = \psi(0_+) & (1) \\ \psi'(0_-) = \psi'(0_+) & (2) \end{array} \right\}$$

Vad är reflektionen $R = |B|^2$? Utnyttja (1) & (2)

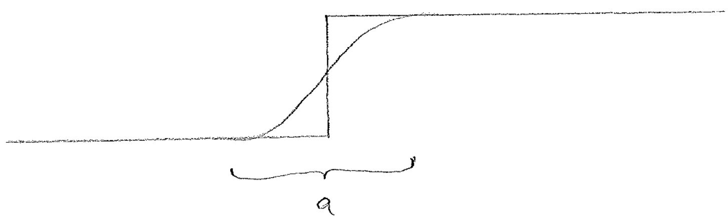
$$R = \left(\frac{k-k'}{k+k'} \right)^2 = \left(\frac{\sqrt{E} - \sqrt{E-V_0}}{\sqrt{E} + \sqrt{E-V_0}} \right)^2$$

oberoende av $\hbar$

$\hbar \to 0 \Rightarrow$ klassisk fysik Bohrs Korrespondansprincip
(kan visas m.h.a. Feynmans vägintegralformulering + Sadelpunktsmetoden)

Men, klassisk gräns måste tas mer försiktigt

$\lambda = \frac{\hbar}{p} \ll a$ ← systemets karaktäristiska längd



vi har valt
$a = 0 < \lambda$
därav felet.

Låt oss välja $a > 0$, $V(x) = \frac{V_0}{1 + e^{-x/a}}$

fortfarande asymptotiska vågtalen
$$\begin{cases} k = \frac{\sqrt{2mE}}{\hbar} \\ k' = \frac{\sqrt{2m(E - V_0)}}{\hbar} \end{cases}$$

lös Schrödingerekv.

$$R(\hbar, a) = \left(\frac{\sinh(a\pi(k - k'))}{\sinh(a\pi(k + k'))} \right)^2$$

För fixt a, $\hbar \to 0$

$$\lim_{\substack{\hbar \to 0 \\ a \neq 0}} R(\hbar, a) = R(0, a) = 0$$

För fixt $\hbar$, $a \to 0$

$$\lim_{\substack{a \to 0 \\ \hbar \neq 0}} R(\hbar, a) = R(\hbar, 0) = \left(\frac{k - k'}{k + k'} \right)^2 \neq 0$$

$$\lim_{\hbar \to 0} \left(\lim_{a \to 0} R(\hbar, a) \right) \neq \lim_{a \to 0} \left(\lim_{\hbar \to 0} R(\hbar, a) \right)$$

P.1

$$I_1 = \int_{-\infty}^{\infty} dx \int_0^\infty \frac{e^{-y}}{x^2+y}\, dy = 2\int_0^\infty dy \underbrace{\int_0^\infty dx\, \frac{e^{-y}}{x^2+y}}_{I_2}$$

$$I_2 = e^{-y}\int_0^\infty \frac{dx}{x^2+y} = \left[\begin{array}{l} x=\sqrt{y}\tan\theta \\ dx = \sqrt{y}(1+\tan^2\theta)\,d\theta\end{array}\right]$$

$$= e^{-y}\int_0^{\pi/2} d\theta\, \frac{\sqrt{y}\cdot(1+\tan^2\theta)}{y\cdot(1+\tan^2\theta)} = y^{-1/2}e^{-y}\cdot\frac{\pi}{2}$$

$$I_1 = 2\cdot\int_0^\infty dy\, \frac{\pi}{2} e^{-y} y^{-1/2} = \pi\,\Gamma(1/2) = \pi\sqrt{\pi}$$

P.2

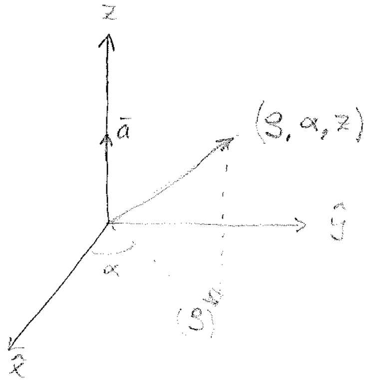
$$I = \lim_{\eta\to 0^+}\int_{-\infty}^\infty dx\, \frac{e^{-x^2}}{x+\eta i} \qquad \left[\lim_{\eta\to 0}\frac{1}{x+\eta i} = P\frac{1}{x} - \pi i\,\delta(x)\right]$$

$$I = \int_{-\infty}^\infty dx\left[P\frac1x - \pi i\,\delta(x)\right]e^{-x^2} = P\int_{-\infty}^\infty \frac{dx}{x} - \pi i \int_{-\infty}^\infty dx\,\delta(x)\, e^{-x^2} = -\pi i\, e^{-0^2} = -\pi i$$

P.3

$$I = \int_{\mathbb{R}^3} d^3x\, \bar{x}\, e^{i\bar a\cdot\bar x}\, e^{-bx^2} = -i\frac{\partial}{\partial \bar a}\int_{\mathbb{R}_3} d^3x\, e^{-i\bar a\cdot\bar x - bx^2} =$$

$$= -i\frac{\partial}{\partial\bar a}\, e^{-\frac{a^2}{2b}} \underbrace{\int_{\mathbb{R}_3} e^{-b\left(\bar x - \frac{\bar a i}{2b}\right)^2} d^3x}_{I_x}$$



$$\left[\left(\bar{x} - \frac{\bar{a}i}{2b}\right)^2 = \left(\varrho\hat{\varrho} + z\hat{z} - \frac{\bar{a}i}{2b}\hat{z} \right) = \hat{\varrho} + \left(z - \frac{ai}{2b}\right)^2 \right]$$

$$I_2 = \int_0^{2\pi} d\alpha \int_0^{\infty} d\varrho \cdot \varrho \int_{-\infty}^{\infty} dz \, e^{-b\left[\left(z - \frac{ai}{2b}\right)^2 + \varrho^2\right]} =$$

$$= 2\pi \int_0^{\infty} d\varrho \cdot \varrho e^{-b\varrho^2} \cdot \int_{-\infty}^{\infty} dz \, e^{-b\left(z - \frac{ai}{2b}\right)^2}$$

$$= \pi \left[\frac{e^{-b\varrho^2}}{-b} \right]_0^{\infty} \cdot I_3 = \frac{\pi}{b} \cdot \sqrt{\frac{\pi}{b}}$$

$$I = -i \frac{\partial}{\partial a} e^{-\frac{a}{4b}} \left(\frac{\pi}{2}\right)^{3/2} = \frac{i}{4b} e^{-\frac{a}{4b}} \left(\frac{\pi}{2}\right)^{3/2}$$

$$I_3 = \int_{-\infty}^{\infty} dx \, e^{-bx^2} = \sqrt{\frac{\pi}{b}}$$

$$(I_3)^2 = \int_{-\infty}^{\infty} dx \, e^{-bx^2} \int_{-\infty}^{\infty} dy \, e^{-by^2}$$

$$= \int dx dy \, e^{-b(x^2+y^2)} =$$

$$= \int_0^{\infty} 2\pi r e^{-br^2} dr =$$

$$= \left[\pi \frac{e^{-br^2}}{-b} \right]_0^{\infty} = \frac{\pi}{b}$$

P.5

$$I = \int d^3p \cdot \bar{p} (\bar{p} \cdot \bar{E}) e^{-\alpha p^2}, \quad \bar{p} = z\hat{z} + \varrho\hat{\varrho}, \quad \bar{E} = E\hat{z}$$

$$I = \int d^3p \left[z\hat{z} + \varrho\hat{\varrho} \right] (zE) e^{-\alpha(z^2+\varrho^2)} =$$

$$= \int d^3p \, E z^2 e^{-\alpha(z^2+\varrho^2)} \hat{z} + \underbrace{\int d^3p \, \varrho z E \, e^{-\alpha(z^2+\varrho^2)} \hat{\varrho}}_{=0}$$

integration över 2π

$$= \int_0^{2\pi} d\varphi E \int_0^{\infty} dp \, p \, e^{-\alpha p^2} \int_{-\infty}^{\infty} dz \, z^2 e^{-\alpha z^2} \hat{z} = \pi E \left[\frac{e^{-\alpha p^2}}{-\alpha} \right]_{p=0}^{\infty} \cdot I_2 \hat{z} = \frac{\pi E}{\alpha} \cdot I_2 \hat{z}$$

$$I_2 = \int_{-\infty}^{\infty} dz \, z^2 e^{-\alpha z^2} = -\frac{d}{d\alpha} \int_{-\infty}^{\infty} dz \, e^{-\alpha z^2} = -\frac{d}{d\alpha} \sqrt{\frac{\pi}{\alpha}} = \frac{\sqrt{\pi}}{2(\alpha)^{3/2}}$$

$$I = \frac{\pi^{3/2}}{2\alpha^{5/2}} E\hat{z} = \frac{\pi^{3/2}}{2\alpha^{5/2}} \bar{E}$$

P. 6 $\quad I = \int_S d^3k \, \frac{1}{1-\frac{1}{3}(\cos k_x + \cos k_y + \cos k_z)} \quad , \quad S = [-\pi, \pi]^3$

(2)

Modifierad Besselfunktion av ordning 0

$$I_0(z) = \frac{1}{\pi}\int_0^{\pi} dt\, e^{z\cos t}$$

$$I = \int_0^{\infty} dz \int_S d^3k\, e^{-z[1-\frac{1}{3}(\cos k_x + \cos k_y + \cos k_z)]} =$$

$$= \int_0^{\infty} dz\, e^{-z} \int_S d^3k\, e^{\frac{z}{3}(\cos k_x + \cos k_y + \cos k_z)} =$$

$$= \int_0^{\infty} dz\, e^{-z} \int_{-\pi}^{\pi} dk_x \int_{-\pi}^{\pi} dk_y \int_{-\pi}^{\pi} dk_z\, e^{\frac{z}{3}(\cos k_x + \cos k_y + \cos k_z)} =$$

$$= \int_0^{\infty} dz e^{-z} \left[\int_{-\pi}^{\pi} dk_x\, e^{\frac{z}{3}\cos k_x} \right]^3 = \int_0^{\infty} dz e^{-z} \left[2\int_0^{\pi} dk_x e^{\frac{z}{3}\cos k_x} \right]^3 =$$

$$= \int_0^{\infty} dz e^{-z} \left[2\pi I_0(z/3) \right] = \int_0^{\infty} dz\, e^{-z}\, I_0^3\left(\frac{z}{3}\right) \cdot (2\pi)^3$$

Hilbertrum

repetition/notation

- V vektorrum $\{|a\rangle, |b\rangle \dots\}$

  $|a\rangle + |b\rangle = |b\rangle + |a\rangle \in V$

  $|a\rangle + |0\rangle = |a\rangle$

  $|a\rangle + (|b\rangle + |c\rangle) = (|a\rangle + |b\rangle) + |c\rangle \in V$

  $\forall |a\rangle, \exists -|a\rangle : |a\rangle + (-|a\rangle) = 0$

  $\alpha(\beta|a\rangle) = \alpha\beta|a\rangle \qquad \alpha, \beta \in \mathbb{C}$

  $1 \cdot |a\rangle = |a\rangle$

  $(\alpha + \beta)|a\rangle = \alpha|a\rangle + \beta|a\rangle$

  $\alpha(|a\rangle + |b\rangle) = \alpha|a\rangle + \alpha|b\rangle$

- Bas

  inreprodukt

  $\langle a|b\rangle = \langle b|a\rangle^*$

  $\langle a|(\beta|b\rangle + \gamma|c\rangle) = \beta\langle a|b\rangle + \gamma\langle a|c\rangle$

  $\langle a|a\rangle \geq 0$

  $\langle a|a\rangle = 0 \Rightarrow |a\rangle = 0$

  $\langle \mathbf{r}|\mathbf{k}\rangle = \psi_k(\mathbf{r})$ $\qquad$ $\mathbf{r}$ - lägestillstånd

  $\mathbf{k}$ - vågtal

Ex. 1) $|a\rangle = (\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n)$

$|b\rangle = (\beta_1, \beta_2, \beta_3 \dots \beta_n)$

$$\langle a|b\rangle = \alpha_1^* \beta_1 + \alpha_2^* \beta_2 + \dots + \alpha_n^* \beta_n = \sum_{i=1}^{n} \alpha_i^* \beta_i$$

2) $\mathbb{R}^3 \Rightarrow |a\rangle, |b\rangle, \ n=3$

$|a\rangle = (a_1, a_2, a_3) \equiv \vec{a}$

$|b\rangle = (\beta_1, \beta_2, \beta_3) \equiv \vec{b}$

$$\langle a|b\rangle = \sum_{i=1}^{3} \alpha_i \beta_i = \vec{a} \cdot \vec{b}$$

3) $f, g \in \mathbb{C}(a,b)$ mängden av komplexvärda funktioner $[a,b]$ vektorrum

$$\langle f|g\rangle \equiv \int_a^b f^*(x) g(x) \underbrace{w(x)}_{\text{viktfunktion}} dx$$

- Ortogonalitet Gram-Schmidt
- Schwarz olikhet

$$\langle a|a\rangle \langle b|b\rangle \geq |\langle a|b\rangle|^2$$ med likhet då $|b\rangle = \alpha |a\rangle$

- Norm

$\|a\| = \sqrt{\langle a|a\rangle}$ inreproduktnorm

$\|a+b\| \leq \|a\| + \|b\|$ triangelolikheten

avstånd $d(a,b) \equiv \|a-b\|$

- Linjär transformation

$T: V \rightarrow W$

$$T(\alpha|a\rangle + \beta|b\rangle) = \alpha T|a\rangle + \beta T|b\rangle$$

Endomorfism

Oändligdimensionella vektorrum

Cauchy-sekvens

$V \ni \{ |a_i\rangle \}_{i=1}^{\infty}$ är en Cauchysekvens om

$$\lim_{\substack{i\to\infty \\ j\to\infty}} \| a_i - a_j \| = 0$$

$V \ni \{ |a_i\rangle \}_{i=0}^{\infty}$ är konvergent om $\exists |a\rangle \in V$

$$\lim_{i\to\infty} \| a_i - a \| = 0$$

Konvergens $\Rightarrow$ Cauchy men Cauchy $\Rightarrow$ Konvergens endast för fullständiga vektorrum

① $\mathbb{R}$ är fullständigt med avseende på $\|\alpha\| = |\alpha|$ $\alpha \in \mathbb{R}$

dvs. varje Cauchysekvens av reella tal har ett gränsvärde i $\mathbb{R}$

② $\mathbb{Q}$ är inte fullständigt m.a.p. $\|\alpha\| = |\alpha|$

$$\underbrace{\left\{ (1 + 1/k)^k \right\}_{k=1}^{\infty}}_{\text{Cauchy}} \longrightarrow e \notin \mathbb{Q}$$

<u>Sats</u>. Varje ändligdimensionellt vektorrum är fullständigt

Matematisk fysik 16/11

$\mathcal{H} \sim \mathcal{L}_w(a,b)$ Bas $\{x^k\}_{k=0}^{\infty} \rightarrow \{c_k(x)\}_{k=0}^{\infty}$ ortogonala polynom

$c_0 = a$, $c_1 = a + bx, \ldots$ Legendrepolynom

Problem: Givet en bas $\{c_k\}_{k=0}^{\infty}$ hur uttrycka en godtycklig funktion $f \in \mathcal{L}^2$?

