2011-03-21
Måndag Lv1

Hållfasthetslära

(ingen dynamik)

Teori; matem. modell
- jämvikt ; samband mellan krafter; inre och yttre } statik
- kinematik; samband mellan förskjutningar och deformationer } geometri
- konstitutiva ekvationer (materialsamband) samband deformation — inre kraft

mtrl.-provning { — materialdata : brotthållfasthet, etc.

Förkunskaper:

Mekanik: allmänna jämviktssamband (statik)

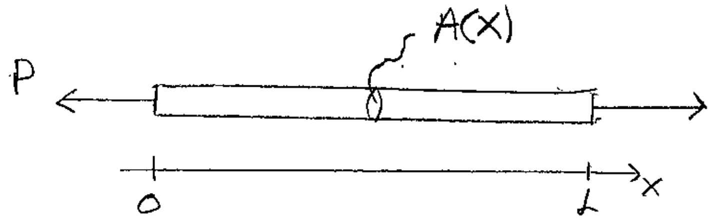
Matematik: integrerings- och deriveringsregler, max/min funk. värden, ordinära diff.ekv., ~~randvillk.~~, egenproblem (kontinuerligt och diskret)

Frivilliga inlämningsuppgifter : 5p

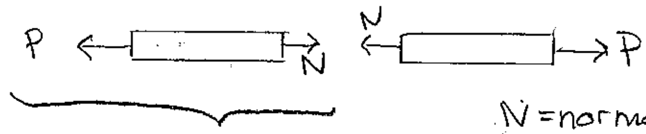
Tentamen : 25 p

Enaxlig elasticitet – stänger

Stång – enaxligt strukturellt element som bär last i längsled



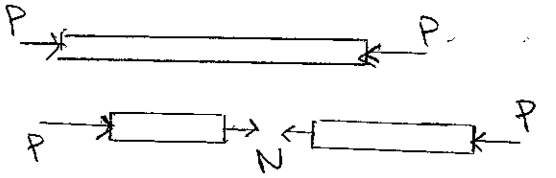
P → deformation
→ inre kraft



N = normalkraft,
$N > 0$ riktad ut från det tänkta snittet

$\leftarrow$: $P - N = 0$

$\underline{N = P}$ drag



$\underline{N = -P}$ tryckkraft

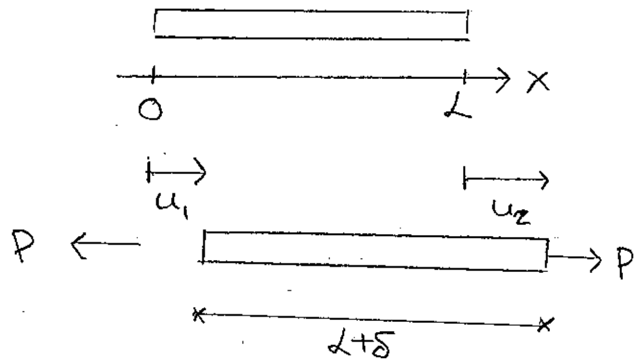
Hur stor kan $N = \pm P$ bli innan stången går av?
Beror av materialets brottspänning (σ_B) och av hur stort A är.

Inte bra: Normalisera: $\sigma = \dfrac{N}{A}$; $\sigma < 0$ tryckspänning etc.

Stången går av då $|\sigma| \geq \sigma_B$

Stål: $\sigma_B = 220 - 1000$ MPa

Deformationen: stången förlängs - $\delta = u_2 - u_1$



Hur stor blir δ? Beror av materialet och längden L. Töjning (deformation)

$$\varepsilon = \text{relativ längdändring} = \frac{\delta}{L}$$

obs genomsnittlig töjning över $(0, L)$.
I allmänhet $\varepsilon = \varepsilon(x)$

Konstitutivt samband: Hookes lag (3:e sambandet)

$$\sigma = E\varepsilon$$

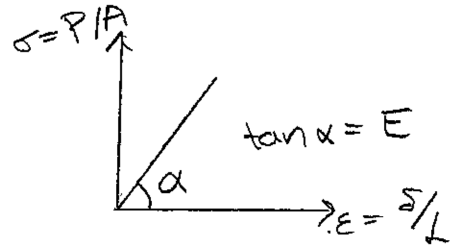
E = elasticitetsmodulen; en materialegenskap

Stål: 180 − 260 GPa

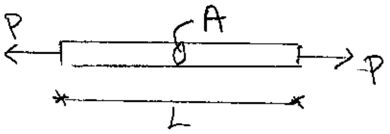
Al : 65–80 GPa

Ti, Cu : ≈110 GPa

Pb : 16 GPa



Sammanställ:



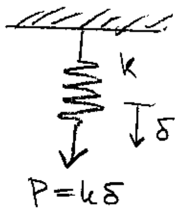
$$\sigma = \frac{N}{A} = \frac{P}{A}$$

$$P = \sigma A = EA\varepsilon = \frac{EA}{L}\delta$$

$$\underline{\underline{P = \frac{EA}{L}\delta}}$$

$\frac{EA}{L}$ = strukturstyvhet

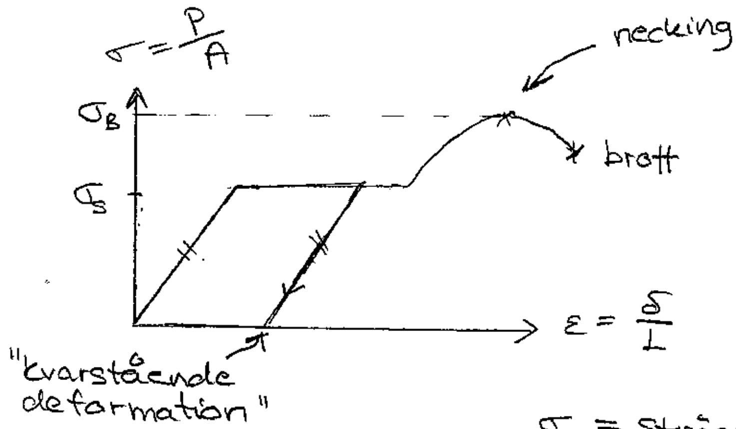
EA = axialstyvhet



$$\underline{\underline{\delta = \frac{PL}{EA}}}$$

OBS: A är här konstant

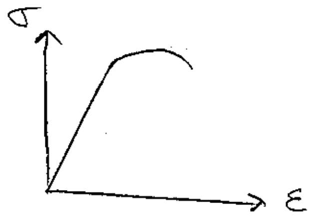
Materialparametrar för stål



mjuk stål

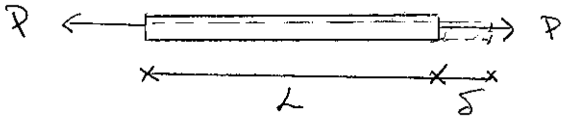
σ_s = sträckgräns

Kallbearbetat stål:



Tvärkontraktion

$$\varepsilon = \frac{\delta}{L}$$



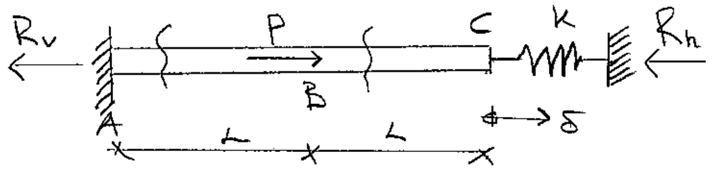
$$\varepsilon_{\text{tvärs}} = \frac{-\Delta d}{D} = -\nu\varepsilon$$

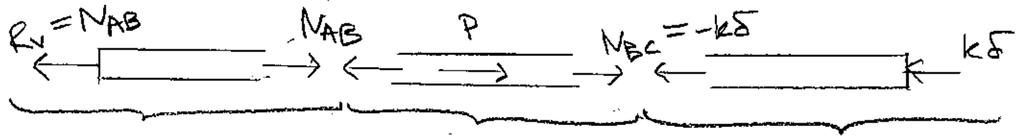
ν = Poissons tal ; $\nu \approx 0,3$ för stål

$-1 < \nu < 0,5$ Gränsfallet $\nu = 0,5$ för inkompressibelt mtrl.

Exempel

E, A givna
(konst.)





$$\longrightarrow: \; N_{BC} + P - N_{AB} = 0$$

$$N_{AB} = P + N_{BC} = P - k\delta$$

$$\delta_{AB} = \frac{N_{AB} \cdot L}{EA} = \frac{(P-k\delta)L}{EA}, \qquad \delta_{BC} = \frac{N_{BC} L}{EA} = \frac{-k\delta L}{EA}$$

$$\delta = \delta_{AB} + \delta_{BC} = \frac{PL}{EA} - \frac{2kL}{EA}\delta, \qquad \underline{\underline{\delta = \frac{PL}{EA + 2kL}}}$$

/end mån

2011-03-23
Onsdag Lv 1

Statiskt bestämda problem:

Alla snittkrafter och stödreaktioner kan bestämmas med enbart jämvikt.
Tvångskrafter p.g.a. t.ex. passningsfel eller temperatur uppkommer aldrig.

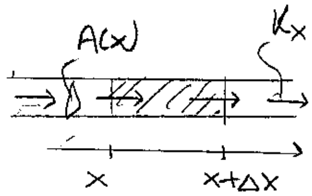
Statiskt obestämda problem:

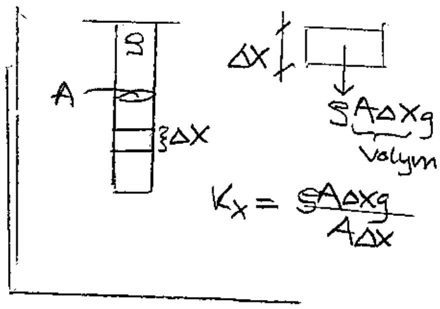
Jämviktsekv. räcker inte till; måste ta hänsyn till kinematik och materialsamband. Tvång kan förekomma.

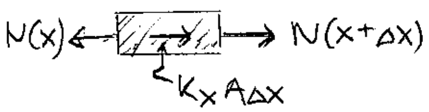
Stångens diff.ekv. (1D elasticitet)

K_x volymlast $\frac{N}{m^3}$ t.ex. egentyngd $K_x = \rho g$

1) Jämvikt







$$\longrightarrow: \quad N(x+\Delta x) - N(x) + K_x A \Delta x = 0$$

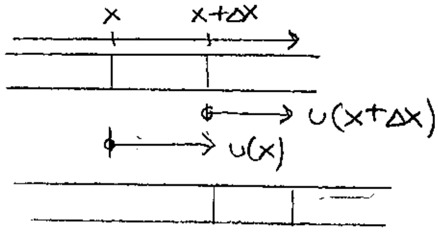
$$-\frac{N(x+\Delta x) - N(x)}{\Delta x} = K_x A$$

$\Delta x \to 0$ ger $-\frac{dN}{dx} = K_x A$

$$N = \sigma A \quad \Rightarrow \quad \boxed{-\frac{d}{dx}[\sigma A] = K_x A}$$

2) Kinematik $u(x)$ = axiell förskjutning

Före belastning:



Def:

$$\varepsilon = \lim_{\Delta x \to 0} \text{(relativ längdförändring)} =$$

$$= \lim_{\Delta x \to 0} \left(\frac{\Delta x + u(x+\Delta x) - u(x) - \Delta x}{\Delta x} \right) = \boxed{\frac{du}{dx} = \varepsilon}$$

3) Konstitutivt samband: $\boxed{\sigma = E\varepsilon}$ (Hooke)

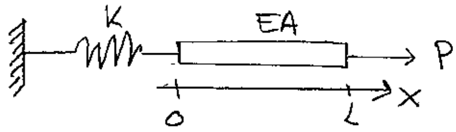
4) Sammanställ

$$K_x A = -\frac{d}{dx}\left[\sigma A\right] \overset{\text{Hooke}}{=} -\frac{d}{dx}\left[EA\varepsilon\right] \overset{\text{Kinematik}}{=} \boxed{-\frac{d}{dx}\left[EA\frac{du}{dx}\right] = K_x A}$$

2 randvillkor behövs, 1 i var ände. Ges på
u (väsentligt r.v.)
och/eller
$\frac{du}{dx}$ (naturligt r.v.)

och/eller komb. av u och $\frac{du}{dx}$ (blandat r.v.)

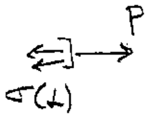
Exempel



$K_x = 0$, EA konst. $\Longrightarrow \; -\dfrac{d^2u}{dx^2} = 0$

Randvillkor

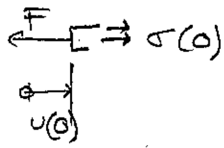
$x = L$:



$\rightarrow$: $P - \sigma(L)A = 0$; $\quad \sigma = E\varepsilon = E\dfrac{du}{dx}$

$$\left.\frac{du}{dx}\right|_{x=L} = \frac{du}{dx}(L) = \frac{P}{EA}$$

$x = 0$:



$\rightarrow$: $\sigma(0)A - F = 0$

$\sigma(0) = E\left.\dfrac{du}{dx}\right|_{x=0}$

$F = ku(0)$

$$\left.\frac{du}{dx}\right|_{x=0} - \frac{k}{EA}u(0) = 0$$

$$\begin{cases} \dfrac{d^2u}{dx^2} = 0 & 0 < x < L \\ \left.\dfrac{du}{dx}\right|_{x=0} - \dfrac{k}{EA}u(0) = 0 & (1) \\ \left.\dfrac{du}{dx}\right|_{x=L} = \dfrac{P}{EA} & (2) \end{cases}$$

$u = C_1 x + C_2$

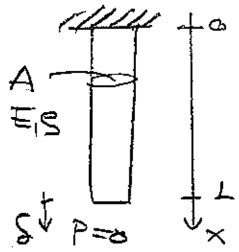
$(2) \Rightarrow C_1 = \dfrac{P}{EA}$

$(1) \Rightarrow C_2 = \dfrac{P}{k}$

$$u(x) = \frac{P}{EA}\left(x + \frac{EA}{k}\right)$$

$$\delta = u(L) = \frac{PL}{EA} + \frac{P}{k}$$

Exempel



$$K_x = \rho g$$

$$-\frac{d}{dx}\left[EA\frac{du}{dx}\right] = \rho g A$$

$$\begin{cases} -\dfrac{d^2u}{dx^2} = \dfrac{\rho g}{E} & 0<x<L \\ u(0) = 0 & \\ \left.\dfrac{du}{dx}\right|_{x=L} = \dfrac{P}{EA} = 0 & P=0 \end{cases}$$

$$\frac{du}{dx} = \frac{-\rho g x}{E} + c_1$$

$$\left.\frac{du}{dx}\right|_{x=L} = 0 \Rightarrow -\frac{\rho g L}{E} + c_1 = 0$$

$$u = \frac{-\rho g x^2}{2E} + \underbrace{\frac{\rho g L}{E}}_{c_1} x + c_2 \; ; \quad u(0) = 0 \Rightarrow c_2 = 0$$

$$u(x) = \frac{\rho g L^2}{2E}\left(2\frac{x}{L} - \left(\frac{x}{L}\right)^2\right)$$

$$\delta = u(L) = \frac{\rho g L^2}{2E}$$

$$\left(\sigma(x) = E\varepsilon = E\frac{du}{dx}\right)$$

ej på tenta, men inlämning

Finit element-metod

g och f givna
h given

$$\begin{cases} -\frac{d}{dx}\left[g(x)\cdot\frac{du}{dx}\right] = f(x) \quad 0<x<L \\ u(0)=0 \\ \left.\frac{du}{dx}\right|_{x=L} = h \end{cases}$$

variationsformulera : inför testfunktion $v=v(x)$

Mult. D.E. med $v(x)$ och integrera:

$$\underbrace{-\int_0^L v\frac{d}{dx}\left[g\frac{du}{dx}\right]dx}_{} = \int_0^L vf\,dx$$

Part. integrera: $-\underbrace{\left[vg\frac{du}{dx}\right]_0^L}_{} + \int_0^L \frac{dv}{dx}g\frac{du}{dx}\,dx$

$$v(L)g(L)\left.\frac{du}{dx}\right|_{x=L} - v(0)g(0)\left.\frac{du}{dx}\right|_{x=0} = \left\{\begin{array}{l}\left.\frac{du}{dx}\right|_{x=L} = h \\ \left.\frac{du}{dx}\right|_{x=0} \text{ är obetydlig}\end{array}\right\} =$$

$= v(L)g(L)h$

välj v så att $v(0)=0$

$$\int_0^L g\frac{dv}{dx}\frac{du}{dx}\,dx = \int_0^L vf\,dx + v(L)g(L)\cdot h, \quad u(0)=0,\ v(0)=0$$

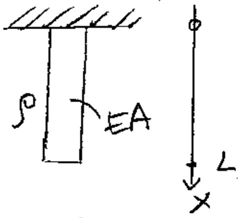
FEM: approximera $u \approx u_h = \sum_{i=1}^{N} a_i\varphi_i(x)$
där $\varphi_i(x)$ är valda basfunk. och a_i är nodvariabler.

