SOLUTIONS
OPTIONS AND MATHEMATICS
(CTH[mve095], GU[MMA700])

May 21, 2012, morning, v.
No aids.
Each problem is worth 3 points.
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1. (Binomial model) Suppose $T = 2$, $e^r = \frac{1}{2}(e^u + e^d)$, and $B(2) = 1$. A derivative of European type pays the amount

$$Y = \mid \frac{S(2)}{S(1)} - \frac{S(1)}{S(0)} \mid$$

at time of maturity T and the self-financing portfolio strategy $h(t) = (h_S(t), h_B(t))$, $t = 0, 1, 2$, replicates Y. Find q_u, $h_S(1)$, and $h_B(2; d)$.

Do not hand in any motivations of the answers!

Answer: $q_u = \frac{1}{2}$, $h_S(1) = 0$, and $h_B(2; d) = -e^d$.

Solution. Computation of q_u. We have that

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{\frac{1}{2}(e^u + e^d) - e^d}{e^u - e^d} = \frac{1}{2}.$$

Computation of $h_S(1)$. Set $\Pi_Y(t) = v(t)$. Then

$$\begin{cases} v(2)_{\mid X_1=u,\ X_2=u} = 0 \\ v(2)_{\mid X_1=u,\ X_2=d} = e^u - e^d \\ v(2)_{\mid X_1=d,\ X_2=u} = e^u - e^d \\ v(2)_{\mid X_1=d,\ X_2=d} = 0 \end{cases}$$

and it follows that $v(1) = \frac{1}{2}e^{-r}(e^u - e^d)$. Thus $h_S(1) = 0$.

Computation of $h_B(2;d)$. For short, set $\kappa_S = h_S(2;d)$ and $\kappa_B = h_B(2;d)$. Now

$$\begin{cases} \kappa_S S(0)e^d e^u + \kappa_B B(0)e^r e^r = e^u - e^d \\ \kappa_S S(0)e^d e^d + \kappa_B B(0)e^r e^r = 0 \end{cases}$$

and

$$\kappa_B = h_B(2;d) = -\frac{e^d}{B(0)e^{2r}} = -e^d.$$

2. (Black-Scholes model) Suppose $K, T > 0$ are constants. A financial derivative of European type has the payoff

$$Y = \begin{cases} 1 \text{ if } S(T) > K, \\ -1 \text{ if } S(T) \leq K, \end{cases}$$

at time of maturity T. Determine K such that $\Pi_Y(0) = 0$.

Solution. Put

$$g(x) = \begin{cases} 1 \text{ if } x > K, \\ -1 \text{ if } x \leq K, \end{cases}$$

and note that $Y = g(S(T))$. Now

$$\Pi_Y(0) = e^{-rT} E\left[g(se^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}G})\right]$$

where $s = S(0)$ and $G \in N(0,1)$ and, hence,

$$\Pi_Y(0) = e^{-rT}\left(P\left[se^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}G} > K\right] - P\left[se^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}G} \leq K\right]\right)$$

$$= e^{-rT}\left(2P\left[se^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}G} > K\right] - 1\right)$$

$$= e^{-rT}\left(2P\left[se^{(r-\frac{\sigma^2}{2})T-\sigma\sqrt{T}G} > K\right] - 1\right)$$

$$= e^{-rT}\left(2P\left[G \leq \frac{1}{\sigma\sqrt{T}}(\ln\frac{s}{K} + (r - \frac{\sigma^2}{2})T)\right] - 1\right)$$

$$= e^{-rT}\left(2\Phi(\frac{1}{\sigma\sqrt{T}}(\ln\frac{s}{K} + (r - \frac{\sigma^2}{2})T)) - 1\right).$$

Accordingly from this, $\Pi_Y(0)=0$ if and only if

$$\ln\frac{s}{K}+(r-\frac{\sigma^2}{2})T=0$$

that is,

$$K=S(0)e^{(r-\frac{\sigma^2}{2})T}.$$

3. (Black-Scholes model) Suppose $T>0$. A financial derivative of European type pays the amount

$$Y=\max(\frac{S(\frac{T}{2})}{S(0)},\frac{S(T)}{S(\frac{T}{2})})$$

at time of maturity T. Find $\Pi_Y(0)$.