Bilda $S(a_1, a_2, \ldots a_n) \equiv \int_a^b \left[f(x) - \sum_{k=0}^{n} a_k c_k(x) \right]^2 w(x)dx$

minimera S för varje a_j

$$0 = \frac{\partial S}{\partial a_j} \int_a^b -2c_j \left[f(x) - \sum_{k=0}^{n} a_k c_k(x) \right] w(x) dx$$

$$\Rightarrow \sum_{k=0}^{n} a_k \int_a^b c_j(x) c_k(x) w(x) dx = \int_a^b c_j(x) f(x) w(x) dx$$

$$\Rightarrow a_j = \frac{\int_a^b c_j(x) f(x) w(x) dx}{\int_a^b [c_j(x)]^2 w(x) dx}, \quad j = 0, 1, 2 \ldots$$

Hur konstruera $\{c_k(x)\}_{k=0}^{\infty}$ på ett effektivt sätt

1 "Sturm-Liouville System"

2 Rodriquez generaliserade formel

$Lu = \lambda u$

$L = \alpha(x) \frac{d^2}{dx^2} + \beta(x) \frac{d}{dx} + \gamma(x)$

Oändligt-dimensionella vektorrum

1) uppräknelig

2) icke uppräknelig – Hur representera en vektor?

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1) Ändligdimensionellt V_n : Bas $\{|e_i\rangle\}_{i=1}^{N}$

$$|f\rangle = f_1|e_1\rangle + f_2|e_2\rangle + \dots + f_n|e_n\rangle = (f_1, f_2 \dots f_n)$$

2) uppräkneligt oändligdimensionellt vektorrum V : bas $\{|e_i\rangle\}_{i=1}^{\infty}$

$$|f\rangle = (f_1, f_2 \dots\dots)$$

3) icke uppräkneligt oändligdimensionellt vektorrum

bas $\{|e_x\rangle\}_{x\in\mathbb{R}}$ $\left(= \{|x\rangle\}_{x\in\mathbb{R}} \right)$ fysikspråk   generaliserad bas

problem $|f\rangle = \dots ?$

$$\langle x|f\rangle = f(x) \quad \text{jfr} \quad \langle e_i|f\rangle = f_i$$

$$\langle g|f\rangle = \int_a^b g^*(x) f(x) w(x)dx = \int_a^b \langle g\underbrace{|x\rangle\langle x|}_{\text{yttre produkt}}f\rangle\, w(x)dx$$

$$g^*(x) = \langle x|g^*\rangle = \langle g|x\rangle$$

$$= \langle g| \underbrace{\left\{ \int_a^b |x\rangle\langle x| w(x) dx \right\}}_{\equiv \mathbb{1}} |f\rangle$$

$$\Rightarrow \mathbb{1} = \int_a^b |x\rangle\langle x| w(x) dx \qquad \text{"resolution of identity"}$$

$$\Rightarrow \quad |f\rangle = \mathbb{1}|f\rangle = \left(\int_a^b |x\rangle\langle x|\, w(x)\,dx\right)|f\rangle =$$

$$= \int_a^b |x\rangle\langle x|f\rangle\, w(x)\,dx = \int_a^b f(x)\,|x\rangle\, w(x)\,dx$$

Tag inre produkten med $|x'\rangle$

$$\underbrace{\langle x'|f\rangle}_{f(x')} = \int_a^b f(x)\langle x'|x\rangle\, w(x)\,dx$$

$$\Rightarrow f(x') = \int_a^b f(x)\langle x'|x\rangle\, w(x)\,dx \qquad (*)$$

antag välj en $f(x')=0$

för alla oändligt många funktioner f med egenskapen

$f(x')=0 \; \exists\, x'$ så gäller $(*)$

Inför $w(x)\langle x'|x\rangle \equiv \delta(x-x')$ Diracfunktion, exempel på distribution.

$$\int_a^b f(x)\,\delta(x-x')\,dx = \begin{cases} f(x'), & x'=[a,b] \\ 0, & \text{annars} \end{cases}$$

Dirac 1929?

Matematisk grund Laurent Schwartz 1944.

Diracs deltafunktion kan representeras som gräns av funktionssekvenser

1) $$\delta(x-x') = \lim_{\epsilon\to 0} \frac{1}{\sqrt{\pi\epsilon}}\, e^{-(x-x')^2/\epsilon}$$

2) $$\delta(x-x') = \lim_{t\to\infty} \frac{1}{\pi}\, \frac{\sin[t(x-x')]}{x-x'}$$

3) $$\delta(x-x') = \frac{d}{dx}\Theta(x-x') = \frac{d}{dx}\begin{cases} 0, & x<x' \\ 1, & x>x' \end{cases}$$

4) $\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x-x')t}\,dt$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx, \quad f(x') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'}\tilde{f}(k)\,dk$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx\right]dk$$

$$= \int_{-\infty}^{\infty}\underbrace{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}\,dk\right)}_{\delta(x-x')} f(x)\,dx$$

Hur definiera $\frac{d}{dx}\delta(x-x')$?

$$\frac{d}{dx}\int f(x)\,\delta(x-x')\,dx = \frac{d}{dx} f(x') = 0$$

ok.

$$= \int f'(x)\,\delta(x-x')\,dx + \int f(x)\frac{d}{dx}\delta(x-x')\,dx$$

$$\Rightarrow \int f(x)\frac{d}{dx}\delta(x-x')\,dx = -\int f'(x)\,\delta(x-x')\,dx$$

$$\Rightarrow \text{"}\frac{d}{dx}\delta(x-x') = -\delta(x-x')\frac{d}{dx}\text{"}$$ Differential-operatorer

Operatorer på Hilbertrum

Analogi från ändligtdimensionella vektorrum

$\boxed{A|u\rangle = |v\rangle}$ Välj en ON bas $\{|e_i\rangle\}_{i=1}^{N}$

$\langle e_i|A|u\rangle = \langle e_i|v\rangle \;/\!/\; \langle e_i|A\,\mathbb{1}|u\rangle = \langle e_i|v\rangle$

$$\sum_{j=1}^{N} \underbrace{\langle e_i|A|e_j\rangle}_{A_{ij}}\langle e_j|u\rangle = \langle e_i|v\rangle \Rightarrow \sum_{j=1}^{N} A_{ij}u_j = v_i$$

$$\Downarrow$$

Välj egenbas till A

$$\mathbb{A} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

$\lambda_i u_i = v_i$

$\mathcal{L}^2_w(a,b)$ med generaliserad bas $\{|x\rangle\}_{x\in\mathbb{R}}$

$K|u\rangle = |f\rangle$

operator i $\mathcal{L}^2_w$

$$\Rightarrow \langle x|K \underbrace{\int_a^b |y\rangle\langle y|w(y)dy}_{\mathbb{1}}|u\rangle$$

$$= \int_a^b \underbrace{\langle x|K|y\rangle}_{K(x,y)}\langle y|u\rangle w(y)dy = \langle x|f\rangle$$

"kärna" = matriselement av en operator på $\mathcal{L}^2$ i en generaliserad bas

$$\Rightarrow \int_a^b K(x,y)u(y)w(y)dy = f(x)$$

(Matematisk Fysik Räknestuga)

(1) $D\,\partial_x^2 Q + \partial_x(xQ) - [1 - i(\alpha x + \delta\omega)]Q + g_0(x) = 0$

$$g_0(x) = \frac{1}{\sqrt{2\pi D}} e^{-x^2/2D} \qquad \text{find} \quad \int_{-\infty}^{\infty} dx\, Q(x) = ?$$

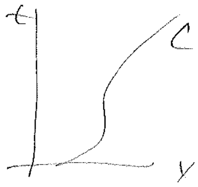
gör om till första ordningen och homogenisera

$$Q(x) = \int_0^{\infty} dt\, e^{-i\delta\omega t}\, \tilde{Q}(x,t)$$

$$(2)\ \begin{cases} \partial_t \tilde{Q} = D\partial_x^2 \tilde{Q} + \partial_x(x\tilde{Q}) - [1 - i\alpha x]\tilde{Q} \ ; \ \tilde{Q}(x, t=0) = g_0(x) \\ \tilde{Q}(x = \pm\infty, t) \to 0 \end{cases}$$

Apply fouriertransform in (2)

$$(*) \quad \partial_t \tilde{Q}(k,t) = -(Dk^2+1)\tilde{Q}(k,t) - \underbrace{(k+\alpha)}_{\frac{d}{dt}k(t)}\partial_k \tilde{Q}(k,t) \ ; \ \tilde{Q}(k,0) = e^{-Dk^2/2}$$



along C

$k(t)$? $\frac{dk}{dt} = k + \alpha \to k(t) = C_0 e^{t} - \alpha$, C_0 = Constant

$$\underline{\frac{d}{dt}\tilde{Q}(k(t),t)} = -(Dk^2+1)\tilde{Q} - \underline{\frac{dk}{dt}\tilde{Q}} \qquad (3a)$$

$$\Rightarrow \frac{d\tilde{Q}}{dt} = -(Dk^2(t)+1)\tilde{Q}(t) \qquad (3b)$$

Solving (3):

$$\tilde{Q} = \tilde{Q}(0)\exp\left[-(D\alpha^2+1)t + 2D\alpha C_0(e^t - 1) - \frac{DC_0^2}{2}(e^{2t}-1)\right]$$

$$\tilde{Q}(0) = e^{-Dk^2(0)/2}$$

$$C_0 = (k+\alpha)e^{-t} \quad \leftarrow \text{eliminating the integration constant } C_0 \text{ in order to find the solution of } (*)$$

$$k(0) = C_0 - \alpha$$

$$\widetilde{Q}(k,t) = \underbrace{e^{-\frac{D}{2}((k+\alpha)e^{-t}-\alpha)^2}}_{\widehat{Q}(0)} \cdot e^{\left[-(D\alpha^2+1)t+2D\alpha(k+1)-2D\alpha(k+\alpha)e^{-t}-\frac{D}{2}(k+\alpha)^2+\frac{D}{2}(k+\alpha)e^{-2t}\right]}$$

simplyfy

$$\widetilde{Q}(k,t) = e^{-Dk^2/2-(D\alpha^2+1)t+\alpha D(k+\alpha)(1-e^{-t})}$$

$$Q(x) \longrightarrow \widehat{Q}(x,t) \xrightarrow{FT} \widetilde{Q}(k,t)$$

$$Q(x) = \int dt\, e^{-i\delta\omega t}\, \widehat{Q}(x,t)$$

we want $\int_{-\infty}^{\infty} Q(x) = Q(k=0) = \int_0^{\infty} dt\, e^{-i\delta\omega t}\, \widetilde{Q}(k=0,t)$

$$= \int_0^{\infty} dt\, e^{-i\delta\omega t}\, e^{-t}\, e^{D\alpha[1-t-e^{-t}]}$$

$$I = \int_{-1}^{1} dt\, \sinh t\, \sin[x(t-\sin t)]$$

obtain leading order behaviour I for $x \to \infty$

$$I = 2\,\mathrm{Im} \int_0^1 dt\, \sinh t\, e^{ix(t-\sin t)}$$

$t_0 : (t-\sin t) = 0 \Rightarrow 1-\cos t_0 = 0\,;\ t_0 = 0$

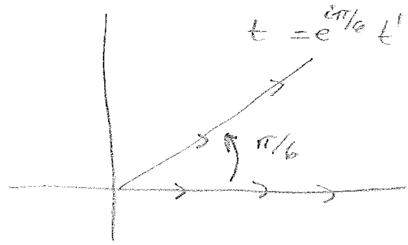
$$I \approx 2\,\mathrm{Im}\left\{ \int_0^{\epsilon^{+}} dt\, t\, e^{ixt^3/6} \right.$$

$\epsilon^{+} > 0$ libel

$t - \sin t = \frac{t^3}{6} + O(t^5)$, $\sinh t \approx t$

$$\int_0^{\infty} dt\, t\, e^{ixt^3/6}$$

$$\int_0^\infty dt\, t\, e^{ix\, t^3/6} \longrightarrow$$ change of integration contour to C_1



$$\Rightarrow \int_0^\infty dt'\, e^{\frac{\pi}{6}i}\, t'\, e^{\frac{\pi}{6}i}\, e^{-\frac{t'^3 x}{6}} =$$

$$= e^{\frac{\pi}{3}i}\int_0^\infty dt'\, t'\, e^{-t'^3 x/6} \qquad \left[\tfrac{x}{6}t'^3 = \tau\right]$$

$$4 \qquad I = 2\,\frac{\sqrt{3}}{2}\int_0^\infty dt'\, t'\, e^{-x t'^3/6} = 2^{2/3}\, 3^{1/6}\, \Gamma(2/3)\, x^{-2/3}$$

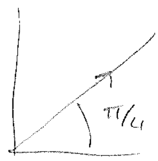
$$I = \int_0^1 \cos(x t^4)\tan t\, dt$$

Find leading order behavior for $x \to \infty^+$

$$I = \mathrm{Re}\int_0^1 e^{x t^4 i}\tan t\, dt$$

$$I \approx \mathrm{Re}\int_0^{\varepsilon>0} e^{x t^4 i}\, t\, dt \sim \int_0^\infty dt\, t\, e^{x t^4 i} = \int_0^\infty dt\, e^{x t^2 i}$$

$$= \frac{1}{2}\int_0^\infty dt\, e^{\pi i/4}\, e^{-x t^2} = \frac{e^{\pi i/4}}{4}\sqrt{\frac{\pi}{x}}$$



$$I \sim \frac{\sqrt{2}}{8}\sqrt{\frac{\pi}{x}}$$

Evaluate I for $x \gg 1$

$$I(x) = \int_0^\infty dt \cdot e^{xt} \cdot e^{-e^t}$$

$f(t) = xt - e^t$

- let's find t_{max} where f peaks

$$f'(t_{max}) = 0 \Rightarrow x - e^{t_{max}} = 0 \; ; \; t_{max} = \ln(x)$$

1) $t = \ln x \cdot s$

2) expand $f(t)$ for $t \approx t_{max}$

$$f(t_{max} + \delta) = e^{x(t_{max}+\delta)} \cdot e^{-e^{t_{max}+\delta}}$$

$$\delta \ll t_{max} \quad \sim e^{-x} x^x e^{-x\delta^2/2}$$

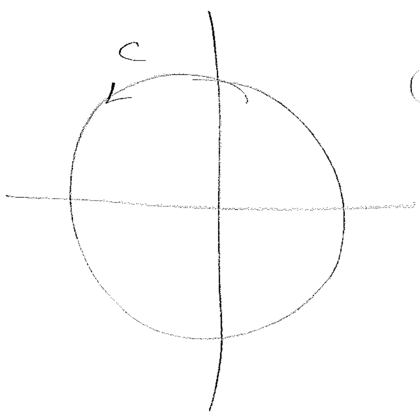
$$I(x) \sim e^{-x} x^x \int_{-\infty}^{\infty} d\delta \, e^{-x\delta^2/2} = e^{-x} x^x \sqrt{\frac{2\pi}{x}}$$

test: $x = 10$ $\quad I(x=10) = 3.62 \cdot 10^5$

$\sim I(x=10) \approx 3.59 \cdot 10^5$

Evaluate:

$$S = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^4 + a^4} \quad , \; a \in \mathbb{R}$$



$$I = \frac{1}{2\pi i} \oint_S \frac{\pi}{\sin(\pi z)} \cdot f(z) = 0$$

poles: $n \in \mathbb{Z}$ & poles from $f(z)$

$$I = \underbrace{\sum_{n \in \mathbb{Z}} f(n) \cdot (-1)^n}_{S} + \dots = 0$$

$$\frac{\pi}{(z-n)\pi \cos \pi n} = (-1)^n$$

Matematisk Fysik 19/11

Operatorer på Hilbertrum (oändligdimensionellt)

$K|u\rangle = |v\rangle$

$\langle x|K\mathbb{1}|u\rangle = \langle x|v\rangle$

$\langle x|K\int_a^b |y\rangle\langle x| w(y)\,dy|u\rangle = \langle x|v\rangle$

(1) $\int_a^b \underbrace{\langle x|K|y\rangle}_{K(x,y)} \underbrace{\langle y|u\rangle}_{u(x)} w(y)\,dy = \underbrace{\langle x|v\rangle}_{v(x)}$

"kärna" till operatorn

Välj en egenbas till K, dvs $K(x,y)$ diagonal

$K(x,y) = \begin{cases} 0, & x \neq y \\ \infty, & x = y \end{cases}$ Påstående!