$\Rightarrow$ N obekanta, a_i, $i=1,2,\dots,N$
Välj v på N olika sätt $\Rightarrow$ N ekvationer

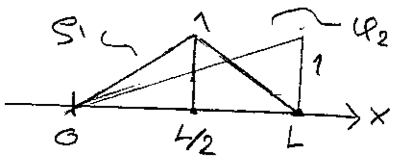
Galerkin: välj $v=\varphi_1$, $v=\varphi_2$, ..., $v=\varphi_N$ $\Rightarrow$

$$\int_0^L g\frac{d\varphi_j}{dx}\underbrace{\sum_{i=1}^{N}a_i\frac{d\varphi_i}{dx}}_{\frac{du}{dx}\approx\frac{du_h}{dx}}dx = \int_0^L \varphi_j f\,dx + \varphi_j(L)g(L)h$$

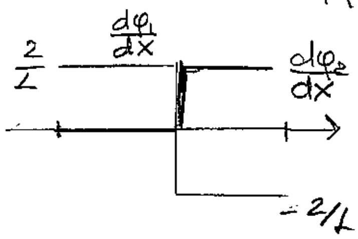
$j=1,2,\dots,N.$

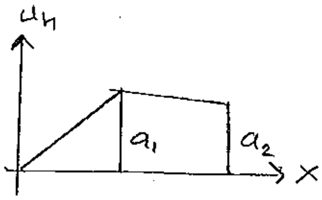


$$g = EA, \quad h = 0 \qquad f = K_x A = \rho g A$$



$$u \approx a_1 \varphi_1 + a_2 \varphi_2$$





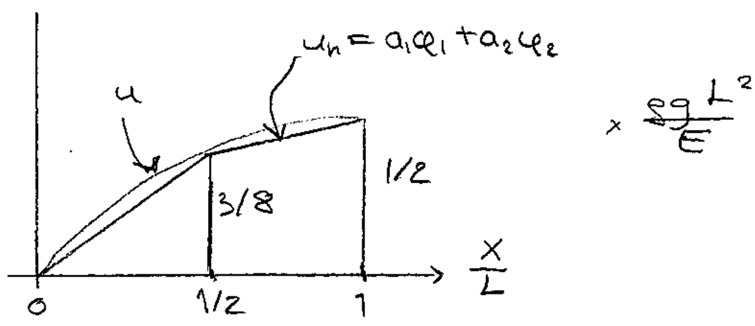
$$v = \varphi_1 : \quad \int_0^{L/2} EA \frac{2}{L}\left[\frac{2}{L} \quad 0\right] dx \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \int_{L/2}^{L} EA \frac{2}{L}\left[-\frac{2}{L} \quad \frac{2}{L}\right] dx \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} =$$

$$= \rho g A \underbrace{\int_0^L \varphi_1 \, dx}_{L/2}$$

$$v = \varphi_2 : \quad \int_0^{L/2} EA \quad 0 \left[\quad \right] dx \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \int_{L/2}^{L} EA \frac{2}{L}\left[\frac{-2}{L} \quad \frac{2}{L}\right] dx \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} =$$

$$= \rho g A \underbrace{\int_0^L \varphi_2 \, dx}_{L/4}$$

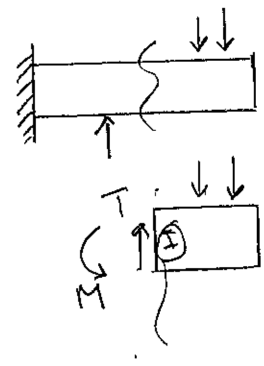
$$\frac{2EA}{L} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{\rho g L A}{4} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{\rho g L^2}{8E} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



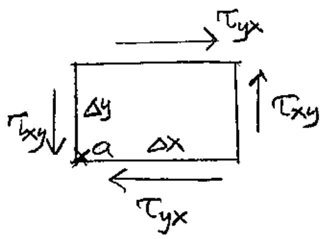
/end ons

Skjuvspänning och skjuvdeformation

2011-03-28
Måndag Lv 2



$$\tau = \frac{T}{A}, \qquad T = \int_A \tau \, dA = \bar{\tau} A$$



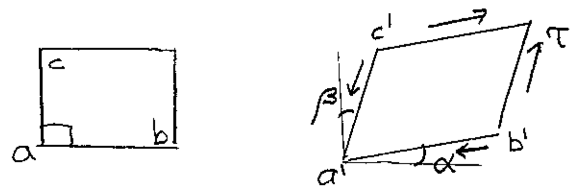
τ skjuvspänning.

tjocklek i z-led

$$\curvearrowleft a: \quad \underbrace{\tau_{xy}\Delta y\, t}_{\text{kraft}}\, \underset{\text{hävarm}}{\Delta x} - \tau_{yx}\Delta x\, t \cdot \Delta y = 0$$

$$\Rightarrow \tau_{xy} = \tau_{yx}$$

Skjuvspänning på ortogonala plan är lika i planens skärningslinje.



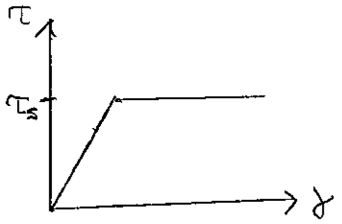
$\gamma = \alpha + \beta$
$= \text{skjuvdeformationen}$

Lineärt elastiskt material: $\tau = G\gamma$ (Hookes lag)
G = skjuvmodulen

Isotropt material (samma egenskaper i alla riktn.):

$$G = \frac{E}{2(1+\nu)} \qquad \text{(ekv. 10-24)}$$

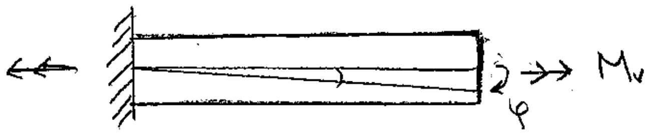
stål: $G \approx 80$ GPa

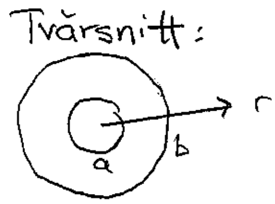


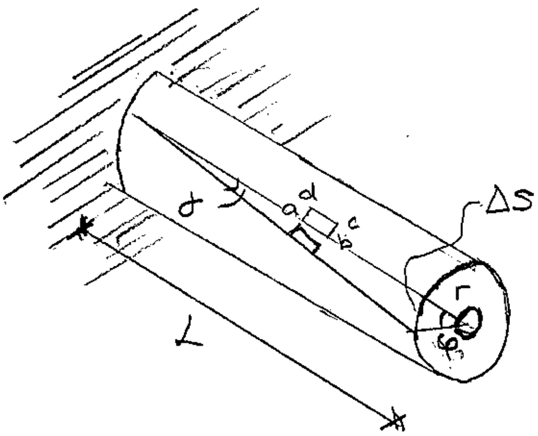
τ_s = sträckgräns vid ren skjuvning $\approx \frac{\sigma_s}{2}$

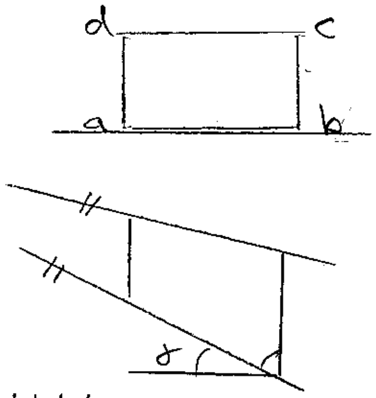
<u>Vridning</u> Vridande moment → vridningsvinkel + böjning
→ τ → σ

Vissa typer av tvärsnitt vrids utan böjning. Dit hör slutna cirkulära tvärsnitt samt slutna tunnväggiga.
Plana tvärsnitt förblir plana; generatriser förblir raka.









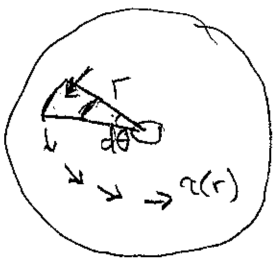
$\Delta S = r\varphi = \gamma L$ ← kinematiskt samband

$$\boxed{\gamma = \frac{\varphi}{L} \cdot r}$$

(jmf. $\varepsilon = \frac{\delta}{L}$)

materialsamband: $\boxed{\tau = G\gamma}$

Sambandet $\tau - M_v$



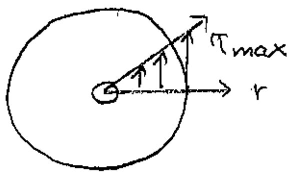
$$dM_v = \overset{\text{hävarm}}{r}\,\underbrace{\tau\,\underbrace{r\,d\theta\,dr}_{\text{yta}}}_{\text{kraft}}$$

$$M_v = \int_0^{2\pi}\int_a^b \tau\, r^2\,dr\,d\theta = 2\pi\int_a^b \tau\, r^2\,dr =$$

$$= \left\{\tau = G\gamma = \frac{G\varphi}{L}r\right\} = \frac{2\pi G\varphi}{L}\int_a^b r^3\,dr =$$

$$= \frac{\pi(b^4 - a^4)}{2}\cdot\frac{G\varphi}{L} = M_v \qquad \boxed{\varphi = \frac{M_v L}{G K}}$$

$$\tau = G\gamma = \frac{G\varphi}{L}\cdot r = \boxed{\frac{M_v\cdot r}{K} = \tau}$$



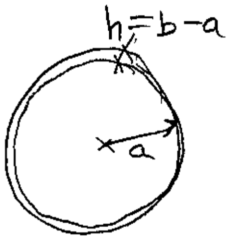
$$\tau_{max} = \frac{M_v b}{K} = \frac{M_v}{W_v}, \quad W_v = \frac{K}{b} = \text{vridmotståndet}$$

Specialfall: tunnväggigt tvärsnitt

$b \approx a$

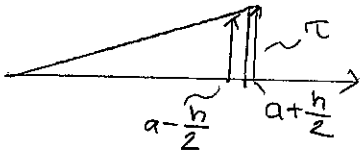
$b+a \approx 2a$

$b-a=h$

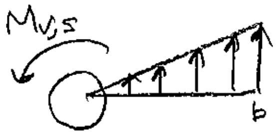


$$K = \frac{\pi}{2}(b^4-a^4) = \frac{\pi}{2}(b^2+a^2)(b^2-a^2) =$$
$$= \frac{\pi}{2}\left((a+b)^2-2ab\right)(b+a)(b-a)$$
$$\approx \frac{\pi}{2}\left((2a)^2-2a^2\right)\cdot 2a\cdot h = 2\pi a^3 h$$

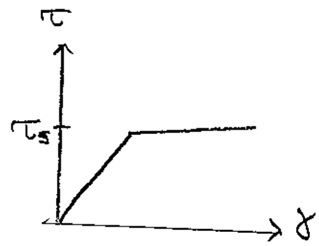
$$\varphi = \frac{M_v L}{GK} = \underline{\frac{M_v L}{2\pi G a^3 h}} \qquad \tau = \frac{M_v r}{K} \approx \frac{M_v a}{2\pi a^3 h} = \frac{M_v}{2\pi a^2 h}$$



Plasticering vid vridning



$\tau_{max} = \tau_s$ vid begynnande plasticering

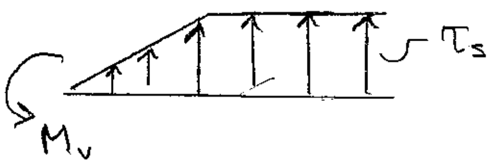


$$\tau_{max} = \frac{M_{v,s}\cdot b}{\frac{\pi}{2}(b^4-a^4)} = \tau_s$$

$$M_{v,s} = \frac{\pi}{2}\,\frac{b^4-a^4}{b}\cdot\tau_s$$

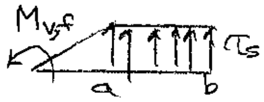
vridmom. vid beg. plast.

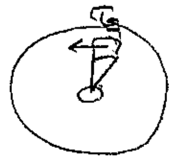
$M_v > M_{v,s}$



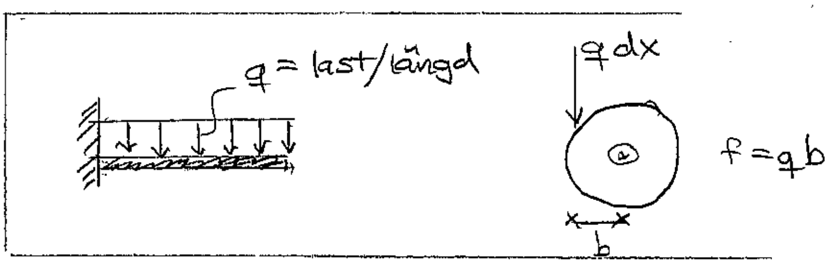
$M_v = M_{v,f}$ då tvärsnittet genomplasticerat

$$M_{v,f} = \int_0^{2\pi}\int_a^b r\,\tau_s\, r\, d\theta\, dr = \underline{2\pi \cdot \frac{b^3-a^3}{3} \cdot \tau_s}$$

