OPTIONS AND MATHEMATICS
(CTH[mve095], GU[man690])

May 26, 2007, morning (4 hours), v
No aids.
Examiner: Christer Borell, telephone number 0704 063 461
Each problem is worth 3 points.

Solutions

1. (The binomial model with $u > 0$, $d = -u$, $r = \frac{1}{2}u$, and $T = 2$). Suppose $g(x) = 1$ if $x = 0$ and $g(x) = 0$ if $x \neq 0$. A derivative of European type has the payoff $g(S(T) - S(0))$ at time of maturity T. (a) Find the price of the derivative at time 0. (b) Suppose the strategy h replicates the derivative. Find $h_S(0)$. The answers in Parts (a) and (b) may contain the martingale probabilities q_u and q_d.

Solution. (a) We have

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{e^{u/2} - e^{-u}}{e^u - e^{-u}}$$

and

$$q_d = 1 - q_u = \frac{e^u - e^{u/2}}{e^u - e^{-u}}.$$

Thus if $v(t)$ denotes the price of the derivative at time t,

$$\begin{aligned} v(2)_{|X_1=u,X_2=u} &= 0 \\ v(2)_{|X_1=u,X_2=d} &= 1 \\ v(2)_{|X_1=d,X_2=u} &= 1 \\ v(2)_{|X_1=d,X_2=d} &= 0 \end{aligned}$$

and

$$\begin{aligned} v(1)_{|X_1=u} &= e^{-r}(q_u 0 + q_d 1) = e^{-r} q_d \\ v(1)_{|X_1=d} &= e^{-r}(q_u 1 + q_d 0) = e^{-r} q_u. \end{aligned}$$

Now

$$v(0) = e^{-r}(q_u e^{-r} q_d + q_d e^{-r} q_u)$$
$$= 2e^{-2r} q_u q_d = 2e^{-u} q_u q_d.$$

(b) Recall that $h(0) = h(1)$ and

$$h_S(1)S(1) + h_B(1)B(1) = v(1)$$

or

$$\begin{aligned} h_S(1)S(0)e^u + h_B(1)B(0)e^r &= e^{-r} q_d \\ h_S(1)S(0)e^d + h_B(1)B(0)e^r &= e^{-r} q_u. \end{aligned}$$

Hence

$$h_S(0) = h_S(1) = e^{-u/2} \frac{1}{S(0)} \frac{q_d - q_u}{e^u - e^{-u}}$$

2. Suppose $Z = (Z_1(t), Z_2(t))_{t \geq 0}$ is a standard Brownian motion in the plane. Find

$$E\left[e^{|Z_1(t)+Z_2(t)|}\right] .$$

Solution. The process $X(t) = \frac{1}{\sqrt{2}} Z_1(t) + \frac{1}{\sqrt{2}} Z_2(t)$, $t \geq 0$, is a standard Brownian motion since $(\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2 = 1$. Hence $X(t) \in N(0,t)$ and it follows that

$$E\left[e^{|Z_1(t)+Z_2(t)|}\right] = E\left[e^{\sqrt{2}|X(t)|}\right] = E\left[e^{\sqrt{2t}|G|}\right]$$

where $G \in N(0,1)$. Thus

$$E\left[e^{|Z_1(t)+Z_2(t)|}\right] = \int_{-\infty}^{\infty} e^{\sqrt{2t}|x| - \frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = 2\int_{-\infty}^{0} e^{-\sqrt{2t}x - \frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$= 2e^t \int_{-\infty}^{0} e^{-\frac{1}{2}(x+\sqrt{2t})^2} \frac{dx}{\sqrt{2\pi}} = 2e^t \int_{-\infty}^{\sqrt{2t}} e^{-\frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}} = 2e^t \Phi(\sqrt{2t}).$$

3. (Black-Scholes model) Suppose $0 < T_0 < T$ and consider a simple derivative of European type with the payoff $Y = \min(S(T_0), S(T))$ at time of maturity T. Find $\Pi_Y(t)$ for all $t \in [0, T_0]$.

Solution. If a and b are real numbers $\min(a,b) + \max(a,b) = a + b$ and, consequently, $\min(a,b) = a + b - \max(a,b) = b - \max(0, b-a)$. Therefore $Y = S(T) - \max(0, S(T) - S(T_0))$ and it follows that

$$\Pi_Y(T_0) = S(T_0) - c(T_0, S(T_0), S(T_0), T).$$

But $c(T_0, S(T_0), S(T_0), T) = S(T_0)c(T_0, 1, 1, T)$, where

$$c(T_0, 1, 1, T) = \Phi(\frac{r + \frac{\sigma^2}{2}}{\sigma}\sqrt{T - T_0}) - e^{-r(T-T_0)}\Phi(\frac{r - \frac{\sigma^2}{2}}{\sigma}\sqrt{T - T_0}).$$

Hence, if we define $a = 1 - c(T_0, 1, 1, T)$,

$$\Pi_Y(T_0) = aS(T_0)$$

and it follows that

$$\Pi_Y(t) = aS(t) \text{ if } 0 \le t \le T_0.$$

4. (Dominance Principle) Show that the European call price $c(t, S(t), K, T)$ is a convex function of K.

5. (Black-Scholes model) Assume $t, T \in \mathbf{R}$, $\tau = T - t > 0$, and $g \in \mathcal{P}$.

(a) Define the price $\Pi_Y(t)$ at time t of a European derivative with payoff $g(S(T))$ at time of maturity T.

(b) Let

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{K} + (r + \frac{\sigma^2}{2})\tau),$$

and $d_2 = d_1 - \sigma\sqrt{\tau}$. Show that

$$c(t, s, K, T) = s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2).$$