T.ex. kan vi välja $K(x,y) = L(x)\delta(x-y)/w(x)$

insättning i (1) $\Rightarrow$ $L(x)v(x) = v(x)$

antag $L(x)$ differentialoperator $\Rightarrow$ diff.ekvation

Egenskaper hos K

T.ex. $K: \mathcal{H}_1 \to \mathcal{H}_2$

$\max\left\{\frac{\|Kx\|_2}{\|x\|_1}, \; |x\rangle \neq |0\rangle\right\}$ operatornorm $< \infty$

$\Rightarrow K$ begränsad

$\Rightarrow K$ linjär

K självadjungerad om $K = K^+$, dvs. $K^+(x,y) = K^*(y,x)$

K Hermitisk $\langle \varphi | K \psi \rangle = \langle K\varphi | \psi \rangle$

K Självadjungerad $\Rightarrow$ K hermitsk
(och begränsad $\Leftarrow$ om K definierad på hela Hilbertrummet)

$\langle \varphi | K \psi \rangle = \langle \varphi | K | \psi \rangle$

$= \langle \varphi | K^+ | \psi \rangle = \langle K \varphi | \psi \rangle$

K Hermitesk $\Rightarrow$

- $\langle \varphi | K \psi \rangle \in \mathbb{R}$
- K:s spektrum $\in \mathbb{R}$
- Egenvektorer till K med distinkta egenvärden är ortogonala

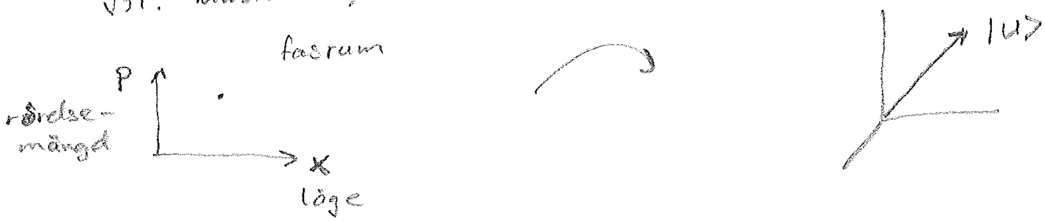
- Spektralteoremet

Varje Hermitsk begränsad operator på $\mathcal{L}^2$ har en ortogonal egenbas av funktioner med reella egenvärden.

Paul Dirac "Principals of Quantum Mechanics"
(Cambridge University Press)

Postulat I

Jfr. klassisk fysik



Postulat II

Läge, rörelsemängd
rörelsemängdmoment, spin
Hur hitta operatorn?
— Kanonisk kvantisering
Betrakta ett oändligdimensionellt Hilbertrum
Välj generaliserad bas $\{|x\rangle\}$
Där $\hat{x}|x\rangle = x|x\rangle$

$$p \to \hat{p} \equiv -i\hbar \frac{d}{dx} \qquad p = \hbar k$$

Ekvivalent med
$$\langle x|\hat{x}|x'\rangle = x\,\delta(x-x')$$
$$\langle x|\hat{p}|x'\rangle = -i\hbar\,\delta'(x-x')$$

Välj $\{|k\rangle\}$: $\hat{k}|k\rangle = k|k\rangle$

$$\langle x|\hat{k}|k\rangle = k\langle x|k\rangle$$

$w(x)=1$

$$\Rightarrow \int dx' \underbrace{\langle x|\hat{k}|x'\rangle}_{-i\delta(x-x') = -i\delta(x-x')\frac{d}{dx'}} \langle x'|k\rangle = k\langle x|k\rangle$$

$$-i\frac{d}{dx}\langle x|k\rangle = k\langle x|k\rangle \Rightarrow \hat{p} = -i\hbar\frac{d}{dx}$$

Dirac

Dynamiska variabler $\omega(x_i, p_i)$, $\lambda(x_i, p_i)$ — observabler

Poisson parenteser

$$\{\omega, \lambda\} = \sum_i \left(\frac{\partial \omega}{\partial x_i} \frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial x_i} \right)$$

kvantisering

$$[\hat{\Omega}, \hat{\Lambda}] = \hat{\Omega}\hat{\Lambda} - \hat{\Lambda}\hat{\Omega}$$

$\to \omega$ $\quad \to \lambda$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$\{x, p\} = 1$$

$\hat{x}$ och $\hat{p}$ är fundamentala

$\hat{p}$: $\hat{p}$ – "generator" för translationer i rummet

$$T(\Delta)|x\rangle = |x+\Delta\rangle$$

T – translationsoperator

$$T(\Delta)\mathbb{1}|\psi\rangle \stackrel{?}{=} T(\Delta)\underbrace{\int_{-\infty}^{\infty} dx\,|x\rangle\langle x|}_{\mathbb{1}}\psi\rangle$$

$$= \int_{-\infty}^{\infty} dx\,|x+\Delta\rangle\langle x|\psi\rangle$$

$$= \int_{-\infty}^{\infty} dx'\,|x'\rangle\langle x'-\Delta|\psi\rangle$$

slå på $\langle x|$ till vänster

$$\langle x|T(\Delta)|\psi\rangle = \int_{-\infty}^{\infty} dx' \underbrace{\langle x|x'\rangle}_{=\delta(x-x')}\langle x'-\Delta|\psi\rangle = \langle x-\Delta|\psi\rangle = \psi(x-\Delta)$$

$$T: \psi(x) \rightarrow \psi(x-\Delta)$$

välj $\Delta = \varepsilon \ll 1$

$$T(\varepsilon) = T(0) \underbrace{- \frac{i\varepsilon}{\hbar}}_{\text{konvention}} \overset{\uparrow \text{"generator"}}{G} + \mathcal{O}(\varepsilon^2)$$

$$\approx \mathbb{1} - \frac{i\varepsilon}{\hbar} G$$

① $\Rightarrow \langle x | T(\varepsilon) | \psi \rangle = \psi(x-\varepsilon)$

taylor

$$= \psi(x) - \varepsilon \frac{d\psi}{dx} + \cancel{\mathcal{O}(\varepsilon^2)}$$

② $\Rightarrow \langle x | T(\varepsilon) | \psi \rangle = \langle x | \mathbb{1} - \frac{i\varepsilon}{\hbar} G | \psi \rangle$

$$= \underbrace{\langle x | \psi \rangle}_{\psi(x)} - \frac{i\varepsilon}{\hbar} \langle x | G | \psi \rangle + \cancel{\mathcal{O}(\varepsilon^2)}$$

Identifiera ① och ②

$$\Rightarrow \langle x | G | \psi \rangle = -i\hbar \frac{d\psi}{dx} = \langle x | \hat{k} | \psi \rangle$$

$$\Rightarrow G = \hat{k}$$

dvs. $\hat{k}$ genererar translationer i rummet.

jfr. Noeters teorem

Matematisk Fysik 25/11

$\delta'(-x) = -\delta'(x)$

$\delta'(ax) = \frac{1}{a|a|}\delta'(x)$

$$\int_{-\infty}^{\infty} \delta'(\underbrace{ax}_{x'})\varphi(\underbrace{x}_{x'/a})\,dx = \begin{cases} a>0 & \frac{1}{a}\int_{-\infty}^{\infty}\delta'(x')\varphi\left(\frac{x'}{a}\right)dx' \\ a<0 & \frac{1}{a}\int_{\infty}^{-\infty}\delta'(x')\varphi\left(\frac{x'}{a'}\right)dx' = -\frac{1}{a}\int_{-\infty}^{\infty}\delta'(x')\varphi\left(\frac{x'}{a}\right)dx' \end{cases}$$

$$\Big\} = \frac{1}{|a|}\int_{-\infty}^{\infty}\delta'(x')\varphi\left(\frac{x'}{a}\right)dx = -\frac{1}{|a|}\int_{-\infty}^{\infty}\delta(x')\underbrace{\frac{d}{dx'}\varphi\left(\frac{x'}{a}\right)}dx$$

$$= -\frac{1}{|a|}\,\frac{1}{a}\,\varphi'(0) = -\frac{1}{|a|a}\int_{-\infty}^{\infty}\delta(x)\varphi(x)\,dx$$

---

postulat III

$\Omega|\omega_i\rangle = \omega_i|\omega_i\rangle$

möjliga mätdata

operator som representerar vår fysikaliska storhet (rörelsemängd, spinn...)

Sannolikheten att mäta $\omega_i = P(\omega_i) = |\langle\omega_i|\psi\rangle|^2$

systemets tillstånd

$$|\psi\rangle \longrightarrow \boxed{}^{\,\Omega} \longrightarrow |\omega_i\rangle$$

Operatiot: 4-stegsprogram

1. Konstruera sin Ω (självadjungerad operator)

$$\Omega = \omega(x \rightarrow \hat{x},\ p \rightarrow \hat{p} = -i\hbar \frac{d}{dx})$$ [kanonisk kvantisering]

i basen $\{|x\rangle : \hat{x}|x\rangle = x|x\rangle$

2. Hitta $\{|\omega_i\rangle$ och $\omega_i\}$ till Ω

3. Utveckla $|\Psi\rangle$ i Ω:s egenbas $\{|\omega_i\rangle\}$

$$|\Psi\rangle = \sum_i \alpha_i |\omega_i\rangle$$

ω i klassisk dynamisk storhet:

$\omega = \omega(x, p)$

$D=3$: $\omega = (x_1, x_2, x_3, p_1, p_2, p_3)$

4. Sannolikheten att mäta ω_i

$$P(\omega_i) = |\langle \omega_i | \Psi \rangle|^2 =$$

$$= \langle \Psi | \underbrace{\omega_i \rangle \langle \omega_i}_{\text{Projektionsoperator } P_{\omega_i}} | \Psi \rangle$$

$$= \langle \Psi | P_{\omega_i} | \Psi \rangle$$

5. $|\Psi\rangle \xrightarrow{M} |\omega_i\rangle$ Vågfunktionskollaps

Några implikationer av III:e postulatet

(A) III $\rightarrow$ Kvantmekanik är en probabilistisk teori

Men $g = 2.0023193043\underbrace{[8620\ldots]}_{\text{experimentell osäkerhet}}$

$= 2\frac{\mu_s}{\mu_B}\hbar$

$|\psi\rangle = |\omega_i\rangle \rightarrow \underset{\Omega}{\Box} \rightarrow |\omega_i\rangle$

(B) Sannolikheter kräver normering

$$P(\omega_i) = \frac{|\langle\omega_i|\psi\rangle|^2}{\sum_i |\langle\omega_i|\psi\rangle|^2} = \frac{|\langle\omega_i|\psi\rangle|^2}{\langle\psi|\psi\rangle}$$

$|\psi\rangle = \sum_i \alpha_i |\psi\rangle$

Låt oss kolla den generaliserade baserna $\{|x\rangle\}$, $\{|k\rangle\}$ $(\hat{p} = \hbar\hat{k})$

$$\hat{k}|k\rangle = k|k \implies \langle x|\hat{k}|k\rangle = k\underbrace{\langle x|k\rangle}_{k(x) = \psi_k(x)}$$

$$\langle x|\hat{k}\,\mathbb{1}|k\rangle = \psi_k(x)$$

$$= \int dx' \underbrace{\langle x|\hat{k}|x'\rangle}_{-i\delta'(x-x')}\underbrace{\langle x'|k\rangle}_{\psi_k(x')} = -i\int dx'\,\delta(x-x')\frac{d}{dx'} = -i\frac{d}{dx}\psi_k(x)$$

$$\implies -i\frac{d}{dx}\psi_k(x) = k\,\psi_k(x)$$

$$\implies \psi_k(x) = Ae^{ikx} = \frac{1}{\sqrt{2\pi}}e^{ikx}$$ Planväg

$\frac{1}{\sqrt{2\pi}}$

Antag att vi vill mäta läget x hos en partikel

i tillståndet $|k\rangle$, dvs. med vågfunktionen $\frac{1}{\sqrt{2\pi}} e^{ikx}$

4. stegsprogrammet: sannolikheten att hitta partikeln i x

$$\cancel{|\langle x|k\rangle|^2 = \frac{\left|\frac{1}{\sqrt{2\pi}} e^{ikx}\right|^2}{\langle k|k\rangle}}$$

normera

$$P(\omega_i) = \frac{|\langle \omega_i|\psi\rangle|^2}{\langle\psi|\psi\rangle}$$

$$\langle k|k'\rangle = \delta(k-k')$$

normerbarhet: $\langle\psi|\psi\rangle$ ändligt

$$= \frac{\frac{1}{2\pi}}{\delta(0)} \to 0$$

Försiktighet anbefalles ty

$\{|x\rangle\}, \{|k\rangle\}$ __ej__ normerbara

$$\to |\langle x|k\rangle|^2 = \frac{\left|\frac{1}{\sqrt{2\pi}} e^{ikx}\right|^2}{\langle k|k\rangle}$$

c) Hur göra om Ω inte har någon klassisk motsvarighet

t.ex. __spinn__ hos partiklar? Ny fysik

Postulat IV

Tidsutveckling av Schrödingerekvationen

$$i\hbar \frac{d}{dt}|\Psi\rangle = \hat{H}|\Psi\rangle$$

→ antag H tidsoberoende

$$\Rightarrow |\Psi(t)\rangle = \underbrace{e^{-i\hat{H}t/\hbar}}_{U(t)}|\Psi(0)\rangle$$

$U(t)$ - tidsutvecklingsoperator

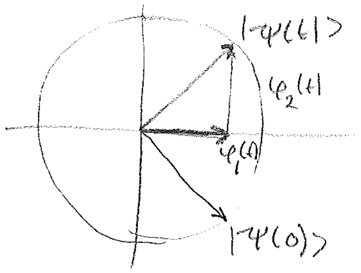
$H = H^{+} \Rightarrow U^{-1}(t) = U^{+}(t)$ unitär

$U(t) : |\Psi(0)\rangle \rightarrow |\Psi(t)\rangle$

$$\langle\Psi(t)|\Psi(t)\rangle = \langle\Psi(0)|\underbrace{U^{+}(t)U(t)}_{\mathbb{1}}|\Psi(0)\rangle = \langle\Psi(0)|\Psi(0)\rangle$$

Tidsutveckling är rotation i Hilbertrummet

$$|\Psi(0)\rangle \xrightarrow{U(t)} |\Psi(t)\rangle \xrightarrow{\Omega\text{-mätning}} |\varphi_1\rangle$$ ej unitär



φ_1, φ_2 möjliga egentillstånd till Ω

Mätproblemet

Mätningar är ej unitära!