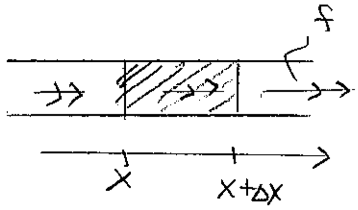


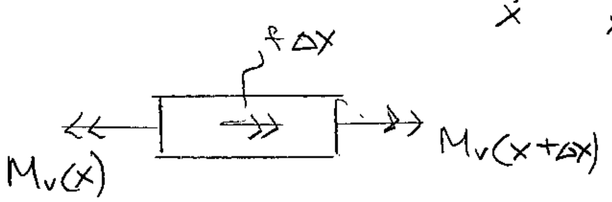


Axelns diffekv.: $f(x)$ moment/längd; yttre belastning



1) Jämvikt





$$\twoheadrightarrow: \; M_v(x+\Delta x) - M_v(x) + f\Delta x = 0$$

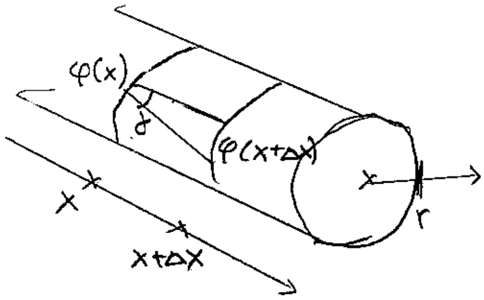
$$-\frac{M_v(x+\Delta x) - M_v(x)}{\Delta x} = f$$

$$\Delta x \to 0 \quad \Rightarrow \quad \boxed{-\frac{dM_v}{dx} = f(x)}$$

2) Kinematik

$$\tan\gamma = \frac{\varphi(x+\Delta x)\cdot r - \varphi(x)\cdot r}{\Delta x} =$$

$$= \frac{\varphi(x+\Delta x) - \varphi(x)}{\Delta x}\cdot r \approx \gamma$$



$\Delta x \to 0 \quad \Rightarrow$

$$\Rightarrow \boxed{\gamma = \frac{d\varphi}{dx}\cdot r}$$

3) Konstitutivt samband $\boxed{\tau = G\gamma}$

4) Sammanställ:

$$f(x) = -\frac{dM_v}{dx} = \left\{\tau = \frac{M_v r}{K},\ M_v = \frac{\tau K}{r}\right\} =$$

$$= -\frac{d}{dx}\left(\frac{\tau K}{r}\right) \underset{\uparrow \text{Hooke}}{=} -\frac{d}{dx}\left[\frac{GK}{r}\gamma\right] = \quad \text{kinematik}$$

$$= -\frac{d}{dx}\left[\frac{GK}{r}\cdot\frac{d\varphi}{dx}\cdot r\right] = \boxed{-\frac{d}{dx}\left[\underbrace{GK\frac{d\varphi}{dx}}_{M_v(x)}\right] = f(x)}$$

2 RV behövs; ett i var ände

Exempel (r.v)

$\varphi(0) = 0$

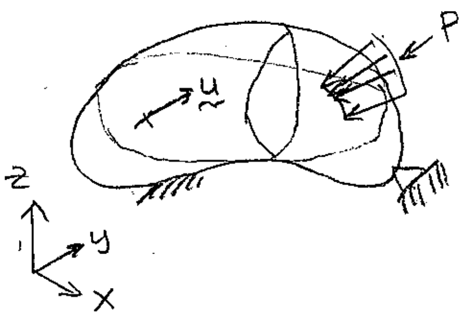
$\twoheadrightarrow$: $M_v(L) = T$, $M_v = GK \frac{d\varphi}{dx}$

$$\left.\frac{d\varphi}{dx}\right|_{x=L} = \frac{T}{GK}$$

/end mån

2011-03-30
Onsdag Lv2

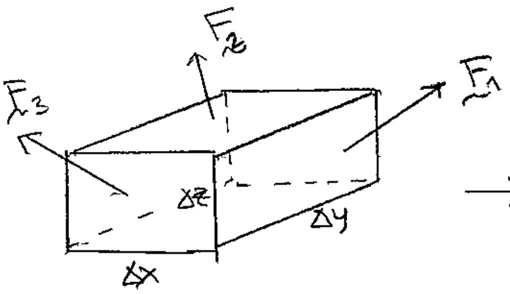
Elasticitetsteori (2- à 3P)
(Kap. 9, 10)



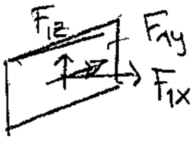
$\underset{\sim}{K} = [K_x \; K_y \; K_z]^T$
(kraft/volym);
$K_i = K_i(x,y,z)$ givna

$\underset{\sim}{u} = [u \; v \; w]^T$ obekanta förskjutningar

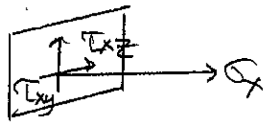
(s. 155)



$\longrightarrow$



$\Delta y, \Delta z \longrightarrow 0$



τ_{ij} verkar på yta med i-axeln som normal och pekar i positiv j-riktning om positiv i-axel är normal.

$\tau_{ij} = \tau_{ji}$ visas med momentekv. (se sid. 155)

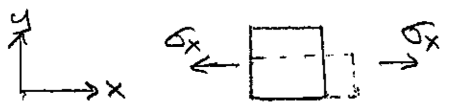
$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix} \qquad \underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix}$$

associerade deformationer.

Jämvikt	$\underset{\sim}{\sigma} \leftrightarrow \underset{\sim}{K}$	(9-9.2.2)
Kinematik	$\underset{\sim}{\varepsilon} \leftrightarrow \underset{\sim}{u}$	(9.3.0)
Material	$\underset{\sim}{\varepsilon} \leftrightarrow \underset{\sim}{\sigma}$	(10.2, 10.4)

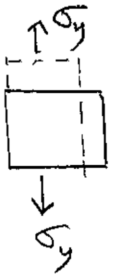
1) Konstitutivt samband

10.2, 10.4 $\qquad \underset{\sim}{\sigma} = \underset{\sim}{D}\,\underset{\sim}{\varepsilon}$



(+)

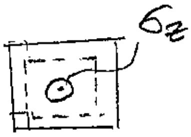
$$\varepsilon_x = \frac{\sigma_x}{E}, \quad \varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = \frac{-\nu\sigma_x}{E}$$



$$\varepsilon_y = \frac{\sigma_y}{E}, \quad \varepsilon_x = \varepsilon_z = \frac{-\nu\sigma_y}{E}$$

(+)

(+)

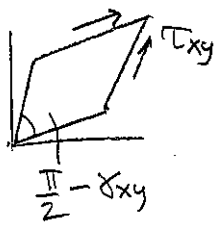


$$\varepsilon_z = \frac{\sigma_z}{E}, \quad \varepsilon_x = \varepsilon_y = \frac{-\nu\sigma_z}{E}$$

(=)

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu(\sigma_y + \sigma_z)\right) \quad (+\alpha\Delta T)$$
$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu(\sigma_x + \sigma_z)\right)$$
$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu(\sigma_x + \sigma_y)\right)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}, \quad G = \frac{E}{2(1+\nu)}$$
$$\gamma_{xz} = \frac{\tau_{xz}}{G} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G}$$



$\underset{\sim}{\varepsilon} = \underset{\sim}{C}\,\underset{\sim}{\sigma}$, $\quad \underset{\sim}{\sigma} = \underset{\sim}{D}\,\underset{\sim}{\varepsilon}$ $\quad$ Hookes lag

(flexibilitet) $\quad$ (styvhet)

$\frac{1}{2}k\delta^2$

Elastisk energi: $\frac{1}{2}\int_V \underset{\sim}{\sigma}^T \underset{\sim}{\varepsilon}\, dV = \frac{1}{2}\int_V \underset{\sim}{\varepsilon}^T \underset{\sim}{D}\,\underset{\sim}{\varepsilon}\, dV > 0$

$\Rightarrow \underset{\sim}{\varepsilon}^T \underset{\sim}{D}\,\underset{\sim}{\varepsilon} > 0 \quad \therefore \underset{\sim}{D}$ måste vara pos. definit.

$\Rightarrow \underline{E > 0}, \; \underline{-1 < \nu < \frac{1}{2}}$ krävs

Reduktion till 2D (10.4)

a) Plan deformation: antar $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

$\Rightarrow \tau_{xz} = \tau_{yz} = 0$

$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu(\sigma_x + \sigma_y)\right) = 0 \Rightarrow \sigma_z = \nu(\sigma_x + \sigma_y)$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \underset{\sim}{D} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \qquad \boxed{\underset{\sim}{\sigma} = \underset{\sim}{D}\,\underset{\sim}{\varepsilon}}$$

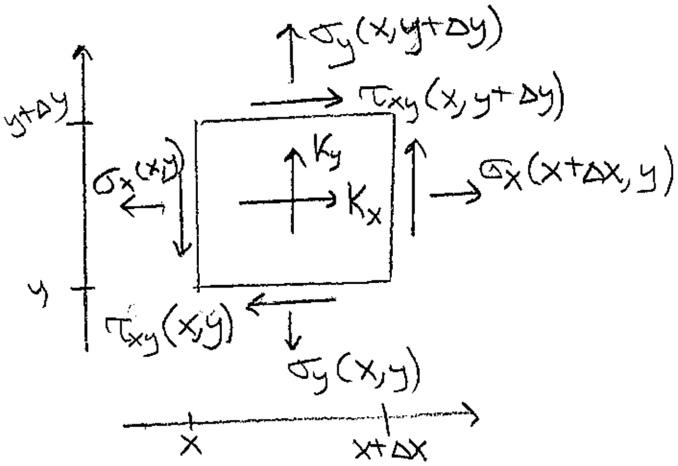
b) Plan spänning: antag att

$\sigma_z = \tau_{yz} = \tau_{xz} = 0 \Rightarrow \gamma_{xz} = \gamma_{yz} = 0$

$$\varepsilon_z = \frac{-\nu}{E}(\sigma_x + \sigma_y) \qquad \Rightarrow \boxed{\underset{\sim}{\sigma} = \underset{\sim}{D}\,\underset{\sim}{\varepsilon}}$$

2) Jämvikt – 2D (se 9.2.2 för 3D) $-\tilde{\nabla}^T \underset{\sim}{\sigma} = \underset{\sim}{K}$

t = tjocklek i z-led. Antas konstant.



$$\rightarrow: \; K_x \cdot \Delta x \cdot \Delta y \cdot t + \sigma_x(x+\Delta x, y) \cdot \Delta y \cdot t$$
$$- \sigma_x(x,y) \cdot \Delta y \cdot t + \tau_{xy}(x, y+\Delta y) \cdot \Delta x \cdot t - \tau_{xy}(x,y) \cdot \Delta x \cdot t = 0$$

$$\times \frac{1}{\Delta x \Delta y \cdot t}: \quad K_x = -\left(\frac{\sigma_x(x+\Delta x, y) - \sigma_x(x,y)}{\Delta x} + \frac{\tau_{xy}(x, y+\Delta y) - \tau_{xy}(x,y)}{\Delta y} \right)$$

$$\Delta x, \Delta y \to 0 \implies K_x = -\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right)$$

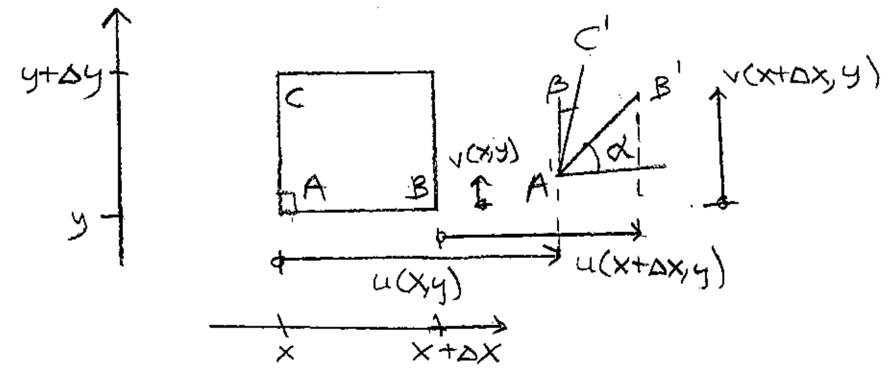
$$\text{p.s.s.} \uparrow: \; -\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} \right) = K_y$$

$$\left[3D: \; -\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) = K_x \text{ etc} \right]$$

$$\underset{\sim}{K} = \begin{bmatrix} K_x \\ K_y \end{bmatrix}, \quad \underset{\sim}{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}; \quad -\tilde{\nabla}^T \underset{\sim}{\sigma} = \underset{\sim}{K}$$

$$\tilde{\nabla}^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \qquad \text{om } t = t(x,y) \quad -\tilde{\nabla}^T(\underset{\sim}{\sigma} t) = \underset{\sim}{K} t$$

3) Kinematik (9.3.0) $\underset{\sim}{u} = [u\ v]^T, \quad \underset{\sim}{\varepsilon} = \tilde{\nabla} \underset{\sim}{u}$



$\alpha \ll 1$

$$\varepsilon_x = \lim_{\Delta x \to 0} \frac{|\overline{A'B'}| - |\overline{AB}|}{|\overline{AB}|} \approx \lim_{\Delta x \to 0} \frac{\Delta x + u(x+\Delta x, y) - u(x,y) - \Delta x}{\Delta x}$$

$$= \underline{\frac{\partial u}{\partial x} = \varepsilon_x}$$

p.s.s. $\varepsilon_y = \dfrac{\partial v}{\partial y}$

$$\tan\alpha \approx \alpha = \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\underbrace{(1+\varepsilon_x)}_{\approx 1 \text{ ty } \varepsilon_x \ll 1} \cdot \Delta x} = \frac{\partial v}{\partial x}$$

P.s.s. $\beta = \dfrac{\partial u}{\partial y}$ $\quad \gamma_{xy} = \alpha + \beta = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

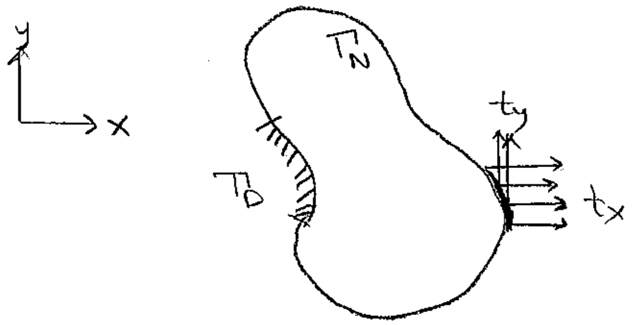
4) Sammantaget:

$$\underset{\sim}{K} = -\tilde{\nabla}^T \underset{\sim}{\sigma} = -\tilde{\nabla}^T \underset{\sim}{D}\, \underset{\sim}{\varepsilon} = \boxed{-\underbrace{\tilde{\nabla}^T \underset{\sim}{D}\, \tilde{\nabla} \underset{\sim}{u}}_{\underset{\sim}{\sigma}} = \underset{\sim}{K}}$$

2 (3) 2:a ord. pde med 2 (3) obekanta u, v $(, w)$

Givet lösningen $\underset{\sim}{u}$ fås $\underset{\sim}{\sigma} = \underset{\sim}{D}\,\underset{\sim}{\varepsilon} = \underset{\sim}{D}\,\tilde{\nabla}\underset{\sim}{u}$

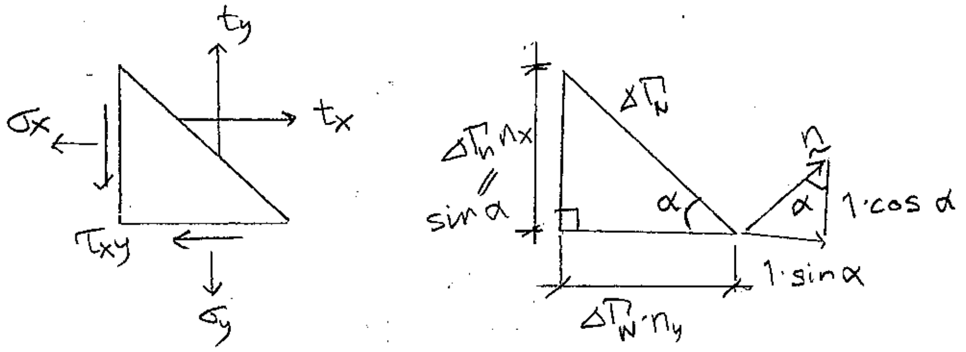
Randvillkor



$\underset{\sim}{t} = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$ last/yta

– tractionvektorn (känd)

$\underset{\sim}{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \underset{\sim}{0}$ på Γ_D



$\underset{\sim}{n} = \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \begin{bmatrix} \sin\alpha \\ \cos\alpha \end{bmatrix}$ utåtnormal $\|\underset{\sim}{n}\|_2 = 1$