Proof. Put $S(0)=s$ and $a=\frac{T}{2}$. We have

$$\Pi_Y(0)=e^{-rT}E\left[\max(\frac{se^{(r-\frac{\sigma^2}{2})a+\sigma W(a)}}{s},\frac{se^{(r-\frac{\sigma^2}{2})T+\sigma W(T)}}{se^{(r-\frac{\sigma^2}{2})a+\sigma W(a)}})\right]$$

$$=e^{-rT}E\left[\max(e^{(r-\frac{\sigma^2}{2})a+\sigma W(a)},e^{(r-\frac{\sigma^2}{2})a+\sigma(W(T)-W(a))})\right]$$

$$=e^{-(r+\frac{\sigma^2}{2})a}E\left[\max(e^{\sigma\sqrt{a}G},e^{\sigma\sqrt{a}H})\right]$$

$$=e^{-(r+\frac{\sigma^2}{2})a}E\left[e^{\sigma\sqrt{a}\max(G,H)}\right],$$

where $G,H\in N(0,1)$ are independent. Moreover,

$$P\left[\max(G,H)\leq x\right]=P\left[G\leq x,\ H\leq x\right]$$

$$=P\left[G\leq x\right]P\left[\ H\leq x\right]=\Phi^2(x).$$

Hence

$$E\left[e^{\sigma\sqrt{a}\max(G,H)}\right]=\int_{-\infty}^{\infty}e^{\sigma\sqrt{a}x}\frac{d}{dx}\Phi^2(x)dx$$

$$=2\int_{-\infty}^{\infty}e^{\sigma\sqrt{a}x}\Phi(x)\varphi(x)dx.$$

Now introduce $b = \sigma\sqrt{a}$ and note that

$$\int_{-\infty}^{\infty} e^{bx}\Phi(x)\varphi(x)dx = e^{\frac{b^2}{2}}\int_{-\infty}^{\infty}\Phi(x)\varphi(x-b)dx$$

$$= e^{\frac{b^2}{2}}\int_{-\infty}^{\infty}\Phi(b-x)\varphi(x)dx.$$

But

$$\int_{-\infty}^{\infty}\varphi(y-x)\varphi(x)dx = \frac{1}{\sqrt{2}}\varphi(\frac{y}{\sqrt{2}})$$

since $G + H \in N(0,2)$ and by integration from $y = -\infty$ to $y = b$ we get

$$\int_{-\infty}^{\infty}\Phi(b-x)\varphi(x)dx = \int_{-\infty}^{b}\frac{1}{\sqrt{2}}\varphi(\frac{y}{\sqrt{2}})dy = \Phi(\frac{b}{\sqrt{2}}).$$

Hence

$$\int_{-\infty}^{\infty} e^{bx}\Phi(x)\varphi(x)dx = e^{\frac{b^2}{2}}\Phi(\frac{b}{\sqrt{2}})$$

and

$$\Pi_Y(0) = 2e^{-(r+\frac{\sigma^2}{2})a}e^{\frac{\sigma^2 a}{2}}\Phi(\frac{\sigma\sqrt{a}}{\sqrt{2}})$$

$$= 2e^{-\frac{rT}{2}}\Phi(\frac{\sigma\sqrt{T}}{2}).$$

4. Let $(X_n)_{n=1}^{\infty}$ be an i.i.d. such that $P[X_1 = 1] = P[X_1 = -1] = \frac{1}{2}$ and set

$$Y_n = \frac{1}{\sqrt{n}}(X_1 + ... + X_n),\ n \in \mathbf{N}_+.$$

Prove that $Y_n \to G$, where $G \in N(0,1)$.

5. Consider two stock price processes $(S_1(t))_{t\geq 0}$ and $(S_2(t))_{t\geq 0}$ and suppose the stochastic process $S = (S_1(t), S_2(t))_{t\geq 0}$ is a bivariate geometric Brownian motion with volatility (σ_1, σ_2) and correlation ρ, where $\sigma_1, \sigma_2 > 0$ and $-1 < \rho < 1$.

(a) Suppose $T > 0$ and $Y = g(S_1(T), S_2(T))$, where the payoff function $g \in \mathcal{P}_2$ is positively homogeneous of degree one.

For any fixed $t < T$, state the time t price $\Pi_Y(t) = u(t, S_1(t), S_2(t))$ of a European derivative with payoff Y at time of maturity T. Do not hand in any motivations of the price formula!

(b) Find $\Pi_Y(t)$ if $t < T$ and $Y = (S_1(T) - S_2(T))^+$.