Möjliga vägar ut

1) Bohrs köpenhamnstolkning:
Skarp gräns mellan observatör (klassisk) och mätobjekt (kvantmekanisk)

2) Hugh Everett: "Many world interpretation"

3) Dekoherensteori (Zurek)

Kvantsammanflätning = tillstånd som ej kan skrivas som en produkt av ett tillstånd för S_1 och ett tillstånd för S_2

EPR 1935:

spinn-½ $|\tfrac{1}{2}\rangle$ $|-\tfrac{1}{2}\rangle$

Spinn-1

$|1\rangle \; |0\rangle \; |-1\rangle$

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

$$\alpha_1 |1\rangle|\tfrac{1}{2}\rangle + \alpha_2 |1\rangle|-\tfrac{1}{2}\rangle + \alpha_3 |0\rangle|\tfrac{1}{2}\rangle \; \ldots \qquad 6 \text{ möjligheter}$$

t.ex. $|\Psi\rangle = \alpha_1 |1\rangle|\tfrac{1}{2}\rangle + \alpha_6 |-1\rangle|-\tfrac{1}{2}\rangle$

$$= \alpha_1 |1\rangle \otimes |\tfrac{1}{2}\rangle + \alpha_6 |-1\rangle \otimes |-\tfrac{1}{2}\rangle$$

$$= \alpha_1 |1, \tfrac{1}{2}\rangle + \alpha_6 |-1, -\tfrac{1}{2}\rangle$$

$$|p\rangle \longleftarrow \pi\text{-meson} \longrightarrow |e\rangle$$

$$|\Psi^{(s)}_{ep}\rangle = \frac{1}{\sqrt{2\pi}}\left(|\uparrow\rangle_e |\downarrow\rangle_p - |\downarrow\rangle_e |\uparrow\rangle_p\right)$$

(s) – spinn

Icke-lokalitet ! — kvantinformationsteori

Greensfunktioner

$L_x[u] = f(x)$ $\qquad L = \mathcal{L}(\Omega)$

linjär differentialoperator t.ex $\alpha(x)\frac{d^2}{dx^2} + \beta(x)\frac{d}{dx} + q(x) = L_x$

Dirac $\rightarrow$ $L|u\rangle = |f\rangle$

Antag $\exists G : G = L^{-1}$ Greensoperator

$\langle x|\underbrace{GL}_{\mathbb{1}}|u\rangle = \langle x|G\mathbb{1}|f\rangle$

$$\langle x|u\rangle = \int_\Omega dy \underbrace{\langle x|G|y\rangle}_{G(x,y)}\underbrace{\langle y|f\rangle}_{f(y)}$$

$$\Rightarrow u(x) = \int_\Omega G(x,y) f(y)\, dy \qquad (\star)$$

Greensfunktion

Hur hitta $G(x,y)$?

Undersök $\langle x|\underbrace{L\,G}_{=1}|y\rangle = \langle x|y\rangle = \delta(x-y)$

$$\langle x|L\mathbb{1}G|y\rangle = \int dx' \underbrace{\langle x|L|x'\rangle}_{L(x,x')}\underbrace{\langle x'|G|y\rangle}_{G(x',y)} = \delta(x-y)$$

antag L-lokal dvs $L(x,x') = L_x\delta(x-x')$

$\Rightarrow L_x G(x,y) = \delta(x-y)$

välj $y=0 \Rightarrow L_x G(x) = \delta(x) \qquad (\star\star)$

Lös $G(x)$ från $(\star\star)$ sätt in i $(\star) \Rightarrow u(x)$

Centralt i fysik

$$L_x[u] = f(x)$$

input/"Störning"

output/respons

kodar teorin/modellen för systemet

Dämpad tvungen harmonisk oscillator

$$L_t = \frac{d^2}{dt^2} + \gamma \frac{d}{dt} + \omega_0^2$$

$f(t) = F(t)$ - kraften

$u = x$ - läget

Greensfunktioner beror på randvillkor

Ex. $L_x = \frac{d}{dx} \Rightarrow \frac{du}{dx} = f(x) \Rightarrow \frac{d}{dx} G(x,y) = \delta(x-y)$

$$\Rightarrow G(x,y) = \underbrace{\int \delta(x-y)\,dx}_{\text{primitiv}} = \Theta(x-y) + \alpha(y)$$

integrationskonstant

Randvillkor $\quad u(x=0) = 0 \quad , x \in [0, \infty]$

$$u(x) = \int_0^\infty G(x,y) f(y)\,dy = \int_0^\infty \Theta(x-y) f(y)\,dy + \int_0^\infty \alpha(y) f(y)\,dy$$

$$= \int_0^x f(y)\,dy + \underbrace{\int_0^\infty \alpha(y) f(y)\,dy}_{=0}$$

ty $u(0) = 0$

Standardteknik för beräkning av
Greensfunktioner i fysiken:
Fouriertransformer

Ex. Driven harmonisk oscillator

$$\frac{d^2}{dt^2} x(t) + \omega_0^2 x(t) = F(t)$$

$$\Rightarrow \quad G''(t,t') + \omega_0^2 G(t,t') = \delta(t-t')$$

$= \frac{d^2G}{dt^2}$

antag translationsinvarians $\Rightarrow G(t,t') = G(t-t') \equiv G(t)$
sätt $t'=0$

$$\Rightarrow \quad G''(t) + \omega_0^2 G(t) = \delta(t)$$

F.T

$$\Rightarrow \quad G(t) = \int \tilde{G}(\nu) e^{2\pi i \nu t} d\nu$$

OBS faktorer $\frac{1}{\sqrt{2\pi}}$ i F.T

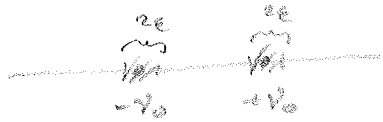
$$\boxed{\Rightarrow \quad (-4\pi^2\nu^2 + \omega_0^2)\, \tilde{G}(\nu) = 1}$$

$$\Rightarrow \quad \tilde{G}(\nu) = \frac{1}{\omega_0^2 - 4\pi^2\nu^2}$$

$$\Rightarrow \quad G(t) = \int_{-\infty}^{\infty} \frac{1}{\omega_0^2 - 4\pi^2\nu^2} \cdot e^{2\pi i \nu t} d\nu$$

Poler vid $\nu = \pm \nu_0 = \pm \omega_0 / 2\pi$
vi måste definiera integralen!

I) Principalvärde



$$P\int_{-\infty}^{\infty} \ldots = \lim_{\varepsilon \to 0} \left(\int_{-\infty}^{-\nu_0-\varepsilon} + \int_{-\nu_0+\varepsilon}^{\nu_0-\varepsilon} + \int_{\nu_0+\varepsilon}^{\infty} \right)$$

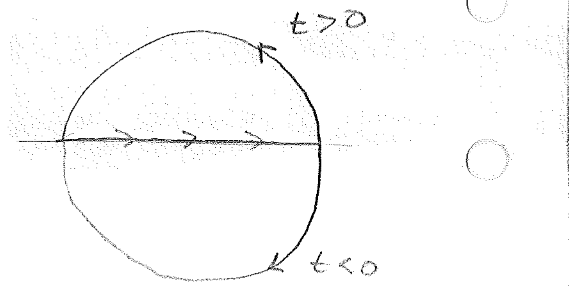
II 

III 

Kom ihåg: $P\int = \mathrm{I} = \frac{1}{2}(\mathrm{II} + \mathrm{III})$

Vilken av I, II, III ska vi välja

Låt oss först undersöka III



retarderad

$$G(t) \longrightarrow G^{(r)}(t) =$$

$(\pm \nu_0 \rightarrow \pm \nu_0 + i\varepsilon)$

$$= \lim_{\varepsilon \to 0} \left(-\frac{1}{4\pi^2} \int \frac{e^{2\pi i \nu t}}{[\nu + (\nu_0 - i\varepsilon)][\nu - (\nu_0 + i\varepsilon)]} \, d\nu \right)$$

$t>0$

$=$ Residysatsen

$$\lim_{\varepsilon \to 0} \left(-\frac{1}{4\pi^2} \cdot 2\pi i \left(\operatorname{Res} f(\nu_0 + i\varepsilon) + \operatorname{Res} f(-\nu_0 + i\varepsilon) \right) \right)$$

$$= \lim_{\varepsilon \to 0} \left(-\frac{2\pi i}{4\pi^2} \left(\frac{e^{2\pi i(\nu_0 + i\varepsilon)t}}{(\nu_0 + i\varepsilon) + (\nu_0 - i\varepsilon)} + \frac{e^{2\pi i(-\nu_0 + i\varepsilon)t}}{(-\nu_0 + i\varepsilon) + (-\nu_0 - i\varepsilon)} \right) \right)$$

$$= \frac{1}{\omega_0} \sin(\omega_0 t)\, \Theta(t)$$

Stoppa in stegfunktionen för att täcka både $t>0$ och $t<0$

Samma räkning för II:

avancerad

$$G(t) \longrightarrow G^{(a)}(t) = -\frac{1}{\omega_0} \sin(\omega_0 t)\, \Theta(-t)$$

$(\pm \nu_0 \rightarrow \pm \nu_0 - i\varepsilon)$

Principalvärdet $= \frac{1}{2}(\mathrm{II}+\mathrm{III})$

symmetrisk

$$G(t) = G^{(s)}(t) = \frac{1}{2\omega_0}\sin(\omega_0|t|)$$

$$x(t) = \int_0^\infty G(t-s)F(s)\,ds$$

$x(0)=x'(0)=0$

$F(s)=0 \quad , s<0$

$G^{(s)} \qquad G^{(a)} \qquad G^{(r)}$

$G^{(a)} \Rightarrow$

$$x(t) = \int_0^\infty G^{(a)}(t-s)F(s)\,ds = -\frac{1}{\omega_0}\int_0^\infty \sin(\omega_0(t-s))\underbrace{\Theta(s-t)}_{s>t}F(s)\,ds$$

$$= -\frac{1}{\omega_0}\int_t^\infty \sin\omega_0(t-s)F(s)\,ds$$

icke causalt

Funkar ej

$\Rightarrow G^{(s)}$ funkar ej

läget av partikeln beror på vad som händer efteråt. Verkan före orsak.

$G^{(r)} \Rightarrow$

[Gör räkningarna själv]

<u>Svar</u> Ja!

Varför inte välja Laplace transform?

$$\tilde{G}(p) = \int_0^\infty G(t)\, e^{-pt}\, dt$$

$$G(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{G}(p)\, e^{pt}\, dp \qquad c = \mathrm{Re}\, p$$

$$\Rightarrow \quad (p^2 + \omega_0^2)\, \tilde{G}(p) = 1$$

$$G(p) = \frac{1}{\omega_0} \sin(\omega_0 t)\, \Theta(t)$$

Detta är den retarderade Greensfunktionen

Så varför F.T?

① F.T. mer allmän

② Ibland vill vi kunna räkna på icke-kausala G.F.
dvs. $G^{(a)}$ faktiskt också användbara

Feynman diagram

G.F. teknik användbart(!) också för linjära P.D.E. $(D>1)$
men lite besvärligare p.g.a. randvillkoren

Men några enkla typfall: • Laplace / D'Alemberts operator
• Schrödingeroperatorn

D'Alembert operator

$$\square \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

Ex. Maxwells ekvationer $\Rightarrow$ Lorentz gauge $\nabla\cdot\vec{A} - \nabla\varphi = 0$

$$\left.\begin{aligned} -\square\varphi &= \frac{\rho}{\varepsilon_0} \\ -\square A &= \mu_0 j \end{aligned}\right\} \quad -\square\begin{pmatrix}\varphi \\ \vec{A}\end{pmatrix} = \begin{pmatrix}\rho/\varepsilon_0 \\ \mu_0 j\end{pmatrix} \qquad (*)$$

4-vektornotation

$$A^\mu \equiv (\varphi, \vec{A})$$

$$j^\mu \equiv (\rho, \vec{j}) \qquad \Rightarrow \partial_\mu \partial^\mu A^\mu = j^\mu$$

$$\partial^\mu \equiv \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$$

$$\partial_\mu \equiv \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right)$$

Greensfunktionen för Maxwellteori?

$$-\square G(t,\vec{r}) = \delta(t)\underbrace{\delta(\vec{r})}_{\delta(x)\delta(y)\delta(z)\,,\ \vec{r}=(x,y,z)} \quad (✡)$$ Translationsinvarians både i rum och tid

lösning

$$✡ \Rightarrow A^\mu(t,\vec{r}) = \int_{R^3}\int_{-\infty}^{\infty} G(t-t', r-r')\, j^\mu(t',\vec{r}')\, dt'\, d^3\vec{r} \qquad (\square)$$

$(\Box)$ $\varphi(\vec{r},t) = \iint G \ldots \rho(\vec{r},t')dt'd^3\vec{r}\,'$

potential Laddningsfördelning

$$\vec{A}(\vec{r},t) = \iint G \quad \vec{j}(\vec{r},t')dt'd^3\vec{r}$$

Hur hitta lösningen till (∗) ? Svar F.T. !

$$\tilde{G}(\vec{k},\omega) = \int_{\mathbb{R}^3} \int_{-\infty}^{\infty} G(\vec{r},t)\, e^{-i(\vec{k}\cdot\vec{r}-\omega\cdot t)} dt\, d^3r$$

konvention

Tillbaka till
diffekv. definierad
av: $-\Box G(\vec{r},t) = \delta(\vec{r})\delta(t)$

$\Longrightarrow$

multiplicera med $e^{-i(\vec{k}\cdot\vec{r}-\omega\cdot t)}$ integrera
partiell integration på vänsterledet

$$\tilde{G}(\vec{k},\omega) = \frac{c^2}{c^2k^2-\omega^2} \quad \Longrightarrow \quad G(\vec{r},t) = \frac{1}{(2\pi)^4}\int_{\mathbb{R}^3}\int_{-\infty}^{\infty} \frac{c^2}{c^2k^2-\omega^2}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} d^3k\, d\omega$$

poler i $\omega = \pm\omega_0 = \pm ck$!