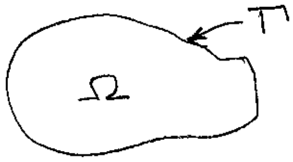
$\rightarrow$: $t_x \cdot \Delta\Gamma_n \cdot t - \sigma_x \cdot \Delta\Gamma_n n_x \cdot t - \tau_{xy} \Delta\Gamma_N n_y \cdot t = 0$

$$\left.\begin{aligned} &\sigma_x n_x + \tau_{xy} n_y = t_x \\ \uparrow: \quad &\tau_{xy} n_x + \sigma_y n_y = t_y \end{aligned}\right\} \text{på } \Gamma_N$$

/end ons Lv 2

$$\left.\begin{aligned} -\tilde{\nabla}^T \underset{\sim}{\sigma} &= \underset{\sim}{K} \\ \underset{\sim}{\sigma} &= \underset{\sim}{D}\underset{\sim}{\varepsilon} \\ \underset{\sim}{\varepsilon} &= \tilde{\nabla}\underset{\sim}{u} \end{aligned}\right\} \rightarrow \left.\begin{aligned} -\tilde{\nabla}^T \underset{\sim}{D}\tilde{\nabla}\underset{\sim}{u} = \underset{\sim}{K} \\ \text{i } \Omega \\ + \text{R.V. på } \Gamma \end{aligned}\right\} \rightarrow \underset{\sim}{u}(x,y,z)$$

2011-04-04
Måndag Lv3



$$\underset{\sim}{\sigma} = \underset{\sim}{D}\tilde{\nabla}\underset{\sim}{u} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix}$$

Plan spänning: $\sigma_z = \tau_{xz} = \tau_{yz} = 0$

Plan töjning: $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

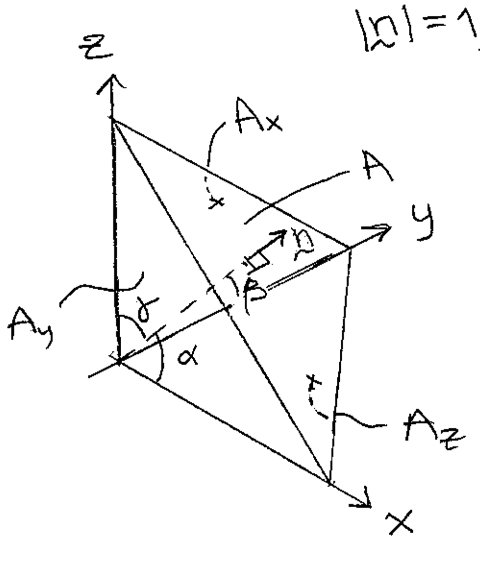
$$\Longrightarrow \tau_{xz} = \tau_{yz} = 0, \quad \sigma_z = \nu(\sigma_x + \sigma_y)$$

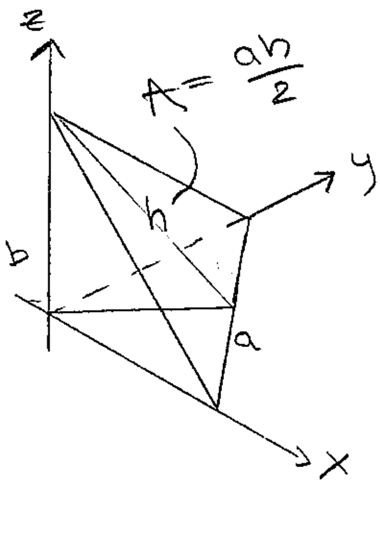
$\underset{\sim}{\sigma} \longleftrightarrow \sigma_s$?

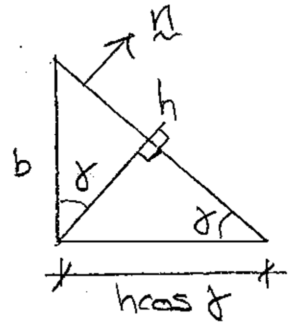
Flythypoteser $\leftarrow$ Huvudspänningar

Spänning i godt. riktning 9.2.3

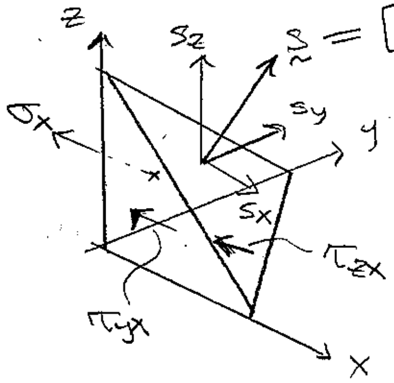
$$|\underset{\sim}{n}| = 1, \quad n = \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix}$$







$$A_z = \frac{ah\cos\gamma}{2} = A\cos\gamma = A n_z$$
$$A_y = A n_y$$
$$A_x = A n_x$$



$\underset{\sim}{s} = [s_x \;\; s_y \;\; s_z]^T$ – spänningsvektor

$\searrow x: \; s_x A - \sigma_x A_x - \tau_{yx} A_y - \tau_{zx} A_z = 0$

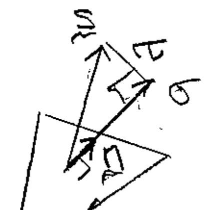
$\Rightarrow \; s_x = \sigma_x n_x + \tau_{yx} n_y + \tau_{zx} n_z$

$\nearrow y: \; s_y = \tau_{xy} n_x + \sigma_y n_y + \tau_{zy} n_z$

$\uparrow z: \; s_z = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z$

$$\underset{\sim}{s} = \underset{\approx}{S}\,\underset{\sim}{n}, \qquad \underset{\approx}{S} = \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

spänningstensorn



Spänningen ortogonalt med A (σ) fås då som

$$\sigma = \underset{\sim}{n}^T \underset{\sim}{s} = \underset{\sim}{n}^T \underset{\approx}{S}\,\underset{\sim}{n}\,; \quad \underset{\sim}{\sigma} = \sigma\,\underset{\sim}{n}$$

$$\underset{\sim}{s} = \sigma\underset{\sim}{n} + \underset{\sim}{\tau}, \quad \underset{\sim}{\tau} = \underset{\sim}{s} - \sigma\underset{\sim}{n}$$

$$\tau = \sqrt{|\underset{\sim}{s}|^2 - \sigma^2}$$

Huvudspänningar 9.2.4

Om $\underset{\sim}{s} \parallel \underset{\sim}{n}$, dvs $\tau = 0$, så kallas σ för en huvudspänning. Vi har då

$$\underset{\sim}{s} = \sigma \underset{\sim}{n} \Rightarrow \underset{\approx}{S}\,\underset{\sim}{n} = \sigma \underset{\sim}{n}$$

$(\underset{\approx}{S} - \sigma \underset{\approx}{I})\underset{\sim}{n} = \underset{\sim}{0}$. Icke-triviala lösn. kräver

$\det(\underset{\approx}{S} - \sigma \underset{\approx}{I}) = 0$ — 3:e-gradspolynom i σ. De tre rötterna $\sigma_1 \geq \sigma_2 \geq \sigma_3$ är huvudspänningarna.

Riktningarna fås som $(\underset{\approx}{S} - \sigma_i \underset{\approx}{I}) \begin{bmatrix} n_{xi} \\ n_{yi} \\ n_{zi} \end{bmatrix} = \underset{\sim}{0}$

$n_{xi}^2 + n_{yi}^2 + n_{zi}^2 = 1$

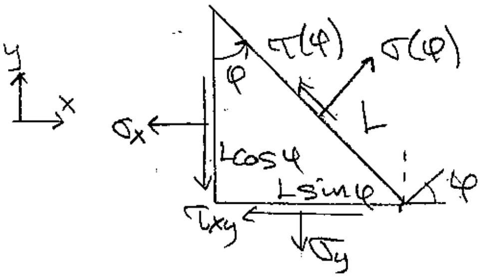
$\sigma_1, \sigma_2, \sigma_3$ är parvis ortogonala.

Spänningar i ett plan ortogonalt mot en huvudspänning

9.2.6 – 9.2.8

(T.ex. plan spänning $\sigma_z = 0$ är en huvudsp.
plan töjning $\sigma_z = \nu(\sigma_x + \sigma_y)$ — " —)

$$S = \begin{bmatrix} \boxed{} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \sigma_z \end{bmatrix}$$



t = tjocklek i z-led

$\nearrow$: $\sigma(\varphi)\cdot Lt - \sigma_x L\cos\varphi\, t\cos\varphi - \sigma_y L\sin\varphi\, t\cdot\sin\varphi$
$- \tau_{xy} L\cos\varphi\, t\sin\varphi - \tau_{xy} L\sin\varphi\cdot t\cdot\cos\varphi = 0$

$$\sigma(\varphi) = \sigma_x\cos^2\varphi + \sigma_y\sin^2\varphi + 2\tau_{xy}\cos\varphi\sin\varphi$$

$$\cos^2\varphi = \frac{1+\cos 2\varphi}{2}, \quad \sin^2\varphi = \frac{1-\cos 2\varphi}{2}, \quad \cos\varphi\sin\varphi = \frac{\sin 2\varphi}{2}$$

$$\Rightarrow \sigma(\varphi) = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\varphi + \tau_{xy}\sin 2\varphi \qquad (1)$$

$$\nwarrow: \ \tau(\varphi) = -\left(\frac{\sigma_x-\sigma_y}{2}\right)\sin 2\varphi + \tau_{xy}\cos 2\varphi \qquad (2)$$

Sök extremvärden på $\sigma(\varphi)$:

$$\frac{d\sigma}{d\varphi} = -\left(\frac{\sigma_x-\sigma_y}{2}\right)\cdot 2\sin 2\varphi + 2\tau_{xy}\cos 2\varphi = 2\tau(\varphi) = 0$$

$\therefore$ σ har max/min då $\tau = 0$
huvudspänningarna är max σ_1 och min σ_3

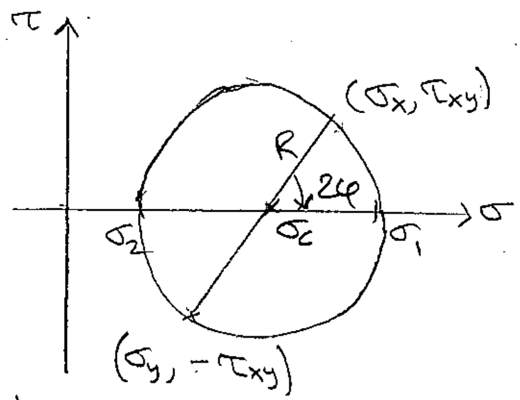
$$(1):\quad \sigma(\varphi)-\left(\frac{\sigma_x+\sigma_y}{2}\right)=\frac{\sigma_x-\sigma_y}{2}\cos 2\varphi+\tau_{xy}\sin 2\varphi$$

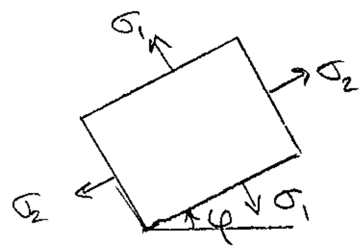
$$(2):\quad \tau(\varphi)=-\left(\frac{\sigma_x-\sigma_y}{2}\right)\sin 2\varphi+\tau_{xy}\cos 2\varphi$$

kvadrera och summera.

$$\left[\sigma(\varphi)-\frac{\sigma_x+\sigma_y}{2}\right]^2+\left[\tau(\varphi)+0\right]^2=\left(\frac{\sigma_x-\sigma_y}{2}\right)^2(\cos^2 2\varphi+\sin^2 2\varphi)$$
$$+\tau_{xy}^2(\sin^2 2\varphi+\cos^2 2\varphi)$$

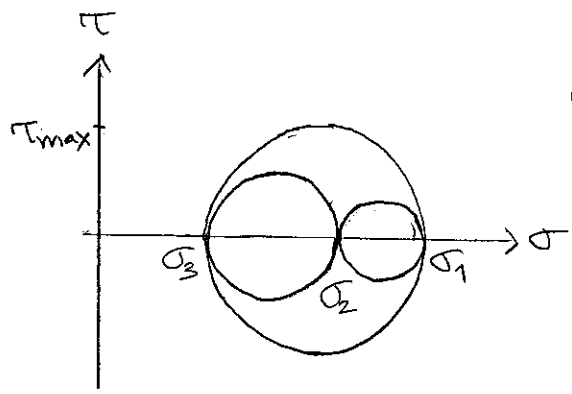
$$(\sigma(\varphi)-\sigma_c)^2+(\tau(\varphi)+0)^2=R^2,\quad \sigma_c=\frac{\sigma_x+\sigma_y}{2}$$
$$R=\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$$





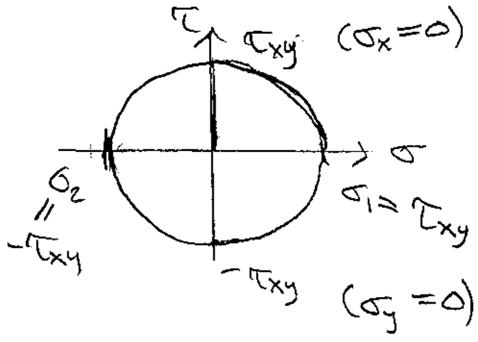
(9-68, 84)

$$\sigma_{1,2}=\sigma_c\pm R=\left(\frac{\sigma_x+\sigma_y}{2}\right)\pm\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$$

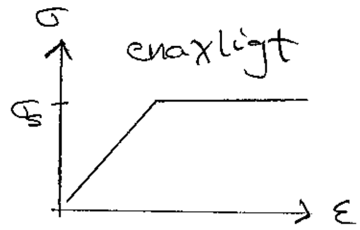


$$\tau_{max}=\frac{1}{2}\max(|\sigma_1-\sigma_2|,|\sigma_2-\sigma_3|,|\sigma_3-\sigma_1|)$$

Ren skjuvning: $\tau_{xy} \neq 0, \quad \sigma_x = \sigma_y = 0$



Flythypoteser (kap. 12)



Fleraxligt: Plasticerar då $\sigma_e = \sigma_s$
σ_e = effektivspänning

$$\sigma_e = f(\sigma_x, \sigma_y, \ldots, \tau_{xz}) \underset{\uparrow}{=} g(\sigma_1, \sigma_2, \sigma_3) =$$

alla riktningar likvärdiga

$$\underset{\uparrow}{=} g(\sigma_1 - p, \sigma_2 - p, \sigma_3 - p)$$

oberoende av hydrostatiskt tryck

$\sigma_e = \sigma_x$ vid enaxlig belastning

Tresca

$$\sigma_e = \underbrace{\max\left(|\sigma_1-\sigma_2|, |\sigma_2-\sigma_3|, |\sigma_1-\sigma_3|\right)}_{2\tau_{max}} = \sigma_s$$

von Mises

$$\sigma_e = \sqrt{\sigma_x^2+\sigma_y^2+\sigma_z^2-\sigma_x\sigma_y-\sigma_x\sigma_z-\sigma_y\sigma_z+3\left(\tau_{xy}^2+\tau_{xz}^2+\tau_{yz}^2\right)}$$

$$= \frac{1}{2}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2}$$

$\sigma_e^T \geqslant \sigma_e^M$; Tresca mer konservativ.

