SOLUTIONS
OPTIONS AND MATHEMATICS
(CTH[mve095], GU[MMA700])

May 25, 2009, morning (4 hours), m
No aids.
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Each problem is worth 3 points.

1. (The one period binomial model, where $0 < d < r < u$) Consider a call with the payoff $Y = \frac{1}{2} \mid \frac{S(1)}{S(0)} - \frac{S(0)}{S(1)} \mid$ at the termination date 1. Find the replicating strategy of the derivative at time 0.

Solution: Let $S(0) = s$ and $S(1) = se^X$, where $X = u$ or d. If (h_S, h_B) denotes the replicating strategy at time 0 we have

$$h_S se^u + h_B B(0)e^r = \sinh u$$

and

$$h_S se^d + h_B B(0)e^r = \sinh d.$$

From this it follows that

$$h_S s(e^u - e^d) = \sinh u - \sinh d$$

and

$$h_S = \frac{\sinh u - \sinh d}{s(e^u - e^d)}.$$

Moreover, we get

$$h_B B(0)(e^{r+u} - e^{r+d}) = e^u \sinh d - e^d \sinh u$$

and

$$h_B = \frac{e^u \sinh d - e^d \sinh u}{B(0)e^r(e^u - e^d)}.$$

$ANSWER:$ $\frac{\sinh u - \sinh d}{S(0)(e^u - e^d)}$ (or $= \frac{1}{2S(0)}(1+e^{-u-d})$) units of the stock and $\frac{e^u \sinh d - e^d \sinh u}{B(0)e^r(e^u - e^d)}$ (or $= -\frac{e^{-r}}{2B(0)}(e^{-u} + e^{-d})$) units of the bond.

2. (Black-Scholes model) Suppose t^*, T_0, T, and δ are positive numbers satisfying the inequalities $T_0 < t^* < T$ and $\delta < 1$. Moreover, suppose $t < t^*$. A stock pays the dividend $\delta S(t^*-)$ at time t^*. Find the price $\Pi_Y(t)$ at time t of a derivative of European type paying the amount

$$Y = (\frac{S(T)}{S(T_0)} - 1)^+$$

at time of maturity T.

Solution. Set $s = S(t)$ and $\tau = T - t$.

To begin with we assume $T_0 \leq t < t^*$. If

$$g(x) = (\frac{x}{S(T_0)} - 1)^+ = \frac{1}{S(T_0)}(x - S(T_0))^+$$

we know that

$$\Pi_Y(t) = e^{-r\tau} E\left[g((1-\delta)se^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}G})\right]$$

where $G \in N(0,1)$. Hence, by the Black-Scholes price formula for a European call,

$$\Pi_Y(t) = \frac{1}{S(T_0)} c(t, (1-\delta)s, S(T_0), T)$$

$$= \frac{1}{S(T_0)}\left\{(1-\delta)S(t)\Phi(D_1(t)) - S(T_0)e^{-r\tau}\Phi((D_2(t))\right\}$$

where

$$D_1(t) = \frac{1}{\sigma\sqrt{\tau}}(\ln\frac{(1-\delta)S(t)}{S(T_0)} + (r + \frac{\sigma^2}{2})\tau)$$

and

$$D_2(t) = \frac{1}{\sigma\sqrt{\tau}}(\ln\frac{(1-\delta)S(t)}{S(T_0)} + (r - \frac{\sigma^2}{2})\tau).$$

In particular,

$$\Pi_Y(T_0) = (1-\delta)\Phi(D_1(T_0)) - e^{-r(T-T_0)}\Phi((D_2(T_0))$$

where

$$D_1(T_0) = \frac{1}{\sigma\sqrt{T-T_0}}(\ln(1-\delta) + (r + \frac{\sigma^2}{2})(T - T_0))$$

and
$$D_2(T_0)=\frac{1}{\sigma\sqrt{T-T_0}}(\ln(1-\delta)+(r-\frac{\sigma^2}{2})(T-T_0)).$$

Since $\Pi_Y(T_0)$ is non-random (= a numerical constant) we conclude that

$$\Pi_Y(t)=e^{-r(T_0-t)}\left\{(1-\delta)\Phi(D_1(T_0))-e^{-r(T-T_0)}\Phi((D_2(T_0))\right\} \text{ if } t<T_0.$$

3. (Black-Scholes model) Let $a, K, T > 0$. A financial derivative of European type pays the amount $Y = (\min(S(T)-K, a))^+$ at time of maturity T. Show that the delta of the derivative is positive and does not exceed

$$\frac{\ln(1+\frac{a}{K})}{\sigma\sqrt{2\pi(T-t)}}$$

at time $t < T$.

Solution. Note that $Y = (S(T)-K)^+ - (S(T)-(a+K))^+$. The delta of a call is standard (see Problem 4) and we get that the delta of Y at time t equals

$$\Delta_Y(t)=\Phi(\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{S(t)}{K}+(r+\frac{\sigma^2}{2})\tau)-\Phi(\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{S(t)}{a+K}+(r+\frac{\sigma^2}{2})\tau)$$

where $\tau = T-t$. Hence $\Delta_Y(t) > 0$ since Φ and ln are increasing in the strict sense. Moreover, if
$$\varphi(u)=\frac{1}{\sqrt{2\pi}}\exp(-u^2/2)$$

we have

$$\Delta_Y(t)=\left\{\frac{1}{\sigma\sqrt{\tau}}((\ln\frac{S(t)}{K}+(r+\frac{\sigma^2}{2})\tau)-(\ln\frac{S(t)}{a+K}+(r+\frac{\sigma^2}{2})\tau))\right\}\varphi(\xi).$$

for an appropriate $\xi \in \left]\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{S(t)}{a+K}+(r+\frac{\sigma^2}{2})\tau, \frac{1}{\sigma\sqrt{\tau}}(\ln\frac{S(t)}{K}+(r+\frac{\sigma^2}{2})\tau)\right[$. But $\varphi(\xi)\leq\frac{1}{\sqrt{2\pi}}$ and we get

$$\Delta_Y(t)\leq\frac{1}{\sigma\sqrt{\tau}}(\ln(a+K)-\ln K)\frac{1}{\sqrt{2\pi}}$$

and the result is immediate.

4. (Black-Scholes model) Suppose $t < T$ and $\tau = T - t$. A simple financial derivative of European type with the payoff function $g \in \mathcal{P}$ has the price

$$\Pi_{g(S(T))}(t) = e^{-r\tau} E\left[g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G})\right]$$

at time t, where $s = S(t)$ is the stock price at time t and $G \in N(0,1)$.

(a) A European call has the strike price K and determination date T. Show that the call price equals $s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$, where

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{K} + (r + \frac{\sigma^2}{2})\tau)$$

and $d_2 = d_1 - \sigma\sqrt{\tau}$.

(b) Show that the delta of the call in Part (a) equals $\Phi(d_1)$.

5. (Black-Scholes model) A European call on a US dollar has the strike strike price K and determination date T. Derive the price of the derivative at time t, if the US interest rate equals r_f and the volatility of the exchange rate process, quoted as crowns per dollar, equals σ. As usual the Swedish interest rate is denoted by r.