Matematisk fysik 30/11

Green's funktioner (forts.)

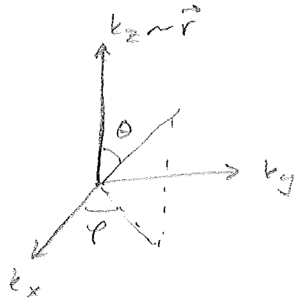
Ex PDE: D'Alemberts operator

$$-\Box u(\vec r,t) = f(\vec r,t)$$

$$\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

$$-\Box G(\vec r,t) = \delta(\vec r,t)$$

$$G(\vec r,t) = \frac{1}{(2\pi)^4}\int_{\vec k}\int_{\omega} \frac{c^2}{c^2k^2-\omega^2} e^{i(\vec k\cdot\vec r-\omega t)}\, d^3k\, d\omega$$



Poler: $\omega = \pm\omega_0 = \pm ck$ Vad göra?

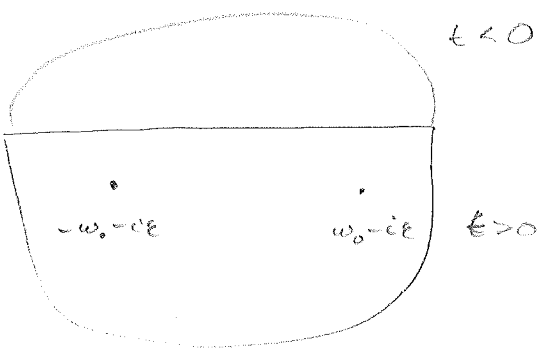
Först i sfäriska koordinater

$$d^3k = k^2 \sin\theta\, d\theta\, d\varphi\, dk = -k^2 d\varphi\, dk\, d(\cos\theta)$$

$$G(\vec r,t) = -\frac{1}{(2\pi)^4}\int_{k=0}^{\infty}\int_{\cos\theta=1}^{-1}\int_{\varphi=0}^{2\pi}\int_{\omega=-\infty}^{\infty} \frac{c^2}{c^2k^2-\omega^2} e^{i(kr\cos\theta-\omega t)} k^2 dk\, d\varphi\, d(\cos\theta)\, d\omega$$

$$= \frac{c^2}{4\pi^3 r}\int_0^\infty \sin(kr)\underbrace{\left(\int_{-\infty}^{\infty}\frac{1}{c^2k^2-\omega^2}e^{-i\omega t}\,d\omega\right)}_{I} dk \qquad (1)$$

(utklar!)



$\pm\omega_0 \to \pm\omega_0 - i\varepsilon$

$$G(\bar r,t) \longrightarrow G^{(r)}(\bar r,t)$$

$$\underset{t>0}{I} = \lim_{\varepsilon\to 0}\left(-2\pi i \operatorname{Res}\left[f,\ \omega=\pm\omega_0 - i\varepsilon\right]\right)$$

$$= -\frac{i\pi}{ck}\left(e^{ickt} - e^{-ickt}\right) \qquad (2)$$

$$I_{t<0} = 0$$

(1)&(2) $$G^{(r)}(\bar r,t) = \frac{c^2}{4\pi^2 r}\int_0^\infty \sin(kr)\left(-\frac{i\pi}{ck}\left(e^{ickt} - e^{-ickt}\right)\right)dk$$

$$= \frac{c}{8\pi^2 r}\int_0^\infty \left(e^{ik(r-ct)} + e^{-ik(r-ct)} - e^{ik(r+ct)} - e^{-ik(r+ct)}\right)dk$$

$$= \frac{c}{8\pi^2 r}\int_{-\infty}^\infty \left(e^{ik(r-ct)} - e^{ik(r+ct)}\right)dk$$

Fouriers integral
$$\frac{1}{2\pi}\int_{-\infty}^\infty e^{ikr}\,dk = \delta(r)$$

$$= \frac{c}{4\pi r}\left(\delta(r-ct) - \cancel{\delta(r+ct)}\right)$$

ty $r, c, t > 0$

$$= \frac{c}{4\pi r}\,\delta(r-ct) \qquad \forall\, t>0 \Rightarrow$$

$$= \frac{c}{4\pi r}\,\delta(r-ct)\,\Theta(t)$$

Jfr tidsoberoende fall

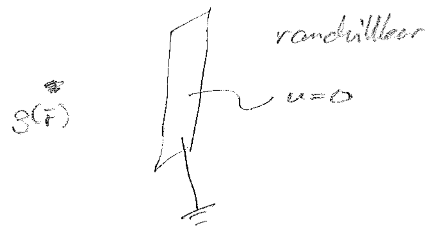
$$-\Box G(\vec r,t) = \delta(\vec r)\cancel{\delta(t)} \quad \longrightarrow \quad -\nabla^2 G(\vec r) = \delta(\vec r)$$

$$\Box = \nabla^2 - \cancel{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}} \qquad G(\vec r) = \frac{\text{konst}}{4\pi r}$$

dvs. potentialen från en punktladdning

Randvillkor?!

$$-\nabla^2 u(\vec r,t) = g(\vec r)$$



i princip enkelt

Definiera en "ny" effektiv Greensfunktion
som inkluderar randvillkor

$$\bar G(\vec r) = G(\vec r) + F(\vec r) \qquad \nabla^2 F(\vec r) = 0$$

$$-\nabla^2 G(\vec r) = \delta(\vec r)$$

Dvs F är lösningen till motsvarande homogena PDE

Välj F så att $\bar G(r)$ uppfyller randvillkoren

Integralekvationer (Inbyggda randvillkor)

I Fredholmekvation av första typen

$$f(x) = \int_a^b K(x,t)\,\varphi(t)\,dt \qquad \text{sök } \varphi(t)$$

känd

II Fredholmekvation av andra typen

$$\varphi(x) = f(x) + \lambda \int K(x,t)\,\varphi(t)\,dt$$

III Volterra av första typen

$$f(x) = \int_a^x K(x,t)\,\varphi(t)\,dt$$

IV Volterra av andra typen

$$\varphi(x) = f(x) + \int_a^x K(x,t)\,\varphi(t)\,dt$$

- Randvillkoren inbyggda i integralekv.
- Ibland, integralekvationer enklare än diffekv.

Vi ska titta på två generella metoder

Metod 1 för separabla kärnor

$$K(x,t) = \sum_{j=1}^{n} M_j(x) N_j(t)$$

Ex $K(x,t)$ polynom, $K(x,t) = \cos(x-t) = \cos x \cos t + \sin x \sin t$

Tillämpning på II:

$$\varphi(x) = f(x) + \lambda \sum M_j(x) \underbrace{\int_a^b N_j(t) \varphi(t)\,dt}_{c_j} \qquad (*)$$

$$\Rightarrow \varphi(x) = f(x) + \lambda \sum_{j=1}^{n} c_j M_j(x) \qquad (**)$$

Multiplicera $(**)$ med $N_i(x)$. Integrera över $[a,b]$

$$\underbrace{\int_a^b N_i(x)\varphi(x)\,dx}_{c_i} = \int_a^b N_i(x) f(x)\,dx + \lambda \sum_{j=1}^{n} c_j \underbrace{\int_a^b N_i(x) M_j(x)\,dx}_{a_{ij}}$$

$$\Rightarrow c_i = b_i + \lambda \sum_{j=1}^{n} a_{ij} c_j \qquad *$$

$*$ komponentform av den algebraiska vektorekvationen

$$c = b + \lambda A c$$

$$\Rightarrow (1-\lambda A)c = b$$

antag $f(x) \equiv 0 \Rightarrow b \equiv 0$ dvs. homogen Fredholm av andra typen

$$(1-\lambda A)c = 0 \xRightarrow{\text{icke triviala lösningar}} \det(1-\lambda A) = 0 \quad \text{etc.}$$

för exempel Arfken 16.3.2

Metod 2 Ej separabel kärna

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,t)\varphi(t)\,dt$$

Gissa en lösning

t.ex. om λ litet $\varphi(x) \approx f(x) \equiv \varphi_0(x)$

$$\Rightarrow \varphi_1(x) = f(x) + \lambda \int_a^b K(x,t) f(t)\,dt$$

iterera

$$\varphi_2(x) = f(x) + \lambda \int_a^b K(x,t)\left(f(t) + \lambda \int K(t,t') f(t')\,dt'\right)dt$$

osv. $\Rightarrow$

$$\varphi_n(x) = \sum_{i=0}^{n} \lambda^i u_i(x)$$

$$u_0(x) = f(x)$$

$$u_n(x) = \int \dots \int K(x,t_1)K(t_1,t_2)\dots K(t,t_n) f(t_n)\,dt_1 dt_2 \dots dt_n$$

$$\sum_{i=0}^{\infty} \lambda^i u_i(x) \quad \text{Neumann Serie}$$

Om $\lim_{n\to\infty} \varphi_n(x)$ konvergerar så är detta lösningen.

Schmidt-Hilbert Teori OK. för symmetriska kärnor $K(x,t) = K(t,x)$

Symmetrisering ibland möjligt

symmetrisk faktor.

$$\varphi(x) = f(x) + \lambda \int_a^b \underbrace{K(x,t)\mu(t)}_{K(x,t)} \varphi(t)\,dt$$

multiplicera med $\sqrt{\mu(x)}$

$$\underbrace{\sqrt{\mu(x)}\,\varphi(x)}_{\psi(x)} = \sqrt{\mu(x)}\, f(x) + \lambda \int_a^b K(x,t)\sqrt{\mu(x)\,\mu(t)}\,\psi(t)\,dt$$

Homogen 2:a typens Fredholm

$$\varphi(x) = \lambda \int_a^b K(x,t)\,\varphi(t)\,dt$$

symmetrisk, reell, kontinuerlig

Courant-Hilbert

$\Longrightarrow$ Egenfunktionerna $\varphi_n(x)$ bildar en "komplett mängd"

definition
komplett mängd: $g(x) = \int_a^b K(x,t)\,h(t)\,dt \Rightarrow g(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x)$

styckvis kontinuerlig

Lösningen till det inhomogena problemet.

$$\varphi(x) = f(x) + \lambda \sum_{i=1}^{\infty} \left\{ \frac{\int_a^b f(t)\varphi_i(t)\,dt}{\lambda_i - \lambda} \right\} \varphi_i(x)$$

För detaljer se A&W 16

Ortogonala polynom (bas i $\mathcal{L}^2_w$)

Sturm-Liouville problem

Rodriguez formel

Teorem

"frö"funktion"

Klassiska ortogonala polynom $F_n(x) = \frac{1}{K_n}\frac{1}{w(x)}\frac{d^n}{dx^n}\left(w(x)\, s^n(x)\right)$, $n = 0, 1, 2 \ldots$

normerings-konstant

- $F_1(x)$ 1:a gradens polynom
- $S(x)$ polynom av graden ≤ 2
- $w(x) > 0 \quad \forall x \in [a,b]$
- $w(a)S(a) = w(b)s(b) = 0$

$\Longrightarrow$ Rodriguez

$F_n(x)$ polynom av grad n
och ortogonalt mot varje annat
polynom $P_k(x)$ av grad $k<n$

dvs.

$$\int_a^b P_k(x) F_n(x)\, dx = 0 \quad k<n$$

P.0 $\quad \left[c\,\partial_t - \mu \partial_x^2\right] G(x,t) = \delta(x,t) = \delta(x)\cdot\delta(t)$

$$G(x,t) \xrightarrow{FT} \tilde{G}(k,\omega) = \int dx\, e^{-ikx} \int dt\, e^{-i\omega t}\, G(x,t)$$

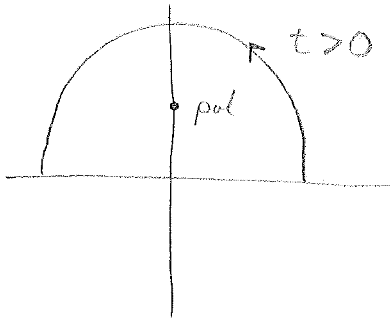
$$\left[ci\omega + \mu k^2\right] \tilde{G}(k,\omega) = 1$$

$$\tilde{G} = \frac{1}{i\omega c + \mu k^2}$$

$$G(x,t) = \int \frac{dk}{2\pi} e^{ikx} \int \frac{d\omega}{2\pi} e^{i\omega t} \underbrace{\tilde{G}(k,\omega)}_{= \frac{1}{i\omega c + \mu k^2}}$$

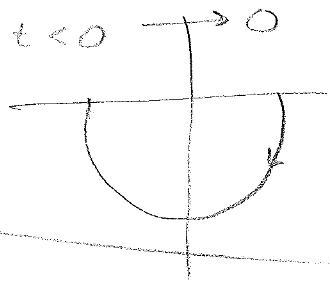
$$\omega_p = \frac{\mu k^2}{c} i$$

$t>0$ $\quad \mu, c > 0$



$$G(x,t) = \int \frac{dk}{2\pi} e^{ikx} \cdot \frac{2\pi i}{2\pi} \frac{e^{i\left(\frac{\mu k^2}{c}i\right)t}}{ic}$$

$$= \int \frac{dk}{2\pi} e^{ikx - \frac{\mu k^2}{c} t} = \frac{1}{2\pi} \frac{1}{\sqrt{4\pi c t}} e^{\frac{-x^2 c}{4\mu t}}$$



$G(x,t) = 0 \quad t<0$

Ny

P.1 $\quad \ddot{y} + \omega^2 y = g(x) \quad , \quad x \in [0, 2\pi]$

$$y(0) = y(2\pi)$$

$$y(x) = \int_0^{2\pi} dx'\, G(x,x')\, g(x')$$

Solve

$$\frac{d^2 G}{dx^2} + \omega^2 G(x,x') = \delta(x-x')$$

1) $0 \le x < x'$: $\quad \frac{d^2 G_<}{dx^2} + \omega^2 G_< = 0$

2) $x' < x \le 2\pi$: $\quad \frac{d^2 G_>}{dx^2} + \omega^2 G_> = 0$

$\Rightarrow \quad G_< = A_<(x')\cos\omega x + B_<(x')\sin\omega x$

$\quad G_> = A_>(x')\cos\omega x + B_>(x')\sin\omega x$

$$\underbrace{\int_{x-\varepsilon}^{x+\varepsilon} \frac{d^2G}{dx^2}}_{\neq 0} + \underbrace{\int \omega^2 G(x,x')}_{=0} = \int \delta(x-x')$$

$=0$ ty G kontinuerlig

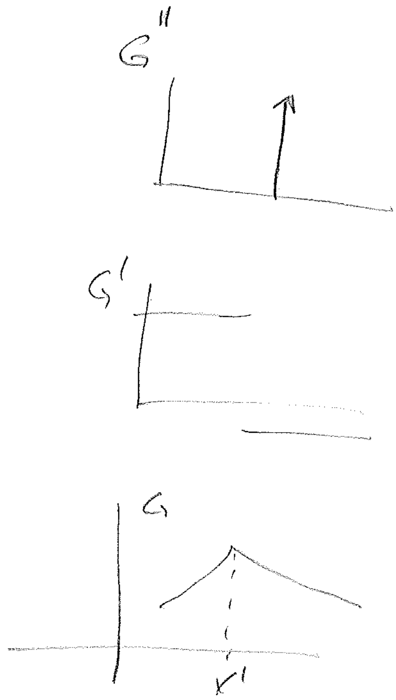
$\neq 0$ ty G' ej kont.