/end mån Lv 3

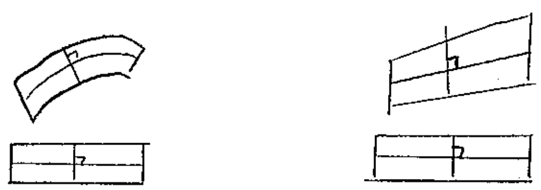
Teknisk balkteori 7.1–7.7

2011-04-06
Onsdag Lv3

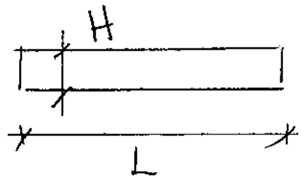
1D strukturelement som belastas transversellt.

Euler – Bernoulliteori:

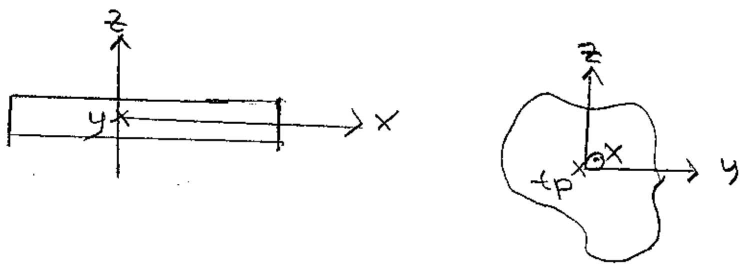
- Plana tvärsnitt ortogonalt mot medellinjen förblir plana
  och ortogonala mot medellinjen
  $\Rightarrow$ skjuvdeformationer försummas

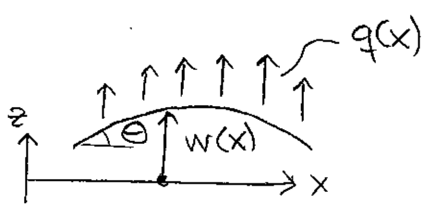


Fungerar om $\frac{L}{H} \geq 5$



Koordinatsystem:



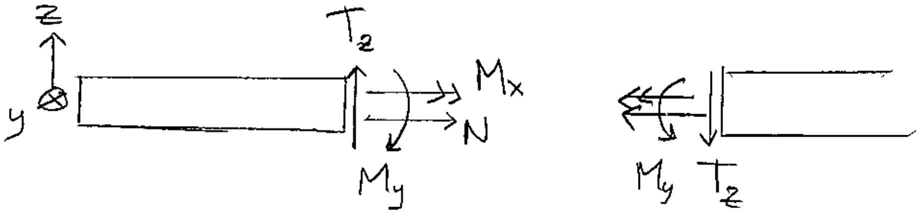


$\frac{\text{kraft}}{\text{längd}}$

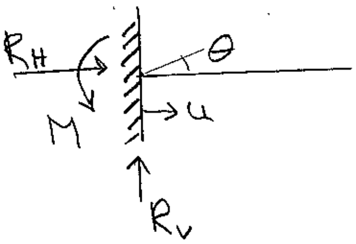
$\tan\theta = w' \approx \theta$

$\theta \ll 1$ antas

Snittkrafter



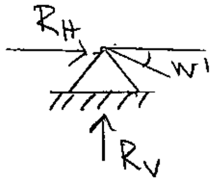
Stödreaktioner (sid. 68, 96)

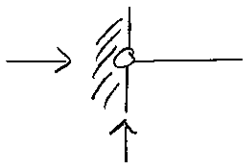


$\theta = w' = 0 \Rightarrow M \neq 0$

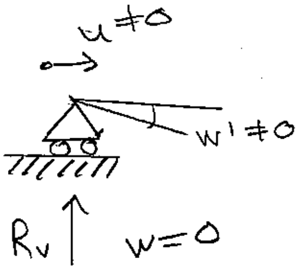
$w = 0 \Rightarrow R_V \neq 0$

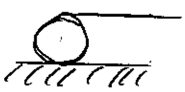
$u = 0 \Rightarrow R_H \neq 0$



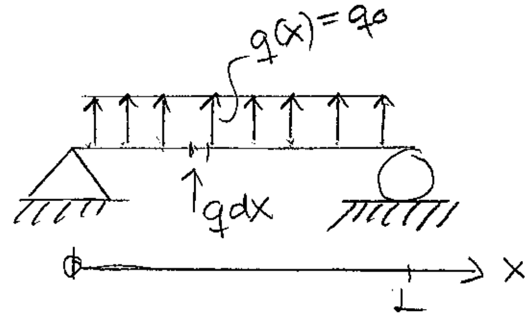


$u = 0$
$w = 0$
$w' \neq 0$



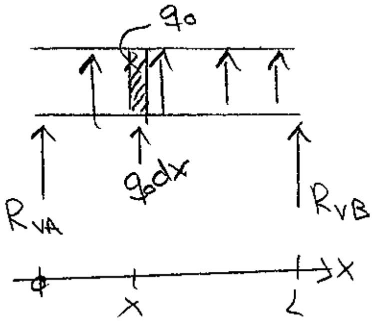


Exempel



Beräkna stödreaktioner, $T(x)$ och $M(x)$

$$Q = \int_0^L q\,dx = q_0 L \quad \text{kraftresultanten}$$

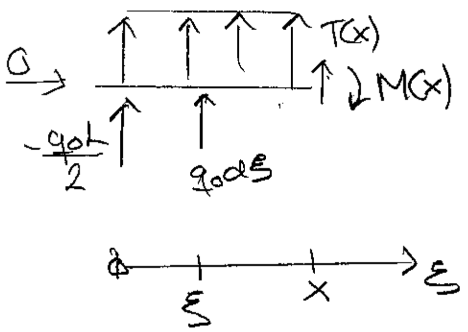


$\rightarrow: \ R_H = 0$

$$\curvearrowleft A: \ R_{VB} \cdot L + \underbrace{\int_0^L x\, q_0\, dx}_{q_0 L \cdot \frac{L}{2}} = 0$$

$$R_{VB} = -\frac{q_0 L}{2} = \frac{-Q}{2}$$

$$\uparrow: \ R_{VA} + R_{VB} + \int_0^L q_0\, dx = 0 \qquad R_{VA} = \frac{-q_0 L}{2} = \frac{-Q}{2}$$



$$\uparrow: \ T(x) - \frac{q_0 L}{2} + \int_0^x q_0\, d\xi = 0$$

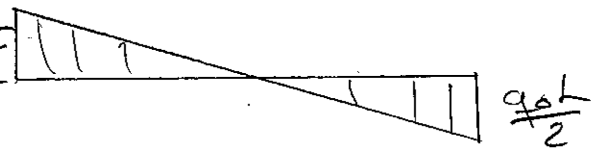
$$T(x) = \frac{q_0 L}{2} - q_0 x$$

$\overset{\curvearrowright}{x}$: $M(x) - \frac{q_0 L}{2}\cdot x + \underbrace{\int_0^x (x-\xi)\, q_0 \, d\xi}_{} = 0$

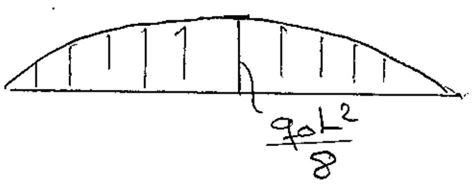
$$\frac{q_0}{-2}\left[(x-\xi)^2\right]_0^x = \frac{q_0 x^2}{2} = q_0 x \cdot \frac{x}{2}$$

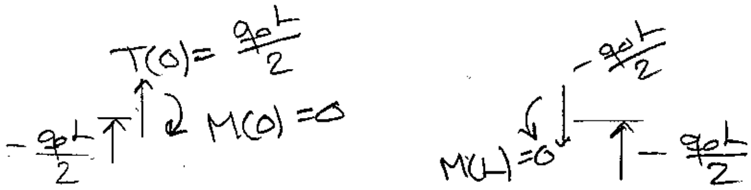
$$\underline{\underline{M(x) = \frac{q_0 L x}{2} - \frac{q_0 x^2}{2}}}$$

$T(x)$:

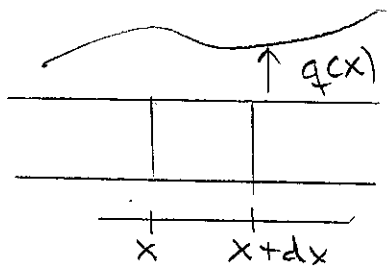


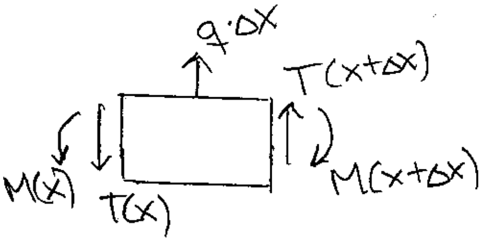
$M(x)$:





$T = \frac{dM}{dx}$?





$\uparrow$: $T(x+\Delta x) - T(x) + q \cdot \Delta x = 0$

$$\frac{T(x+\Delta x) - T(x)}{\Delta x} = -q$$

$$\Delta x \to 0 \quad \Rightarrow \quad \underline{\frac{dT}{dx} = -q}$$

$\overset{\curvearrowright}{x+\Delta x}$: $M(x+\Delta x) - M(x) - T(x)\Delta x + q\Delta x \frac{\Delta x}{2} = 0$

$$\frac{M(x+\Delta x) - M(x)}{\Delta x} = T(x) - \frac{q\Delta x}{2}$$ (Schwedlers satser)

$\Delta x \to 0$: $\dfrac{dM}{dx} = T$

$$\frac{d^2M}{dx^2} = \frac{dT}{dx}$$

$$\boxed{\frac{d^2M}{dx^2} = -q}$$

Exemplet igen: $\dfrac{d^2M}{dx^2} = -q_0 \Rightarrow M(x) = \dfrac{-q_0 x^2}{2} + C_1 x + C_2$

$M(0) = 0 \Rightarrow C_2 = 0$

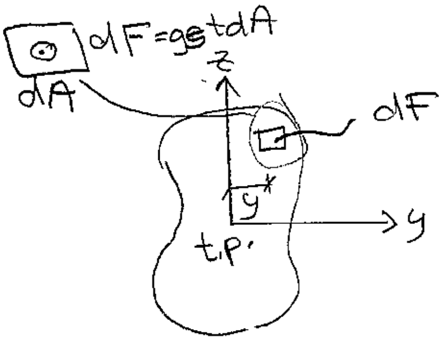
$M(L) = 0 \Rightarrow -\dfrac{q_0 L^2}{2} + C_1 L = 0, \quad C_1 = \dfrac{q_0 L}{2}$

$$M(x) = \frac{q_0 L}{2} x - \frac{q_0 x^2}{2}$$

$$T = \frac{dM}{dx} = \frac{q_0 L}{2} - q_0 x$$

Ytyngdpunkt

t.p. = tyngdpunkt



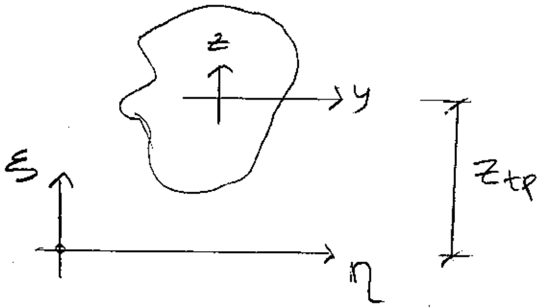
$dF = \varrho g t\, dA$, pekar uppåt (x-led)

$$dM_z = -dF \cdot y$$

$$-M_z = \int_A y \varrho g t\, dA = \varrho g t \underbrace{\int_A y\, dA}_{S_z} = 0$$

$$\boxed{S_z = 0\,, \quad S_y = \int_A z\, dA = 0}$$

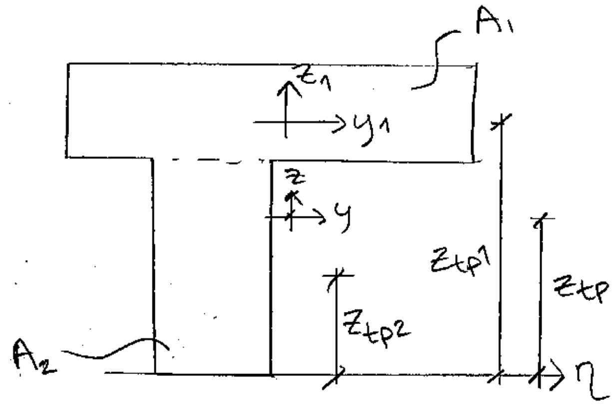
Statiskt moment m.a.p. en godtycklig axel η:



$$S_\eta = \int_A \xi\, dA = \int_A (z + z_{tp})\, dA = \underbrace{\int_A z\, dA}_{S_y = 0} + z_{tp} \int_A dA =$$

$$= \underline{z_{tp} \cdot A}$$

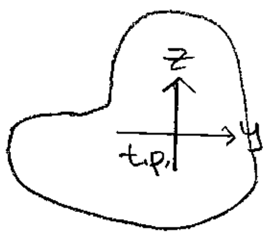
Delareor:



$$S_\eta = \int_{A = A_1 + A_2} \xi \, dA = \underline{(A_1 + A_2) z_{tp}} = \int_{A_1} \xi \, dA + \int_{A_2} \xi \, dA =$$

$$= \underline{A_1 z_{tp1} + A_2 z_{tp2}} \qquad z_{tp} = \frac{\sum_i A_i z_{tpi}}{\sum_i A_i}$$

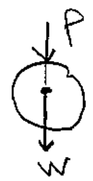
Yttröghetsmoment



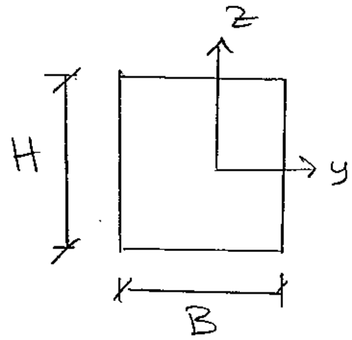
$$I_y = \int_A z^2 \, dA, \quad I_z = \int_A y^2 \, dA$$

$$D_{yz} = \int_A yz \, dA \quad \text{deviationsmoment}$$

Går alltid att hitta ett axelkors (y, z) så att $D_{yz} = 0$. $\Longrightarrow$ plan böjning

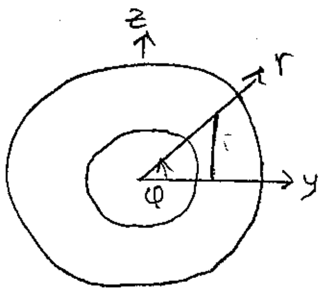


$D_{yz} \neq 0$ ger s.k. skev böjning



$$I_y = \int_{-\frac{H}{2}}^{H/2} \int_{-\frac{B}{2}}^{B/2} z^2 \, dy \, dz = \frac{B}{3}\left[z^3\right]_{-H/2}^{H/2} =$$

$$= \frac{BH^3}{12}, \qquad I_z = \frac{HB^3}{12}$$

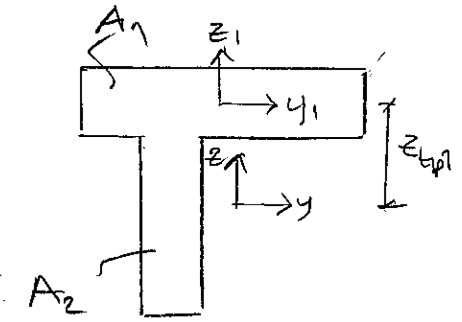


$$I_y = \int_A z^2 dA = \left\{ \begin{array}{l} dA = r\,d\varphi\,dr \\ z = r\sin\varphi \end{array} \right\} =$$

$$= \int_0^{2\pi} \int_a^b r^3 \sin^2\varphi \, dr \, d\varphi =$$

$$= \left[\frac{r^4}{4}\right]_a^b \left[\frac{\varphi}{2} - \frac{\sin 2\varphi}{4}\right]_0^{2\pi} = \underline{\underline{\frac{b^4 - a^4}{4} \cdot \pi}}$$