Kontinuitets kriterium

$G_<(x,x') = G(x,x')$

$x \to x'^- \quad = \quad x \to x'^+$

$$A_< C + B_< S = A_> C + B_> S \qquad (1)$$



diskontinuitetskriterium för G @ $x = x'$

$$\left.\frac{dG_>}{dx}\right|_{x=x'^+} - \left.\frac{dG_<}{dx}\right|_{x=x'^-} = 1$$

$$\Rightarrow \quad -A_> S + B_> C + A_< S - B_< C = \frac{1}{\omega} \qquad (2)$$

More conditions on the coefficients from BC:s

$$G(x=0,x') = G(2\pi,x') \qquad (3)$$

$$G(x=0,x') = G'(2\pi,x') \qquad (4)$$

$$B_< = \frac{\sin(\pi\omega + \omega x')}{2\omega \sin(\pi\omega)} \qquad B_> = \frac{\sin(\omega x' - \omega\pi)}{2\omega\sin(\omega\pi)}$$

$$A_> = \frac{\cos(\omega\pi + \omega x')}{2\omega\sin(\omega\pi)} \qquad A_< = \frac{\cos(\pi\omega - \omega x')}{2\omega\sin(\pi\omega)}$$

$$G(x,x') = \frac{\cos(\pi\omega - \omega|x-x'|)}{2\omega\sin(\pi\omega)}$$

B.3 $\quad EI\frac{\partial^4 u}{\partial x^4} = -q(x) \qquad x\in(0,L)$

$$u(0) = u(L) = 0$$
$$u'(0) = u'(L) = 0$$

$$EI\,\partial_x^4 G(x,x') = \delta(x-x')$$

Conditions of G at $x=x'$

$$@\,x=x' \begin{cases} G & - \text{kontinuerlig} \\ G' & - \text{cont} \\ G'' & - \text{cont} \\ G''' & - \text{discont} \end{cases}$$

$$EI\,G(x,x') = \frac{|x-x'|^3}{12} + C_1(x')(x-x')^3 + C_2(x')(x-x')^2 + C_3(x-x') + C_4(x')$$

Using Bcs: $G(x=0,x')=0$

$$\frac{x'^3}{12}+c_1x'^3+c_2x'^2-c_3x'+c_4=0$$

$G(x=L,x')=0$

$$\frac{(L-x')^3}{12}+c_1(L-x')^3+c_2(L-x')^2+c_3(L-x')+c_4=0$$

$G'(x=0,x')=0$

$$-\frac{x'^2}{4}+3c_1x'^2-2c_2x'+c_3=0$$

$$\partial_x G(x,x')\Big|_{x'=L} : \frac{(L-x')^2}{4}+3c_1(L-x')^2+2c_2(L-x')+c_3=0$$

Solving for c_1, c_2, c_3, c_4

$$EI\,G(x,x')=\frac{(x-x')^3}{6}-(x-x')-\frac{x-3x'+\frac{2xx'}{L}}{6L^2}x^2(L-x')^2$$

$$y''+\omega^2y=g(x) \qquad y=\int G(x,x')g(x')dx'$$

$$(y'G)'-(yG')'=y''G-yG''$$

$$=(g-\cancel{\omega^2y})G-y(\delta(x-x')-\cancel{\omega^2G})$$

integrere

$$(y'G-yG')\Big|_0^{2\pi}=\underline{\int g\cdot G}-\underbrace{\int y\,\delta(x-x')}_{y(x)}$$

P.2 $\quad F(t)=0 \,, \quad t<1$

$$t^2\ddot{y}(t)+t\dot{y}+a^2y(t)=F(t) \qquad t\geq 1 \qquad -(1)$$

BCs: $y(1)=y_0 \,, \quad y'(1)=V_0$

$t=e^s$. $\quad y(t)\Rightarrow y(s) \qquad dt=e^s ds \,, \; dt=t\,ds$

$$y''=\frac{d^2y}{dt^2}=\frac{1}{t}\frac{d}{ds}\left(\frac{1}{t}\frac{dy(s)}{ds}\right) \left[=\frac{d}{dt}\left(\frac{d}{dt}y(t)\right)\right]$$

$$=\frac{1}{t^2}\frac{d^2y}{ds^2}-\frac{1}{t^2}\frac{dy}{ds} \quad \Rightarrow \quad ty'=t\frac{dy}{dt}=\frac{dy}{ds}$$

$$\Rightarrow (1): \quad \frac{d^2y}{ds^2}+a^2y=F(s)$$

$$y(s=0)=y_0 \,, \quad y'(0)=V_0 \,, \quad F=0 \,, \; s<0$$

$$y(s)=\frac{V_0}{a}\operatorname{sen}(as)+y_0\cos(as)+h(s)$$

$$\frac{d^2h}{ds^2}+a^2h=F(s)$$

using the method of GF to find $h(s)$:

$$\frac{d^2G}{ds^2}+a^2G=\delta(s-s')$$

$$G(s,s')=\begin{cases} G_<=A_<(s')\operatorname{sen}(as)+B_<(s')\cos(as) & s<s' \\ G_>=A_>(s')\operatorname{sen}(as)+B_>(s')\cos(as) & s>s' \end{cases}$$

* continuity @ $s=s'$ $\quad G(s,s')\big|_{s=s'^-} = G(s,s')\big|_{s=s'^+}$

$$(A_< - A_>)\, s' + (B_< - B_>)\, c = 0 \qquad (2)$$

* discontinuity in the derivative @ $s=s'$

$$\partial_s G\big|_{s=s'} - \partial_s G\big|_{s=s'} = 1$$

$$A_> \cos(as') - B_> \sin(as') - A_< \cos(as') + B_< \sin(as') = \frac{1}{a} \qquad (3)$$

Boundary conditions

$$G_<(s=0, s') = 0 \quad , \quad G'(s=0, s') = 0$$

$$B_< = 0 \quad , \quad A_< = 0$$

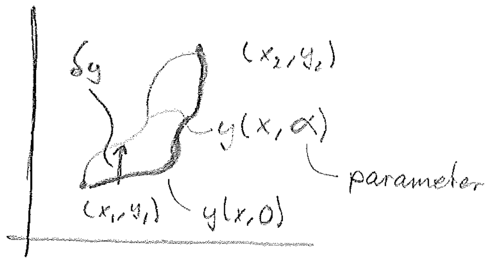
$$\Rightarrow \quad G(s,s') = \frac{\sin a(s-s')}{a} - (s-s')$$

$$h(s) = \int_0^\infty ds' \; F(s')\, G(s',s)\, ds'$$

$$y(s) = \frac{v_0}{a} \sin(as) + y_0 \cos(as) + h(s)$$

Matematisk fysik 3/12

Variationskalkyl:



$$\delta y = y(x,\alpha) - y(x,0) = \alpha\eta(x)$$

$$I(\alpha) = \int_{x_1}^{x_2} F(y(x,\alpha), y'(x,\alpha); x)\, dx$$

problem: hitta den kurva, kalla den $y(x,0)$, som gör $y(\alpha)$ stationär

$$\left.\frac{\partial I(\alpha)}{\partial \alpha}\right|_{\alpha=0} = 0 \quad ☆$$

Med villkoret på $\eta(x)$

- $\eta(x) = \eta(x_2) = 0$
- $\eta(x)$ = deriverbar

$$y(x,\alpha) = y(x,0) + \alpha\eta(x)$$
$$y'(x,\alpha) = y'(x,0) + \alpha\eta'(x)$$

Taylorutveckla $F(y, y'; x)$ kring $\alpha = 0$!

$$F(y+\alpha\eta, \dots) = F(y, \dots) + \frac{\partial F}{\partial y}\alpha\eta + \mathcal{O}(\alpha^2)$$

$$F(\dots, y'+\alpha\eta', \dots) = F(\dots y' \dots) + \frac{\partial F}{\partial y'}\alpha\eta' + \mathcal{O}(\alpha^2)$$

$$I(\alpha) = I(0) + \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\alpha\eta + \frac{\partial F}{\partial y'}\alpha\eta'\right)dx + \mathcal{O}(\alpha^2)$$

$$= I(0) + \alpha\int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right)dx + \mathcal{O}(\alpha^2)$$

$$☆ \Rightarrow \left.\frac{\partial I(\alpha)}{\partial \alpha}\right|_{\alpha=0} = \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right)dx \quad ☆☆$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\frac{d\eta}{dx}\,dx = \underbrace{\cancel{\frac{\partial F}{\partial y'}\eta(x)\Big|_{x_1}^{x_2}}}_{=0} - \int_{x_1}^{x_2} \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right)\eta(x)\,dx$$

$$(**) \Rightarrow \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)\eta(x)\,dx = 0$$

välj η med konstant tecken

$$\Rightarrow \boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0}$$

Nödvändigt men ej tillräckligt villkor för att $I(0)$ stationära

Jämför analytisk mekanik

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)$$

verkan

Lagrange funktion

$S = S(\alpha)$

$q = q(t, \alpha)$

$\dot{q} = \dot{q}(t, \alpha)$

$$\delta S = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

Minsta verkans-principen

Euler-Lagrange ekvationen

Viktigt specialfall: $F(y, \dot{y}, \cancel{x})$

inget explicit x-beroende

$$\frac{dF}{dx} = \underbrace{\cancel{\frac{\partial F}{\partial x}}}_{=0} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial y'}\underbrace{\frac{d^2y}{dx^2}}_{=\frac{dy'}{dx}}$$

Multiplicera Euler med $\frac{dy}{dx}$

$$\frac{dy}{dx}\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y}\right)\right) = 0$$

$$\Rightarrow \frac{dF}{dx} - \frac{\partial F}{\partial y'}\frac{d^2y}{dx^2} - \frac{dy}{dx}\frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

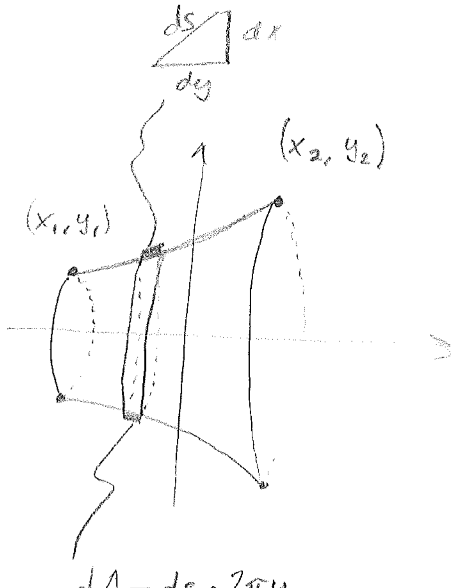
$$\Rightarrow \frac{d}{dx}\left(F - \frac{\partial F}{\partial y'}\frac{dy}{dx}\right) = 0$$

$$\Rightarrow F - \frac{\partial F}{\partial y'}\frac{dy}{dx} = \text{konstant}$$

Klassiskt problem i variationskalkyl

Såpfilmsproblemet

hur välja en kurva som sammanbinder (x_1, y_1) med (x_2, y_2) sådan att dess rotation kring x-axeln ger minimal yta



$$dA = ds \cdot 2\pi y$$
$$= \left((dx)^2 + (dy)^2\right)^{1/2} \cdot 2\pi y$$
$$= dx\left(1 + y'^2\right)^{1/2} \cdot 2\pi y$$

arean $$A = \int_{x_1}^{x_2} dA = 2\pi \int_{x_1}^{x_2} \underbrace{y(1+y'^2)^{1/2}}_{F(y, y', \cancel{x})} dx$$

Euler (utan explicit x-beroende)

$$F - y'\frac{\partial F}{\partial y'} = \text{konstant}$$

$$\Rightarrow y(1+y'^2)^{1/2} - y'\frac{\partial}{\partial y'}\left(y(1+y'^2)^{1/2}\right) = C$$

$$y(1+y'^2)^{1/2} - yy'^2\left(\frac{1}{1+y'^2}\right)^{1/2} = c$$

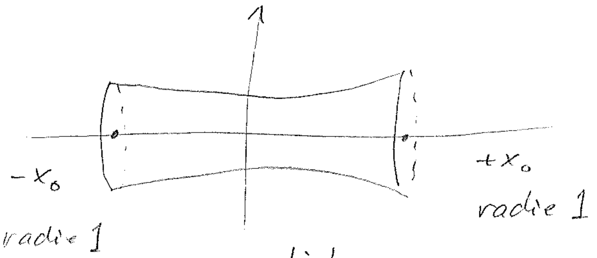
$$\frac{y^2}{1+y'^2} = c^2 \Rightarrow \frac{1}{y'} = \frac{c}{\sqrt{y^2-c^2}} \Rightarrow \frac{dx}{dy} = \frac{1}{\sqrt{(y/c)^2-1}}$$

$$\Rightarrow \int dx = \int dy \frac{1}{\sqrt{(y/c)^2-1}} \Rightarrow \boxed{y = c\cosh\left(\frac{x-c'}{c}\right)}$$

c och c' integrationskonstanter. fran $y_1 = y(x_1)$, $y_2 = y(x_2)$

OBS. Euler Nödvändigt villkor med antagandet att lösningen är deriverbar.

<u>Illustration</u>



Lösning : $y = c\cosh\left(\frac{x}{c}\right)$ ← ty $c'=0$ pga. symmetri

$$1 = c\cosh\left(\frac{\pm x_0}{c}\right)$$

$\Downarrow$ lös ut c där $x_0 = \frac{1}{2}$

$c = 0{,}24$, $c = 0.85$

inte ens ett lokalt minimum! ↑ minsta arean

2) välj $x_0 = 1 \Rightarrow 1 = c\cosh\left(\frac{1}{c}\right)$ Saknar reella lösningar

Lösningen till Euler continuerlig upp till $x_{0_{max}} \approx 0.66$

discontinuitet – Goldschmidt språng!

Generalisering till flera beroende variabler.

$$I = \int_{x_1}^{x_2} F(y_1(x) \dots y_n(x); y_1'(x) \dots y_n'(x); x)\, dx$$

I stationär $\Rightarrow$

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\frac{\partial F}{\partial y_i'} = 0 \qquad i = 1, 2 \dots n$$

Flera oberoende variabler

$$I = \iiint F(u, u_x, u_y, u_z; x, y, z)\, dx dy dz$$

$u = u(x, y, z)$

I stationär $\Rightarrow$

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u_x} - \frac{d}{dy}\frac{\partial F}{\partial u_y} - \frac{d}{dz}\frac{\partial F}{\partial u_z} = 0$$

OBS $\frac{d}{dx}\frac{\partial F}{\partial u_x}(u, u_x, u_y, u_z, x, y, z) = \frac{\partial^2 F}{\partial x \partial u_x} + \frac{\partial^2 F}{\partial u \partial u_x}\frac{du}{dx} + \frac{\partial^2 F}{\partial u \partial u_y}\frac{du_x}{dx}$ etc.

Hemexempel:

V Minimera elektrostatiska energin i V

$$E = \frac{1}{2}\varepsilon \vec{E}^2 = \frac{1}{2}\varepsilon(\nabla\varphi)^2$$

$\Rightarrow$ Euler $\Rightarrow$ Laplace för φ

Variation med tvång → ex. $\varphi(x,y,z)=0$

Lagrange multiplikatorer

uppvärmning vanliga funktioner $f(x,y,z)$

(2) $\quad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0$

(1) $\quad \Longleftrightarrow \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$ dy x,y,z oberoende utan tvång

Låt oss nu införa ett tvång.

$\varphi(x,y,z)=0$ $\quad x,y,z$ ej oberoende! $\quad d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz = 0$

$(2) \not\Rightarrow (1)$

Bilda (□) $df + \lambda d\varphi = \underbrace{\boxed{\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \varphi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \varphi}{\partial y}\right)dy}}_{(\square\square)\ =0} + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \varphi}{\partial z}\right)dz = 0$

välj λ så att t.ex $\left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \varphi}{\partial z}\right) = 0$

Två oberoende variabler kvar

$(\square\square) \Rightarrow \frac{\partial f}{\partial x} + \lambda\frac{\partial \varphi}{\partial x} = \frac{\partial f}{\partial y} + \lambda\frac{\partial \varphi}{\partial y} = 0$

〰 – fyra ekvationer, 4 obekanta

{Matematisk fysik 7/12}

Variationskalkyl med tvång

Problem:

$$I = \int\dots\int F(\underbrace{y_1, y_2 \dots y_n}_{y_i}; \underbrace{\frac{\partial y_1}{\partial x_1}, \dots \frac{\partial y_1}{\partial x_n}}_{\frac{\partial y_i}{\partial x_i}}; \dots; \frac{\partial y_n}{\partial x_1} \dots \frac{\partial y_m}{\partial x_n}; \underbrace{x_1, x_2 \dots x_n}_{x_j}) \, dx_1 \dots dx_n$$

$$= \int F\left(y_i, \frac{\partial y_i}{\partial x_j}, x_j\right) dx_j$$

n-st	x
m-st	y

$\delta I = 0$ under tvånget $\int \varphi_k\left(y_i, \frac{\partial y_i}{\partial x_j}, x_j\right) dx_j = \text{konstant.}$

"Isoperimetriskt problem" $(k = 1 \dots l)$

Hur göra?

$$\delta \int \underbrace{\left[F\left(y_i, \frac{\partial y_i}{\partial x_j}, x_i\right) + \sum_{k=1}^{l} \lambda_k \varphi_k\left(y_i, \frac{\partial y_i}{\partial x_j}, x_j\right) \right]}_{\equiv g\left(y_i, \frac{\partial y_i}{\partial x_j}, x_j\right)} dx_j = 0$$

Euler

$$\Rightarrow \frac{\partial g}{\partial y_i} - \sum_j^n \frac{\partial}{\partial x_j} \frac{\partial g}{\partial\left(\frac{\partial y_i}{\partial x_j}\right)} = 0 \qquad i = 1, 2 \dots m$$

Kommentar: $m - l$ oberoende variabler
välj λ_k $(k = 1 \dots l)$ så att
Euler är uppfylld för de
l kvarvarande variablerna

Gruppteori

Hermann Weyl (1928)
Eugene Wigner (1936)
etc.

Ordning n

$G = \{g_1, g_2 \dots g_n\}$ + kompositionsregel ("gruppmultiplikation")

jargong

$g_i \cdot g_j = g_k$

Är en grupp om de fyra gruppaxiomen är uppfyllda

Gruppaxiomen:

- $g_i g_j \in G \quad \forall g_i, g_j$
- $g_i(g_j g_k) = (g_i g_j) g_k \quad \forall g_i, g_j, g_k$
- $\exists e \in G : eg = ge = g \quad \forall g \in G$
- $\forall g \in G \; \exists g^{-1} \quad gg^{-1} = g^{-1}g = e$

- Om $g_i g_j = g_j g_i \quad \forall g_i g_j \in G$ så kallas G Abelsk
- $H \subset G$ är en delmängd till G om H uppfyller Gruppaxiomen

triviala delgrupper $\{e\}, G$

icke-triviala kallas Äkta

- antalet element i gruppen = gruppens ordning

$[G] = [g] = \#$ element = gruppens ordning

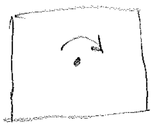
$[g] = n \in \mathbb{Z}$: ändlig grupp

annars oändlig < uppräknelig / icke-uppräknelig ("kontinuerlig")

Intressanta grupper i fysiken: Symmetrigrupper

Grupp av symmetritransformationer

Ex.



$$G = \{ g_1 = \tfrac{\pi}{2}\text{-rot},\ g_2 = \pi\text{-rot},\ g_3 = g_1^{-1} = -\tfrac{\pi}{2}\text{-rot},\ g_4 = -\pi\text{-rot},\ g_5 = e \}$$

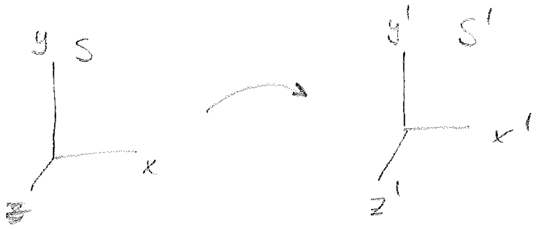
Vanligare i fysiken

Vi är intresserade av symmetritransformationer på ekvationer, "teorier/modeller"

ex. vågekvationen

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0$$

Invariant under Lorentz-transformation



$$g_\gamma \begin{cases} x \to x' = \gamma(x - vt) \\ y \to y' \\ z \to z' \\ t \to t' = \gamma\left(t - \frac{vx}{c^2}\right) \\ \gamma = \dfrac{1}{\sqrt{1-(v/c)^2}} \end{cases}$$

$\mathcal{L} = \{ g_\gamma \}$ Lorentz-gruppen

kontinuerlig ty $\gamma \in \mathbb{R}$

Viktigt fall: Permutationsgruppen $S_n = \{P_1, P_2 \dots P_{n!}\}$

$$\underbrace{P}_{\text{välj en permutation}} = \begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}$$

Ex $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$: $\boxed{1}\,\boxed{2}\,\boxed{3} \to \boxed{3}\,\boxed{1}\,\boxed{2}$

S_n icke abelsk ty $PQ = \begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}\begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix} \neq QP$

Ex
$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$
ej likhet
$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$
$$\cancel{(1)}(2\,3) \quad (1\,2\,3) = (1\,2)\cancel{(3)}$$
$$\Rightarrow \text{"}(2\,3)\,(1\,2\,3) = (1\,2)\text{"} \quad \text{jargong}$$

Cyklerna är disjunkta och kommuterar.

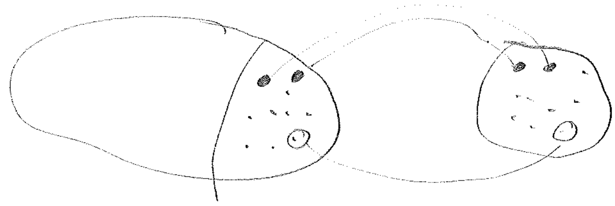
Cayleys sats

Varje grupp av ordning n är isomorf med en

1-1 ok komposition

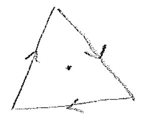
delgrupp till S_n.

1-1



Ex Cykliska gruppen C_n

= Symmetrigruppen av rotationer på en liksidig polygon med n orienterade sidor

C_3: 

$C_3 = \{a, b, e\}$

a: $2\pi/3$-rot.

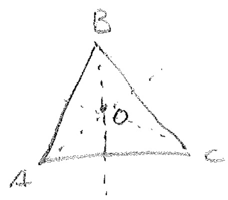
b: $4\pi/3$-rot

"multiplikationstabell"

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Diedergruppen D_n

- Samma som ovan men icke-orienterade sidor

D_3 

$D_3 = \{a, b, e, c_1, c_2, c_3\}$

c_1: π-rotation runt OA

c_2: π-rotation runt OB

c_3: π-rot runt OC

$\simeq S_3$

isomorf

Bra övning ta fram multiplikationstabellen kolla

Några begrepp

- Konjugering

  $a, b \in G$ är konjugerade

  om $\exists g \in G$ så att $b = gag^{-1}$

  exempel på ekvivalensrelation

- Ekvivalensrelation definieras av

  (i) $a \sim a$ reflexivitet

  (ii) $a \sim b \Rightarrow b \sim a$ symmetrisk

  (iii) $a \sim b \sim c \Rightarrow a \sim c$ transitiv

Test: konjugering är en ekvivalensrelation

$$a = eae^{-1}$$

$$a = gbg^{-1} \Rightarrow \underbrace{g^{-1}}_{h} a \underbrace{g}_{h^{-1}} = b \quad \text{ok}$$

$$a = gbg^{-1}, \quad b = hch^{-1}$$

$$\Rightarrow a = g(hch^{-1})g^{-1} = (gh)c(h^{-1}g^{-1})$$

$$\Rightarrow \underbrace{(gh)}_{k} c \underbrace{(gh)^{-1}}_{k^{-1}} = kck^{-1}$$

Ekvivalensklass

välj $a \in G$: $(a) = \{b \mid b \sim a\}$

$b \in G$: $(b) = \{c \mid c \sim b\}$

$b \notin (a) \Rightarrow (a) \cap (b) = \emptyset$

Bevis $(a) \cap (b) \neq \emptyset$ dvs. $\exists d$, $d \sim a$ och $d \sim b$)

$\Rightarrow a \sim b$ VSB.

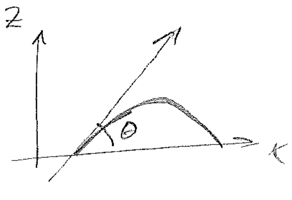
Konjugatklass (till $a \in G$)

$(a) = \{b \mid b = gag^{-1}, \exists g \in G\}$

P.1 Evaluate the functional derivative $\frac{\delta A}{\delta f(x)}$

a) $A[f] = 2f(1) + \int_{-\infty}^{\infty} dx \left\{ 2x[f(x)]^3 - \left[\frac{d^2 f(x)}{dx^2}\right]^2 \right\} + \iint_{-\infty}^{\infty} dx\,dx'\, K(x,x') f(x')$

b) Using Fermat's principle of least times for a ray of light to travel between to points, describe how would you determine the apparent position (angle θ) of object A as seen by an observer P.



$n(z) = n_0(1+\alpha z)$ is the refractive index

* speed of light in medium is $c'(z) = \frac{c}{n(z)}$

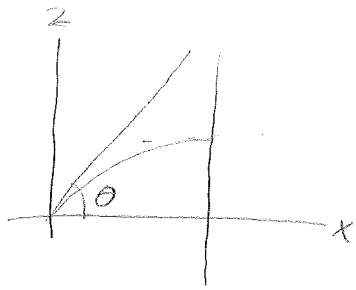
$2f(1) = 2\int dx' \delta(x'-1) f(x)$

Sol.
a) $\frac{\delta A}{\delta f(x)} = 2\delta(x-1) + 6x f^2(x) - 2f^{IV}(x) + \int dx' K(x',x) f(x') +$

$+ \int dx' K(x',x) f(x')$

$\int dx\, 2(f''(x))\, \delta f''(x) \xrightarrow[\text{by parts}]{\text{integrate}} \int dx\, 2(f''''(x))\, \delta(f(x))$

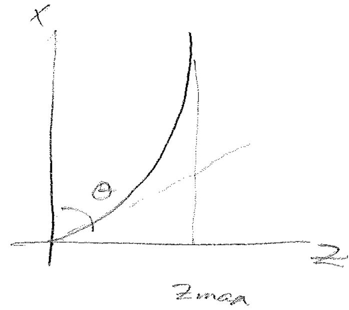
$\left.\frac{dx(z)}{dz}\right|_{z_{max}} = +\infty \quad , \quad x(z=0) = 0$



$x'(z)\Big|_{z=0} = ctg\,\theta$

b) $\frac{T}{2} = \int_0^{z_{max}} dz \frac{\sqrt{1+x'^2}}{c'(z)} \qquad x' = \frac{dx}{dz}$

$$c'(z) = \frac{c}{n(z)} = \frac{c}{n_0(1+\alpha z)}$$



E-L

$$\frac{\partial}{\partial z}\Big(\underbrace{\frac{\partial L}{\partial x'}}_{\text{const.}}\Big) - \cancel{\frac{\partial L}{\partial x}} = 0$$

$$L = \frac{n_0}{c}\sqrt{1+x'^2}\,(1+\alpha z)$$

$$\frac{\partial L}{\partial x'} = \text{const} \Rightarrow \frac{n_0}{c}(1+\alpha z)\frac{x'}{\sqrt{1+x'^2}} = \text{const.}$$

$$x(z=0)=0$$

- $x' = \frac{\partial x}{dz}\Big|_{z\text{-}max} = +\infty \qquad \frac{n_0}{c}(1+\alpha z)\frac{x'}{\sqrt{1+x'^2}}\Big|_{z=z_{max}} = \frac{n_0}{c}(1+\alpha z_{max})$

- $@z=0 \Rightarrow \frac{n_0}{c}\cdot\frac{\cot\theta}{\sqrt{1+\cot^2\theta}} = \frac{n_0}{c}(1+\alpha z_{max}) \Rightarrow \cos\theta = 1-|\alpha z_{max}|$

P2. a) $I[\phi] = \iiint dxdydz\, L(\phi, \nabla\phi, x, y, z)$

what is the condition that $\phi(x,y,z)$ must satisfy such that $I[\phi]$ has an extremum. V is simply connected. three dimensional region

b) Consider the electrostatic energy due to an electric field $E(r)$

$W = \iiint dxdydz\, E^2(x,y,z) \qquad E = -\nabla\phi$ show that ϕ satisfies $\nabla^2\phi = 0$

Sol

a) $$\delta I[\phi] = \iiint \overset{dxdydz}{\downarrow}dv \; \frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial \partial_x \phi}\delta \partial_x \phi + \frac{\partial L}{\partial \partial_y \phi}\delta \partial_y \phi + \frac{\partial L}{\partial \partial_z \phi}\delta \partial_z \phi$$

$$\frac{\delta I[\phi]}{\delta \phi} = 0 \Rightarrow \frac{\partial L}{\partial \phi} - \partial_x \frac{\partial L}{\partial \partial_x \phi} - \partial_y \frac{\partial L}{\partial \partial_y \phi} - \partial_z \frac{\partial L}{\partial \partial_z \phi} = 0 \qquad (1)$$

b) $$W = \int dv \, |\nabla \phi|^2 = \int dv \, (\partial_x \phi)^2 + (\partial_y \phi)^2 + (\partial_z \phi)^2$$

$L = |\nabla \phi|^2$ using (1)

$$\partial_x \partial_x \phi + \partial_y \partial_y \phi + \partial_z \partial_z \phi = 0$$

$$\boxed{\nabla^2 \phi = 0}$$

P.3 a) How wold you solve the equation

$$f(x) = g(x) + \int_{-1}^{1} dy \, (\lambda x - 2x^2 y) f(y)$$

$\lambda = $ const.