Sammansatt tvärsnitt:



$$I_y = \int_A z^2 dA = \int_{A_1} z^2 dA + \int_{A_2} z^2 dA =$$

$$= I_{y1} + I_{y2}$$

$$I_{y1} = \int_{A_1} z^2 dA = \int_{A_1} (z_1 + z_{tp1})^2 dA =$$

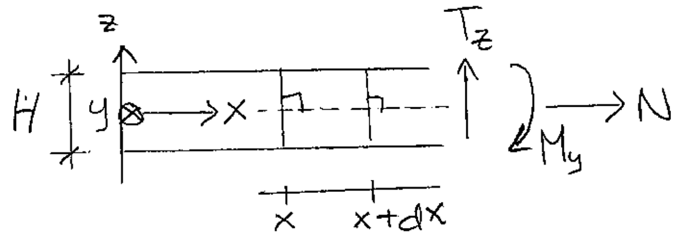
$$= \int_{A_1} z_1^2 dA + z_{tp1}^2 \int_{A_1} dA + 2 z_{tp1} \underbrace{\int_{A_1} z_1 \, dA}_{S_{y1} = 0} =$$

$$= I_{y,tp1} + z_{tp1}^2 \cdot A_1$$

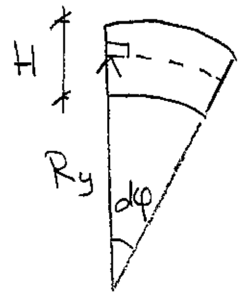
Steinerssats: $\boxed{I_y = \sum_i \left(I_{y,tpi} + z_{tpi}^2 \cdot A_i \right)}$

(Böj) normalspänning σ

2011-04-11
Måndag Lv 4



Plana tvärsnitt ortogonala mot medellinjen, förblir plana och ortogonala. (Euler – Bernoullis antagande)

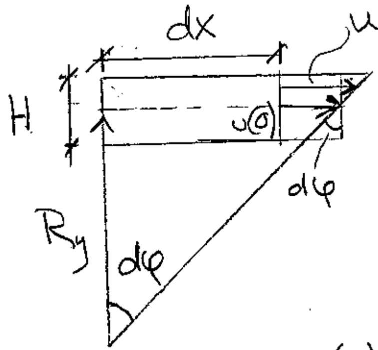


R_y = krökningsradien

$R_y \gg H$

$K_y = \frac{1}{R_y}$ = krökningen.

kappa



$u(z) = u(0) + z\,d\varphi$

$$u(z=0) = R_y d\varphi - dx$$

$$\varepsilon(0) = \frac{u(0)}{dx} = R_y \frac{d\varphi}{dx} - 1$$

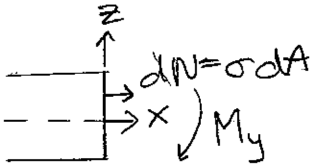
$$\frac{d\varphi}{dx} = \frac{1}{R_y}\left(\varepsilon(0) + 1\right)$$

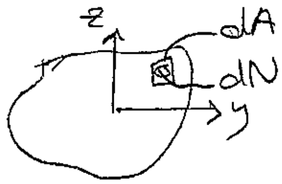
$$\varepsilon(z) = \frac{u(z)}{dx} = \frac{u(0)}{dx} + z\frac{d\varphi}{dx} =$$

$$= \varepsilon(z) = \varepsilon(0) + \frac{z}{R_y}(\varepsilon(0)+1) = \frac{z}{R_y} + \varepsilon(0)\underbrace{\left(1+\frac{z}{R_y}\right)}_{\approx 1} =$$

$$= \{ z \leq H \ll R_y \} = \underline{\varepsilon(0) + K_y z = \varepsilon(z)}$$

Hooke: $\sigma = E\varepsilon$ ger $\sigma(z) = E\varepsilon(0) + E K_y z$ (1)





$$N = \int_A dN = \int_A \sigma dA \underset{(1)}{\overset{\uparrow}{=}} \int_A (E\varepsilon(0) + E K_y z) dA =$$

$$= E\varepsilon(0)\int_A dA + E K_y \int_A z dA = EA\varepsilon(0) + E K_y \underset{=0}{S_y} = EA\varepsilon(0)$$

$$\underline{\varepsilon(0) = \frac{N}{EA} \quad (2)}$$

$$M_y = \int_A z dN = \int_A \sigma z dA \overset{(1)}{=} \int_A E\varepsilon(0)\cdot z dA + E K_y \underbrace{\int_A z^2 dA}_{I_y} =$$

$$= E\varepsilon(0)\cdot \underset{=0}{S_y} + E I_y K_y = E I_y K_y$$

EI_y = böjstyvheten (jmf. EA, GK)

$$\boxed{K_y = \frac{M_y}{EI_y}} \quad (3) \qquad \text{Konstitutivt samband}$$

$$\boxed{\frac{d^2M}{dx^2} = -q} \qquad \text{jämvikt}$$

$$M_z = -\int_A y\,dN = -\int_A \sigma y\,dA \overset{(1)}{=} -E\varepsilon(0)\int_A y\,dA - E\kappa_y \int_A yz\,dA =$$

$$= -E\varepsilon(0)\underbrace{S_z}_{=0} - E\kappa_y D_{yz} = 0$$ omm D_{yz} plan böjning

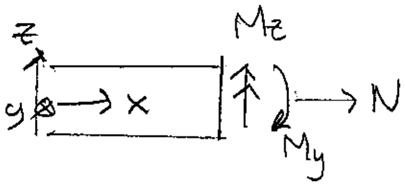
(2) och (3) in i (1) ger: $\sigma = E\dfrac{N}{EA} + E\dfrac{M_y}{EI_y}\cdot z =$

$$= \underline{\frac{N}{A} + \frac{M_y z}{I_y}} \qquad (7\text{-}26)$$

Med $N=0$ fås $|\sigma|_{max} = \dfrac{|M|_{max}|z|_{max}}{I_y} = \dfrac{|M|_{max}}{W_b}$

$W_b = \dfrac{I_y}{|z|_{max}}$ = böjmotståndet

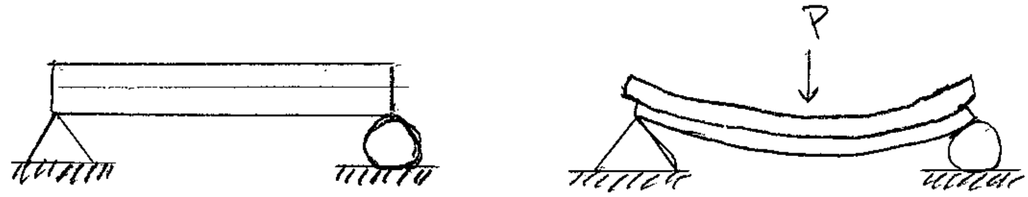
$\underline{M_z \neq 0:}$

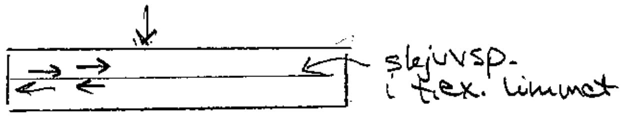


$$\rightarrow \quad \sigma = \frac{N}{A} + \frac{M_y z}{I_y} - \frac{M_z y}{I_z} \qquad (7\text{-}91)$$

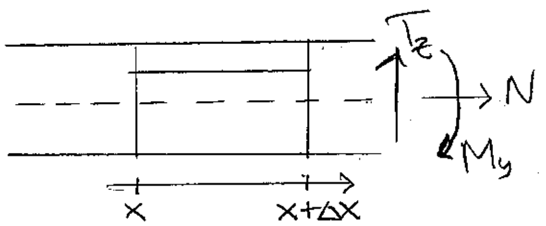
obs: $D_{yz} = 0$ krävs fortfarande

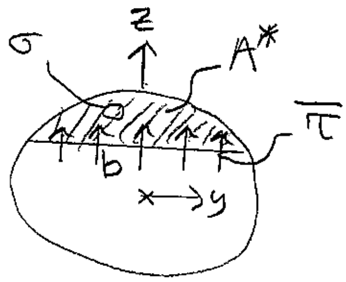
(Böj)skjuvspänning τ

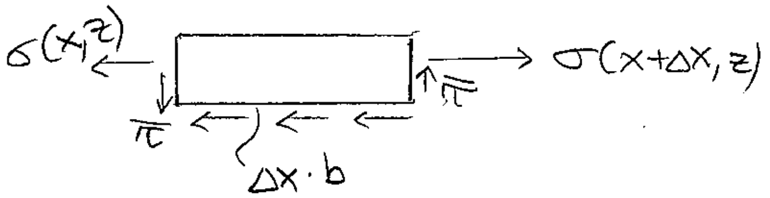




τ är normalt inte dimensionerande ($\sigma \gg \tau$), men kan behövas för att dimensionera lim, svetsar, skruvar etc.







$$\longrightarrow: \quad \int_{A^*} \sigma(x+\Delta x, z)\,dA - \int_{A^*} \sigma(x,z)\,dA - \bar{\tau} b \Delta x = 0$$

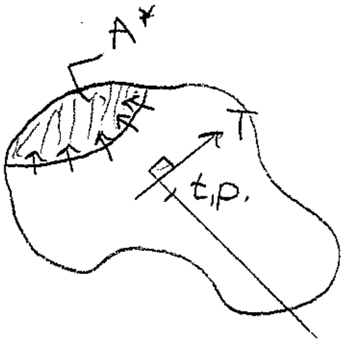
$$\bar{\tau} b = \int_{A^*} \frac{\sigma(x+\Delta x, z) - \sigma(x,z)}{\Delta x}\,dA \,, \quad \Delta x \to 0 \text{ ger}$$

$$\bar{\tau} b = \int_{A^*} \frac{d\sigma}{dx}\,dA = \left\{ \frac{d\sigma}{dx} = \frac{d}{dx}\left[\frac{N}{A} + \frac{M_y z}{I_y}\right] = \right.$$

$$= \left\{ \text{Antag } N, A \text{ o } I_y \text{ konstanta} \right\} = \frac{dM_y}{dx}\frac{z}{I_y} = T_z \frac{z}{I_y} \Big\} =$$

$$= \int_{A^*} \frac{T_z z}{I_y}\, dA = \frac{T_z}{I_y} \cdot \underbrace{\int_{A^*} z\, dA}_{S_{yA^*}} = \frac{T_z S_{yA^*}}{I_y}, \quad \bar{\tau} = \frac{T_z S_{yA^*}}{I_y b}$$

$$\boxed{\sigma = \frac{N}{A} + \frac{M_y z}{I_y}}$$



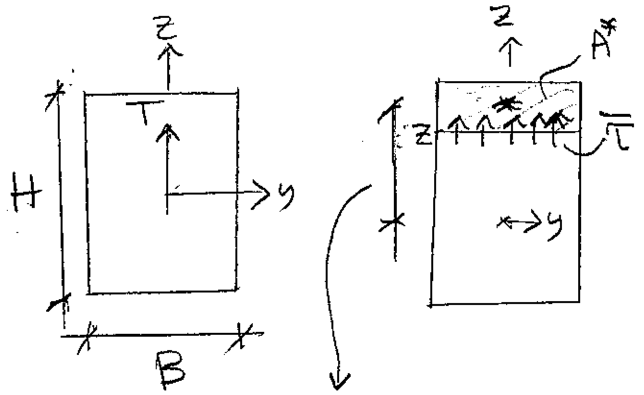
<u>Allmänt:</u> $\bar{\tau} = \dfrac{T S_{A^*}}{I b}$

$\bar{\tau}$ = medelskjuvsp. ⊥ snittlinjen med längd b

I = area-tröghetsmoment m.a.p. en axel genom t.p. och ⊥ T

S_{A^*} stat. moment av A^* m.a.p. samma axel.

<u>Exempel</u> Bestäm $\bar{\tau}(z)$



$b = B$

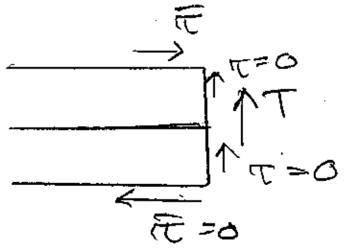
$I = I_y = \dfrac{BH^3}{12}$

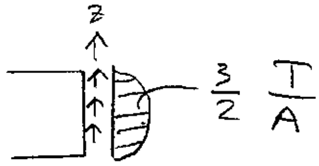
$A^* = B\left(\dfrac{H}{2} - z\right)$

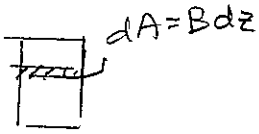
$$S_{A^*} = A^* \overbrace{\left(\frac{H}{2} - \frac{1}{2}\left(\frac{H}{2} - z\right)\right)} = \frac{BH^2}{8}\left(1 - 4\left(\frac{z}{H}\right)^2\right)$$

$$\bar{\tau} = \frac{T \cdot \frac{BH^2}{8}\left(1-4\left(\frac{z}{H}\right)^2\right)}{\frac{BH^3}{12} \cdot B} = \underline{\frac{3T}{2BH}\left(1-4\left(\frac{z}{H}\right)^2\right)}$$

$$\bar{\tau}\left(\pm\frac{H}{2}\right) = 0, \quad \bar{\tau}(0) = \frac{3T}{2BH} = \frac{3}{2}\,\frac{T}{A}$$







$$\int_A \bar{\tau}\,dA = \frac{3T}{2BH}\int_A \left(1-4\left(\frac{z}{H}\right)^2\right)dA = \{dA = Bdz\} =$$

$$= \frac{3T}{2H}\int_{-\frac{H}{2}}^{\frac{H}{2}} \left(1-4\left(\frac{z}{H}\right)^2\right)dz = \frac{3T}{2H}\left[z - \frac{4}{3}\left(\frac{z}{H}\right)^3 \cdot H\right]_{-\frac{H}{2}}^{\frac{H}{2}} =$$

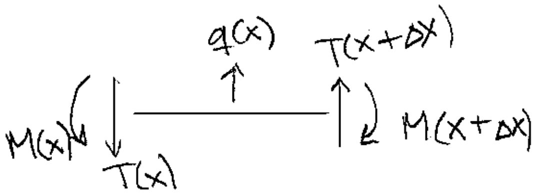
$$= \frac{3T}{2H}\left(H - \frac{H}{6}\cdot 2\right) = \underline{\underline{T}}$$

lend mån Lv 4

Elastiska linjens ekvation

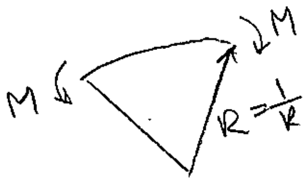
2011-04-13
Onsdag Lv 4

1) Jämvikt



$$\uparrow: \quad \frac{dT}{dx} = -q$$
$$\circlearrowleft: \quad T = \frac{dM}{dx}$$
$$\rightarrow \boxed{-q = \frac{d^2M}{dx^2}}$$

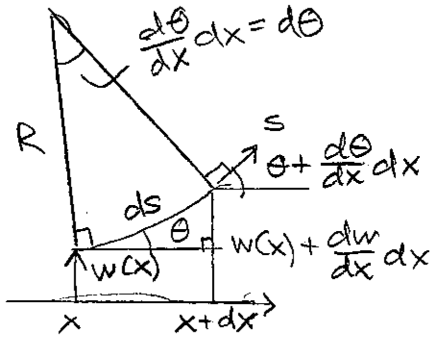
2) Konstitutivt samband



$$\boxed{M = EI \cdot \kappa}$$

3) Kinematiskt samband

$w - \kappa$



$$ds = R\,d\theta, \quad \frac{1}{R} = \frac{d\theta}{ds} = \pm\kappa$$

$$ds = \sqrt{dx^2 + \left(\frac{dw}{dx}dx\right)^2}$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dw}{dx}\right)^2} \quad (1)$$

$$\frac{dw}{dx} = \tan\theta \; ; \quad \frac{d^2w}{dx^2} = \frac{d}{dx}\left[\tan\theta\right] = \frac{d\theta}{dx}\cdot\frac{d}{d\theta}\left[\tan\theta\right] =$$

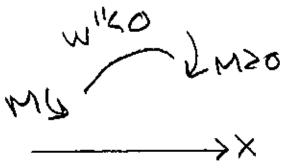
$$= \frac{d\theta}{dx}\left(1+\tan^2\theta\right) = \underline{\frac{d\theta}{dx}\left(1+\left(\frac{dw}{dx}\right)^2\right) = \frac{d^2w}{dx^2}} \qquad (2)$$

$$\frac{d\theta}{dx} = \frac{d\theta}{ds}\cdot\frac{ds}{dx} \underset{(1)}{=} \underline{\frac{d\theta}{ds}\sqrt{1+\left(\frac{dw}{dx}\right)^2} = \frac{d\theta}{dx}} \qquad (3)$$

(3) i (2) ger:

$$\frac{d^2w}{dx^2} = \frac{d\theta}{ds}\sqrt{1+\left(\frac{dw}{dx}\right)^2}\left(1+\left(\frac{dw}{dx}\right)^2\right)$$

$$\frac{d\theta}{ds} \underset{(4)}{=} \underline{\kappa = \frac{d^2w}{dx^2}\Big/\left[1+\left(\frac{dw}{dx}\right)^2\right]^{3/2}}$$



Vill ha $\kappa>0$ då konvext uppåt

Antag $\frac{dw}{dx} \ll 1 \Rightarrow \boxed{\kappa \approx -w''}$

4) <u>Sammanställ</u>:

$$q = -\frac{d^2M}{dx^2} = -\frac{d^2}{dx^2}\left[EI\kappa\right] = \frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right] = q$$

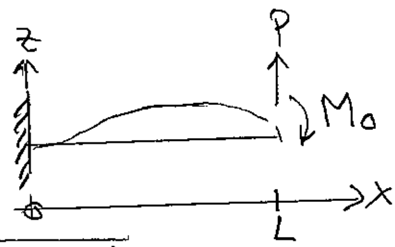
4 randvillkor behövs. Ges w, w', w'', w'''

Lösningar ger $w(x)$ och sedan

$$M = EI\kappa = -EI\frac{d^2w}{dx^2} \qquad \text{och}$$

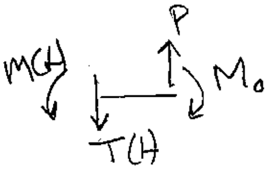
$$T = \frac{dM}{dx} = -\frac{d}{dx}\left[EI\frac{d^2w}{dx^2}\right]$$

Exempel på R.V.



$$\boxed{\begin{matrix} w(0)=0 \\ w'(0)=0 \end{matrix}}$$

EI konstant $\Rightarrow T = -EIw'''$



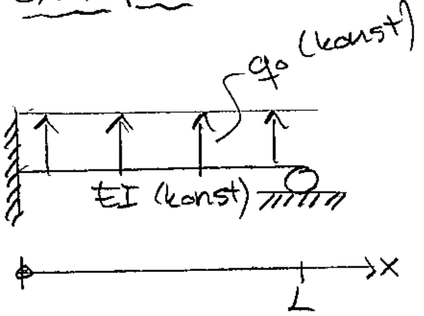
$x=L$:

$\curvearrowright: M_0 = M(L) = -EIw''(L)$

$$\boxed{w''(L) = \frac{-M_0}{EI}}$$

$\uparrow: P = T(L) = -EIw'''(L)$; $\boxed{w'''(L) = \frac{-P}{EI}}$

Exempel



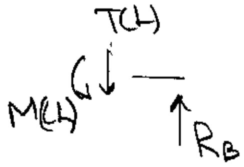
EI konst. $\Rightarrow w^{IV} = \frac{q_0}{EI}$

$$w(x) = \frac{q_0 x^4}{24EI} + C_1 x^3 + C_2 x^2 + C_3 x + C_4$$

$$w'(x) = \frac{q_0 x^3}{6EI} + 3C_1 x^2 + 2C_2 x + C_3$$

$$w''(x) = \frac{q_0 x^2}{2EI} + 6C_1 x + 2C_2$$

$x=L$:



$\curvearrowleft: M(L) = 0 = -EIw''(L)$

$\underline{\text{R.V.}}$ $\quad w(0)=0 \Rightarrow C_4=0 \;;\; w'(0)=0 \Rightarrow C_3=0$

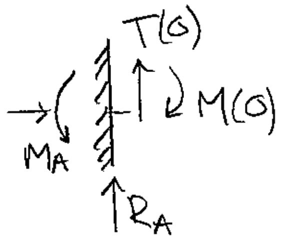
$$\left.\begin{aligned} w(L)=0 &\Rightarrow \frac{q_0L^4}{24EI}+C_1L^3+C_2L^2=0 \\ w''(L)=0 &\Rightarrow \frac{q_0L^2}{2EI}+6C_1L+2C_2=0 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} C_1 &= \frac{-5q_0L}{48EI} \\ C_2 &= \frac{3q_0L^2}{48EI} \end{aligned}$$

$$w(x)=\frac{q_0L^4}{48EI}\left(2\left(\frac{x}{L}\right)^4-5\left(\frac{x}{L}\right)^3+3\left(\frac{x}{L}\right)^2\right);\; w\left(\frac{L}{2}\right)=\frac{q_0L^4}{192EI}$$

$$M(x)=-EIw''=\frac{q_0L^2}{48}\left(-24\left(\frac{x}{L}\right)^2+30\left(\frac{x}{L}\right)-6\right)=$$

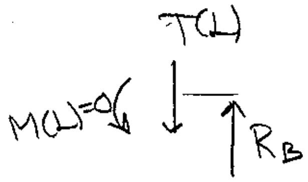
$$=\frac{q_0L^2}{8}\left(-4\left(\frac{x}{L}\right)^2+5\left(\frac{x}{L}\right)-1\right)$$

$$T(x)=\frac{dM}{dx}=\frac{q_0L}{8}\left(-8\frac{x}{L}+5\right)$$



$\uparrow: R_A=-T(0)=\frac{-5q_0L}{8}$

$\curvearrowleft: M_A=M(0)=\frac{-q_0L^2}{8}$



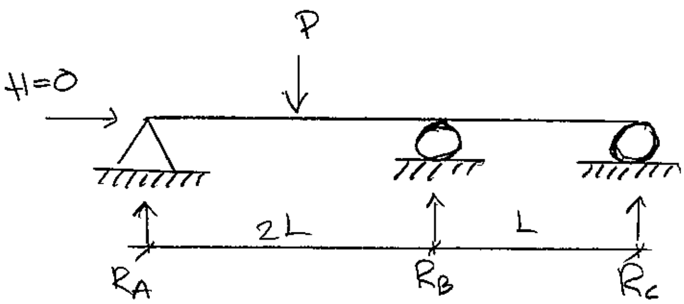
$\uparrow: R_B=T(L)=\frac{-3q_0L}{8}$

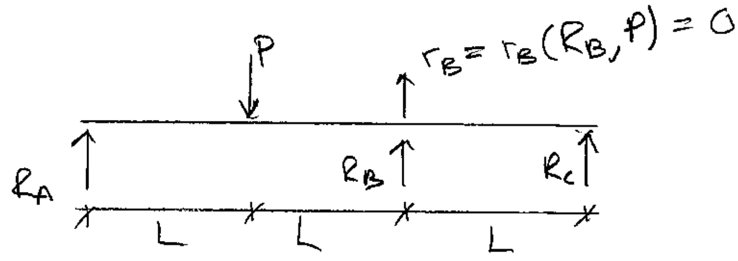
Kraftmetod (för statiskt obestämda bärverk)

- Låtsas att tillräckligt många stödreaktioner och/eller snittkrafter är bekanta yttre laster
- Beräkna associerade förskjutningar/rotationer
- Inför kompatibilitets-villkor

Exempel – kraftmetod

EI konstant





F.s sid. 7 : $L \to 3L$

formelsamling

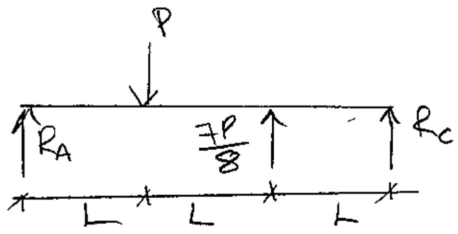
$$r_B = \underbrace{\frac{(3L)^3}{6EI}\left(1-\frac{2L}{3L}\right)\left[\left(2-\frac{2L}{3L}\right)\frac{2L\cdot 2L}{(3L)^2} - \frac{(2L)^3}{(3L)^3}\right]}_{a=2L,\ x=2L} R_B \; -$$

$$- \underbrace{\frac{(3L)^3}{6EI}\left(1-\frac{2L}{3L}\right)\left[\left(2-\frac{2L}{3L}\right)\frac{2L\cdot L}{(3L)^2} - \frac{L^3}{(3L)^3}\right]}_{a=2L,\ x=L} P =$$

$a = 2L$, $x = L$ ← ty $x \leq a$, räkna från punkt C

$$= r_B = \underbrace{\frac{3L^3}{54EI}}_{\text{nånting...}} \underbrace{\left(8R_B - 7P\right)}_{=0}$$

Kinematiskt villkor: $r_B = 0 \Rightarrow R_B = \frac{7P}{8}$



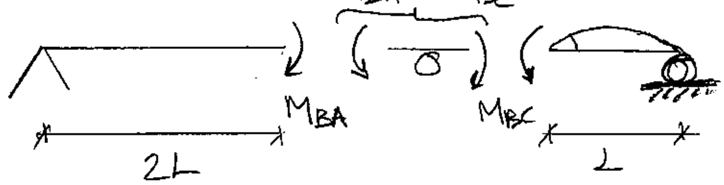
$$\curvearrowleft A: \; R_C \cdot 3L + \frac{7P}{8} \cdot 2L - P \cdot L = 0$$

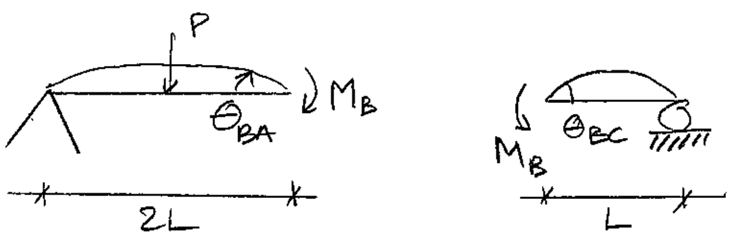
$$R_C = \frac{1}{3} P\left(1 - \frac{14}{8}\right) = \frac{-P}{4}$$

etc.

<u>Alt.</u> Låt snittmomenten vid B vara bekanta.

$M_{BA} = M_{BC} = M_B$



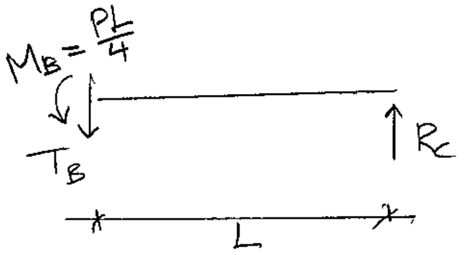


F.s. sid 9: $\theta_{BA} = \frac{M_B \cdot 2L}{3EI} - \frac{P(2L)^2}{16EI}$

$$\theta_{BC} = \frac{M_B L}{3EI}$$

Kompatibilitet: $\theta_{BA} + \theta_{BC} = 0$

$$\frac{L}{EI}\left(\frac{2M_B}{3} - \frac{PL}{4} + \frac{M_B}{3}\right) = \frac{L}{EI}\left(M_B - \frac{PL}{4}\right) = 0, \quad M_B = \frac{PL}{4}$$



$\curvearrowleft B$: $R_C \cdot L + M_B = 0$

$$R_C = \frac{-M_B}{L} = \frac{-P}{4}$$

...

/end ons Lv 4

15.1–15.2

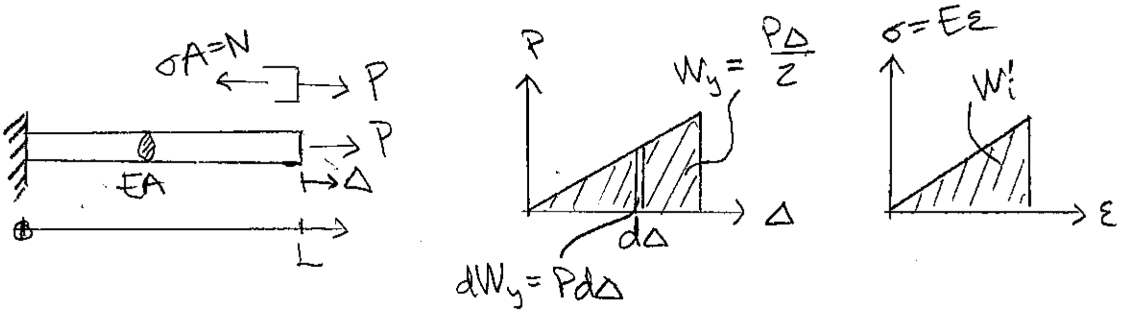
2011-05-02
Måndag Lv 5

Töjningsenergi, elastisk energi, W_i

Krafter som belastar en elastisk kropp uträttar ett arbete, W_y, p.g.a. de förskjutningar de ger upphov till. Detta lagras som elastisk energi i kroppen. Energin återfås då kroppen avlastas (fjädrar tillbaka).

Enaxlig belastning

$0 \to P$ långsamt så att kinetisk energi kan försummas



$$W_y = \int_0^\Delta P d\Delta = A\int_0^\Delta \sigma d\Delta = A\cdot L \int_0^\Delta \sigma \underbrace{\frac{d\Delta}{L}}_{d\varepsilon} = AL\int_0^\varepsilon \sigma d\varepsilon = W_i$$

$$W_i' = \int_0^\varepsilon \sigma d\varepsilon \qquad \left[\frac{N}{m^2} = \frac{Nm}{m^3} = \frac{J}{m^3}\right] \quad \text{energitäthet}$$

$$\Rightarrow \quad W_i' = \frac{\sigma\varepsilon}{2} = \frac{E\varepsilon^2}{2} = \frac{\sigma^2}{2E} \qquad \left[\frac{Nm}{m^3}\right] \quad \text{enaxlig belastning}$$

$$W_i = \int_V W_i' dV \underset{\substack{\uparrow \\ \text{enaxligt} \\ \text{fall}}}{=} AL\int_0^\varepsilon \sigma d\varepsilon = AL\frac{\sigma^2}{2E} = \frac{N^2L}{2EA} = \frac{P^2L}{2EA}$$

$$W_y = W_i \quad \Rightarrow \quad \frac{P\Delta}{2} = \frac{P^2L}{2EA} \qquad \underline{\underline{\Delta = \frac{PL}{EA}}}$$

$A = A(x)$:
$$W_i = \int_V W_i' dV = \int_0^L W_i' A dx$$

Virtuellt arbete: P konstant
$\Delta\delta$ virtuell (tänkt) förskjutning från jämviktsläget.
$\delta W_y = P \cdot \delta\Delta = \delta(P \cdot \Delta)$

$\delta W_y = \delta W_i$ Virtuella arbetets princip
$\delta(W_i - P\Delta) = 0$

Potentiell energi: U = elastisk energi + lastens potential (def)

$U = W_i - P\Delta$; $\delta U = 0$ enl. VAP (vid jämvikt)
Principen om pot. energins minimum.