$g(x) = $ known

b) What conditions must λ, $g(x)$ satisfy in order for the equation to be solvable

c) How would you find the solution of

$$-\frac{\hbar^2}{2m} y''(x) + \frac{V}{\cosh(kx)} y(x) = E y(x) \quad \text{for } E \gg |V| \, , \; k = \sqrt{2mE}/\hbar$$

and subject to the condition $y(x) e^{-ikx} \to 1$ as $x \to \infty$

Sol

$$f(x) = g(x) + \int_{-1}^{1} dy\,(\lambda x - 2x^2 y)\, f(y)$$

- $f(x) = g(x) + \lambda x \cdot A - 2x^2 B$

$$A = \int_{-1}^{1} dy f(y) \quad , \quad B = \int_{-1}^{1} dy\; y f(y)$$

- $A = \underbrace{\int_{-1}^{1} dx\, g(x)}_{g_0} - 2 \cdot \frac{2}{3} \cdot B = g_0 - \frac{4B}{3}$

$$B = \underbrace{\int_{-1}^{1} dx\, x g(x)}_{g_1} + \lambda A \cdot \frac{2}{3} = g_1 + \frac{2\lambda A}{3}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 4/3 \\ 2\lambda/3 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} g_0 \\ g_1 \end{pmatrix}$$

- if $\begin{vmatrix} 1 & 4/3 \\ 2\lambda/3 & -1 \end{vmatrix} \neq 0 \Rightarrow$ solution exists for $\forall g(x)$

$$-1 - \frac{8}{9}\lambda \neq 0 \qquad \lambda \neq -\frac{9}{8}$$

- if $\lambda = -\frac{9}{8}$ $\quad \begin{pmatrix} 1 & \frac{4}{3} \\ -\frac{3}{4} & -1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} g_0 \\ g_1 \end{pmatrix}$

Solution exists if $g_0 = \frac{4}{3} g_1$

$$-\frac{\hbar^2}{2m} y''(x) + \frac{V}{\cosh(kx)} y(x) = E\, y(x)$$

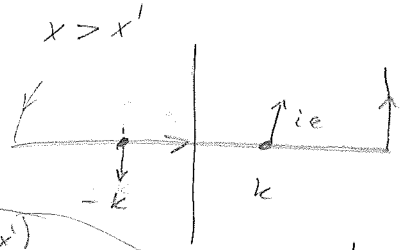
$$y'' + K^2 y = \frac{2m}{\hbar}\,\frac{V}{\cosh(kx)}\, y(x) \qquad K^2 = \frac{2mE}{\hbar}$$

Find Green's function $G(x,x')$

$$\partial_x^2 G + K^2 G = \delta(x-x')$$

Applying F.T.: $-\nu^2 G(\nu) + K^2 G(\nu) = e^{-ix'\nu}$ $\quad \begin{cases} y(\nu) = \int dx\, e^{-ix\nu} y(x) \\ y(x) = \int \frac{d\nu}{2\pi} e^{ix\nu} y(\nu) \end{cases}$

$$G(\nu) = \frac{e^{-i\nu x'}}{k^2 - \nu^2}$$



$$G(x,x') = \frac{1}{2\pi}\int d\nu\, \frac{e^{i\nu(x-x')}}{k^2 - \nu^2}$$

$$G(x,x') = \begin{cases} x < x', & \frac{1}{2\pi}\cdot\frac{1}{-2(-k)}\, e^{-ik(x-x')} \cdot(-2\pi i) = \frac{-i}{2k} e^{-ik(x-x')} \\ x > x', & \frac{-i}{2k}\, e^{ik(x-x')} \end{cases}$$

$$\boxed{G(x,x') = -\frac{i}{k}\, e^{ik|x-x'|}}$$

$$y(x) = e^{ikx} + \int dx'\, G(x,x')\, \frac{V\, y(x')}{\cosh(kx')}\cdot\frac{2m}{\hbar^2}$$

$$y(x) \approx e^{ikx} + \int dx' \left(-\frac{iV}{2k}\right)\cdot\frac{2m}{\hbar^2}\cdot\frac{e^{ik|x-x'|}}{\cosh(kx')}\, e^{ikx'}$$

a) For which value of λ does the equation

$$f(x) = \phi(x) + \lambda \int_0^1 dy\,(1-3xy)\,f(y)$$

have a solution for general ϕ, find the solution

b) For the remainding values of λ, what conditions must $\phi(x)$ satisfy in order for the solution $f(x)$ to exist?

Find the solution

$$f(x) = \phi(x) + \lambda \int_0^1 dy\, f(y) - 3x\lambda \int_0^1 dy\, y\, f(y)$$

- $f(x) = \phi(x) + \lambda A - 3x\lambda B$

$$A = \int_0^1 dy\, f(y) \quad , \quad B = \int_0^1 dy\, y\, f(y)$$

- $$A = \overbrace{\int_0^1 dx\, \phi(x)}^{F_0} + \lambda A - \frac{3}{2}\lambda B \quad \Longrightarrow \begin{pmatrix} 1-\lambda & \frac{3}{2}\lambda \\ -\frac{\lambda}{2} & 1+\lambda \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} F_0 \\ F_1 \end{pmatrix}$$

$$B = \underbrace{\int_0^1 dx\, x\,\phi(x)}_{F_1} + \frac{\lambda A}{2} - \lambda B$$

- For a general $\phi(x)$, solution exist if

$$\begin{vmatrix} 1-\lambda & \frac{3}{2}\lambda \\ -\frac{\lambda}{2} & 1+\lambda \end{vmatrix} \neq 0 \qquad \lambda \neq -2, +2$$

$$A = \frac{(1+\lambda)F_0 - \frac{3}{2}\lambda F_1}{1 - \lambda^2/4}$$

$$B = \frac{(\lambda/2)F_0 + (1-\lambda)F_1}{1 - \lambda^2/4}$$

b) now we study the cases $\lambda = \pm 2$

$$\int_0^1 dx\, \phi(x)\, f^*(x) = 0$$

where f^* is the eigenfunktion of the transpose operator $K(y,x)$

$$f^*(x) = \lambda \int_0^1 K(y,x)\, f^*(x)\, dy$$

$$K(x,y) = 1 - 3xy$$

$\lambda = 2$ $\begin{pmatrix} -1 & 3 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} A^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $A^* = 3B^*$

$$f^*_{\lambda=2}(x) = 6B^* - 6xB^* \sim 1 - x$$

$$\boxed{\int_0^1 dx\, \phi(x)(1-x) = 0 \Rightarrow F_0 = F_1}$$

Solution for $\lambda = +2$ $\begin{pmatrix} -1 & 3 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} F_0 \\ F_1 \end{pmatrix}$, $F_0 = F_1$

$$-A + 3B = F_0$$

$f(x) = \phi(x) + 6B(1-x) - 2F_0$ B is undetermined

(Matematisk Fysik 10/12)

Gruppaxiomen

S_3 Cayleys sats

Ex: C_3, D_3 Ekvivalensrelation/klass

konjugering/klass,

Sidoklass

delgrupp $\rightarrow$ $H \subset G$, $H = \{h_1, h_2, \dots h_r\}$

$g \in G$

Vänster sidoklass av H m.a.p. g

$g_i H = \{g_i h_1, g_i h_2, \dots g_i h_r\}$

(Höger sidoklass $Hg = \{h_1 g, h_2 g, \dots h_r g\}$)

H normal om $gH = Hg$

$\{g_1 H, g_2 H, \dots g_s H\} = G/H$ s-antalet sidoklasser

Är en grupp, kallas kvotgrupp

måste visa att sidoklasserna är disjunkta

dvs. $b \in aH$ $b \sim a$ (dvs. en ekvivalensrelation)

Test reflexivitet (i) $a \in aH$ ok ty $e \in H$

symmetriskt (ii) $a \in bH \Leftarrow b \in aH$

$b \in aH \Rightarrow \exists h \in H, b = ah \Rightarrow a = bh^{-1}, h^{-1} \in H \Rightarrow a \in bH$

transitivitet (iii) $b \in aH, c \in bH \Rightarrow c \in aH$

$\exists h \in H, b = ah \Rightarrow \exists h' \in H \; c = bh'$

$\Rightarrow c = (ah)h' = a\underbrace{(hh')}_{h''} = ah''$ $h'' \in H$ pga. gruppaxiomen

Varje sidoklass innehåller $\underbrace{[H]}_{=r}$ element

$gH = \{gh_1, gh_2 \dots gh_r\}$ hur många element?

Påstående: r st! $(H = \{h_1, h_2 \dots h_r\}$

Bevis Antag $gh_i = gh_j \Rightarrow h_i = h_j$ motsägelse

$S\underbrace{[H]}_{r} = [G]$ Lagranges teorem

sidoklasser · # element i H = # element i G

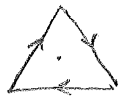
Korrolarium

$\underbrace{\#\text{ element i } H}_{[H]} =$ delare i $[G]$

Antag $[G]$ primtal $\Rightarrow$ G kan inte ha någon äkta delgrupp dvs. enda möjliga delgrupper är $\{e\}$ och G självt

Weinbergs teorem: Garbage in, garbage out

Representationsteori – "oändligt viktig i fysik"

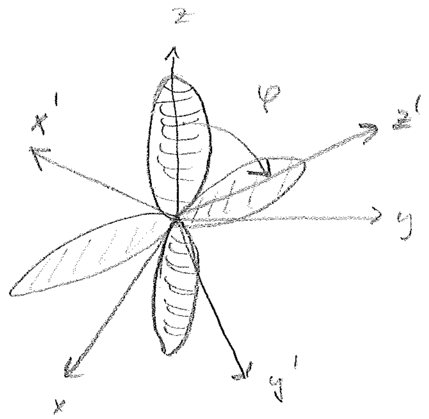
Ex.  $C_3 = \{a, b, e\}$

$$a \to 2\pi/3, \quad b \to 4\pi/3$$

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{cases} a \to R(2\pi/3) \\ b \to R(4\pi/3) \\ e \to R(0) \end{cases}$$

rotations-matris

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Ex 

H (2p tillstånd) : $\Psi(\vec{r}) \to \Psi'(\vec{r})$ 'ny' vågfunktion i gamla koordinatsystem

$\to \Psi'(\vec{r}\,')$ 'ny' vågfunktion i nya koordinatsystem

Rotera koordinatsystemet

$\vec{r} \to \vec{r}\,' = R\vec{r}$

$$\Psi(\vec{r}) = \Psi'(\vec{r}\,') = \Psi(R^{-1}\vec{r}\,')$$

$$\vec{r} = R^{-1}\vec{r}\,'$$

$$\Longrightarrow \underline{\Psi'(\vec{r}\,')} = \Psi(\vec{r}) = \Psi\underline{(R^{-1}\vec{r}\,')}$$

Ta bort primet och betrakta de två understrukna relationerna

$\Psi'(\vec{r}) = \Psi(R^{-1}\vec{r})$ rotationen R^{-1} i konfigurationsrummet har inducerat en transformation i $\mathcal{L}^2$

$\langle \vec{r} | \Psi' \rangle =$

Egenfunktioner för väte

$$u_{nlm}(\vec{r}) = R_n(r)\, Y_{lm}(\theta,\varphi)$$

projektion på z-axeln

invariant, l invariant under R

$$\vec{r} \to \vec{r}\,' = R\vec{r}$$

$$\Rightarrow u'_{nlm}(\vec{r}) = u_{nlm}(R^{-1}\vec{r}) = \sum_{m'} \underbrace{D^{l}_{mm'}(R)}_{\text{representation av } R \text{ i Hilbertrummet.}}\, u_{nlm'}(\vec{r})$$

Representation D av en grupp G

$$G = \{ g_1, g_2 \dots \}$$

$$D = \{ D(g_1), D(g_2) \dots \}$$

isomorfi (1-1) $\Rightarrow$ D är en "trogen" rep. (resentation)

$D(g_i)$ är en $n \times n$ inverterbar matris

$\longrightarrow$ n-dim representation

så att $g_i \cdot g_j = g_k \Rightarrow D(g_i)\cdot D(g_j) = D(g_k)$

D, D' ekvivalenta $\quad \forall g\ \exists S \quad D(g) = S\, D'(g)\, S^{-1}$

D reducerbar om givet $D^{(1)}$ och $D^{(2)}$

$$D = \underbrace{D^{(1)}}_{m} \oplus \underbrace{D^{(2)}}_{m-n} = \left(\begin{array}{c|c} D^{(1)} & 0 \\ \hline 0 & D^{(2)} \end{array}\right) \left(\begin{array}{c} \\ - \\ \\ \end{array}\right) \begin{array}{l} \}\, m \\ \}\, n-m \end{array}$$

mer allmänt

$$D(g) = \begin{pmatrix} D^{(1)}(g) & & & \\ & D^{(2)}(g) & & 0 \\ & & \ddots & \\ 0 & & & \ddots \end{pmatrix}$$

t.ex.

$$= 2D^{(1)}(g) \oplus D^{(2)}(g) \oplus 2$$

Om D ej reducerbar kallas D för irreducibel

hur sönderlägga en given rep i irreps?

Nyckelproblem

Några teorem.

1) Varje rep ekvivalent med en unitär rep.

$D^{+}(g) = D^{-1}(g)$

2) Schurs lemmor

3) Ortogonalitetsteoremet