(Lastens pot. $\pi = -P\Delta$, $-\nabla\pi$ = kraft)

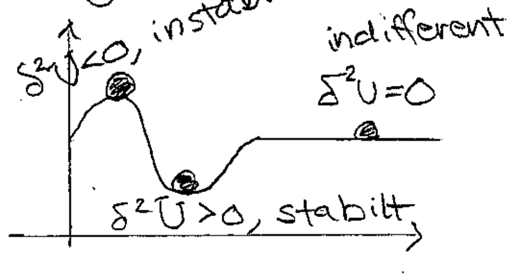
$$U = AL\int_0^{\varepsilon}\sigma d\varepsilon - P\cdot\Delta = EAL\frac{\varepsilon^2}{2} - P\Delta = \left\{\varepsilon = \frac{\Delta}{L}\right\} =$$

$$= \frac{EA}{2L}\Delta^2 - P\Delta$$

$$\delta U = \frac{EA}{2L}\cdot\underbrace{2\Delta\delta\Delta}_{\delta\Delta^2} - P\delta\Delta = \Big(\underbrace{\frac{EA}{L}\Delta - P}_{=0\ \Rightarrow\ \Delta = \frac{PL}{EA}}\Big)\delta\Delta = 0$$

$$\delta^2 U = \delta(\delta U) = \left(\frac{EA}{L}\delta\Delta - 0\right)\delta\Delta$$

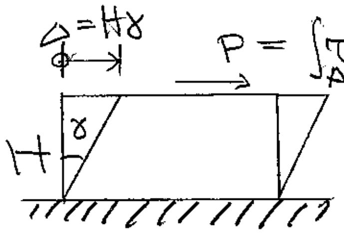
$$= \frac{EA}{L}(\delta\Delta)^2 > 0 \Rightarrow \min$$

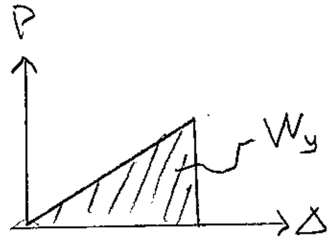


Skjuvning

τ ungefär konstant

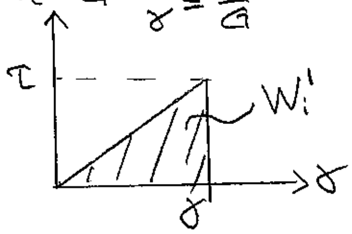
$$P = \int_A \tau \, dA \approx \tau A$$





$$W_y = \int_0^{\Delta} P \, d\Delta = H \int_0^{\gamma} P \, d\gamma = HA \underbrace{\int_0^{\gamma} \tau \, d\gamma}_{W_i'} = W_i$$

$\tau = G\gamma \qquad \gamma = \frac{\tau}{G}$

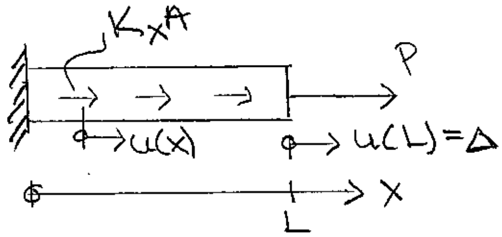


$$W_i' = \int_0^{\gamma} \tau \, d\gamma = \frac{\tau\gamma}{2} = \frac{G\gamma^2}{2} = \frac{\tau^2}{2G}$$

3D: $$W_i' = \int_0^{\underset{\sim}{\varepsilon}} \underset{\sim}{\sigma} \, d\underset{\sim}{\varepsilon} = \frac{1}{2} \underset{\sim}{\sigma}^T \underset{\sim}{\varepsilon} = \{ \underset{\sim}{\sigma} = \underset{\sim}{D} \underset{\sim}{\varepsilon} \} = \frac{1}{2} \underset{\sim}{\varepsilon}^T \underset{\sim}{D} \underset{\sim}{\varepsilon}$$

Elastisk energi i strukturer

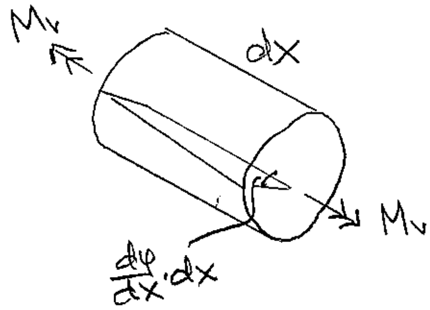
Enaxligt drag/tryck



$$W_i = \int_V W_i' \, dV = \int_0^L A W_i' \, dx =$$

$$= \begin{cases} \int_0^L \frac{EA}{2} \varepsilon^2 dx = \int_0^L \frac{EA}{2} \left(\frac{du}{dx}\right)^2 dx \\ \int_0^L \frac{A}{2E} \sigma^2 dx = \underline{\underline{\int_0^L \frac{N^2}{2EA} dx}} \end{cases}$$

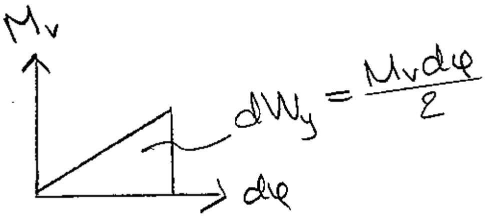
Vridning



$$6\text{-}11: \quad \frac{d\varphi}{dx} dx = \frac{M_v \cdot dx}{GK}$$

$$\frac{d\varphi}{dx} = \frac{M_v}{GK}$$

$$M_v = GK \frac{d\varphi}{dx}$$

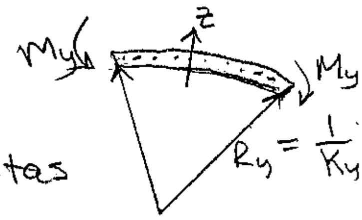


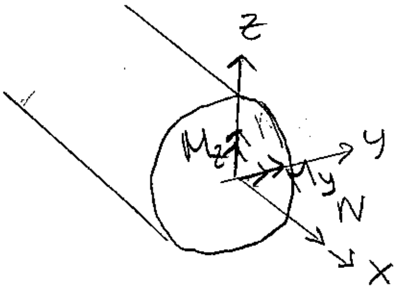
$$dW_i = \frac{M_v}{2} \cdot \frac{d\varphi}{dx} \cdot dx$$

$$W_i = \int_0^L \frac{M_v}{2} \frac{d\varphi}{dx} dx = \int_0^L \frac{GK}{2} \left(\frac{d\varphi}{dx}\right)^2 dx = \underline{\underline{\int_0^L \frac{M_v^2}{2GK} dx}}$$

Böjning –

$D_{yz} = \int_A yz\,dA = 0$ antas





$$\varepsilon(y,z) = \varepsilon(0) + K_y z + K_z y$$

$$\sigma = \frac{N}{A} + \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

$$W_i = \int_V W_i' dV = \int_0^L \int_A \tfrac{1}{2} E\varepsilon^2 dA\,dx = \int_0^L \frac{E}{2} \int_A (\varepsilon(0) + K_y z + K_z y)^2 dA\,dx =$$

$$= \int_0^L \frac{E}{2}\Big[\varepsilon(0)^2 \underbrace{\int_A dA}_{A} + K_y^2 \underbrace{\int_A z^2 dA}_{I_y} + K_z^2 \underbrace{\int_A y^2 dA}_{I_z} + 2\varepsilon(0) K_y \underbrace{\int_A z\,dA}_{S_y = 0} +$$

$$+ 2\varepsilon(0) K_z \underbrace{\int_A y\,dA}_{S_z = 0} + 2 K_y K_z \underbrace{\int_A yz\,dA}_{D_{yz} = 0} \Big] dx$$

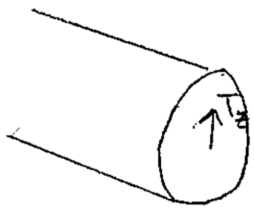
Med $\varepsilon(0) = \frac{du}{dx}$, $K_y = -w''$, $K_z = -v''$ fås då

$$W_i = \int_0^L \frac{EA}{2}(u')^2 + \frac{EI_y}{2}(w'')^2 + \frac{EI_z}{2}(v'')^2 dx$$

eller, med $EAu' = N$, $M_y = -EI_y w''$ och $M_z = -EI_z v''$

$$W_i = \int_0^L \left(\frac{N^2}{2EA} + \frac{M_y^2}{2EI_y} + \frac{M_z^2}{2EI_z} \right) dx$$

Tvärkraft



$$W_i' = \frac{\tau^2}{2G}$$

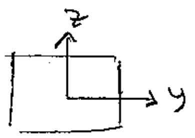
$$W_i = \int_V \frac{\tau^2}{2G} dV = \int_0^L \int_A \frac{\tau^2}{2G} dA\,dx =$$

$$= \int_0^L \frac{\tau^2 A}{2G} dx = \{T_z = \tau A\} = \int_0^L \frac{T_z^2}{2GA} dx$$

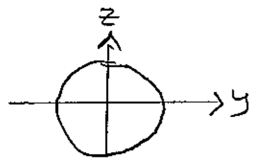
Sammantaget

$$W_i = \int_0^L \frac{N^2}{2EA} + \frac{M_y^2}{2EI_y} + \frac{M_z^2}{2EI_z} + \frac{M_v^2}{2GK} + \beta_y \frac{T_y^2}{2GA} + \beta_z \frac{T_z^2}{2GA}\, dx$$

β_y, β_z tvärsnittsfaktorer som kompenserar för att τ inte är konstant över A.



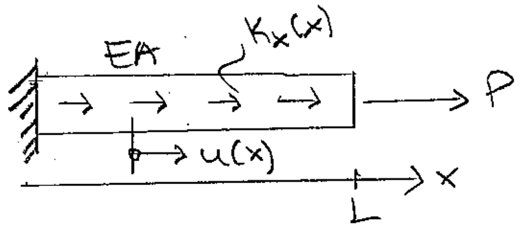
$\beta = \frac{6}{5}$



$\beta = \frac{10}{9}$

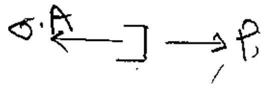
/end mån Lv 5

Virtuella arbetets princip 1D elasticitet



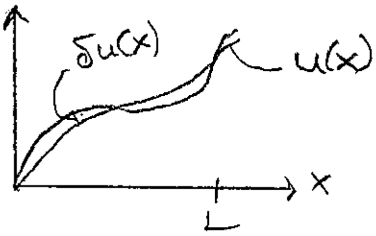
$$\begin{cases} -\frac{d}{dx}\left[EA\frac{du}{dx}\right] = K_x A & 0 < x < L \\ u(0) = 0 & \\ \left.\frac{du}{dx}\right|_{x=L} = \frac{P}{EA} & \end{cases}$$

x = L

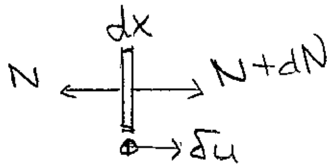


$\sigma A = P$

$\sigma = E\varepsilon = E \cdot \frac{du}{dx}$



$\delta u(x)$ = virtuell förskjutning



$$-\int_0^L \delta u(x)\underbrace{\frac{d}{dx}\left[EA\frac{du}{dx}\right]dx}_{\frac{dN}{dx}\cdot dx = dN} = \int_0^L \delta u(x)\cdot K_x A\,dx$$

$$\underbrace{-\left[\delta u\, EA\frac{du}{dx}\right]_0^L} + \int_0^L \underbrace{\frac{d\delta u}{dx}}_{\delta\varepsilon} EA\frac{du}{dx}\,dx = \int_0^L \delta u\, K_x A\,dx$$

$$-\left(\delta u(L)\cdot \underbrace{EA\frac{du}{dx}\Big|_{x=L}}_{P}\right) + \left(\underbrace{\delta u(0)}_{=0}\cdot EA\frac{du}{dx}\Big|_{x=0}\right) = -P\delta u(x)$$

$$\boxed{\int_0^L \delta\varepsilon\, EA\frac{du}{dx}\,dx = \int_0^L \delta u\cdot K_x A\,dx + P\delta u(L)}$$

$\boxed{\delta W_i = \delta W_y}$ vid jämvikt

Exempel EA konstant, $K_x = 0$

Ansätt $u = Cx$, $\frac{du}{dx} = C$: $\int_0^L \delta\varepsilon\, EAC\,dx = P\delta u(L)$

Välj (t.ex.) $\delta u = x$; $\delta\varepsilon = \frac{d\delta u}{dx} = 1$

$$\Rightarrow \int_0^L 1\cdot EA\cdot C\,dx = P\cdot L$$

$C\cdot EA\cdot L = PL$, $C = \frac{P}{EA}$ $\qquad u(x) = \frac{Px}{EA}$

$u(L) = \frac{PL}{EA}$

/end tis Lv5

Elastisk energi

2011-05-05
Onsdag Lv5

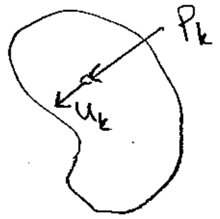
3D: $W_i = \int_V \frac{1}{2} \underset{\sim}{\varepsilon}^T \underset{\sim}{D} \underset{\sim}{\varepsilon} \, dV$

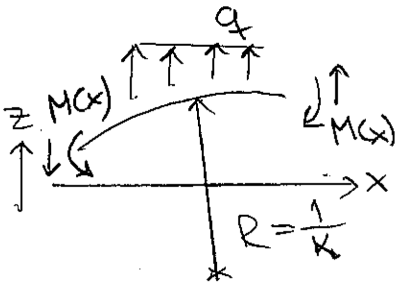
2D: $W_i = \int_A \frac{1}{2} \underset{\sim}{\varepsilon}^T \underset{\sim}{D} \underset{\sim}{\varepsilon} \, t \, dA$

$\left(\underset{\sim}{\sigma} = \underset{\sim}{D} \underset{\sim}{\varepsilon} \Rightarrow \frac{1}{2} \underset{\sim}{\varepsilon}^T \underset{\sim}{D} \underset{\sim}{\varepsilon} = \frac{1}{2} \underset{\sim}{\sigma}^T \underset{\sim}{D}^{-1} \underset{\sim}{\sigma} \right)$

1D: $W_i = \int_0^L \frac{N^2}{2EA} + \frac{M_y^2}{2EI_y} + \frac{M_z^2}{2EI_z} + \frac{M_v^2}{2GK} + \beta_y \frac{T_y^2}{2GA} + \beta_z \frac{T_z^2}{2GA} \, dx$

$\frac{\partial W_i}{\partial P_k} = u_k$

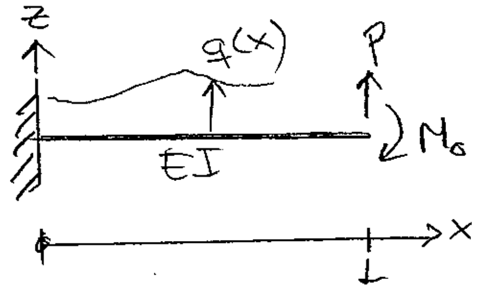




$$\left. \begin{aligned} -\frac{d^2M}{dx^2} &= q \\ M &= EI\kappa \\ \kappa &= -w'' \end{aligned} \right\} \Rightarrow \frac{d^2}{dx^2}\left[EI \frac{d^2 w}{dx^2} \right] = q$$

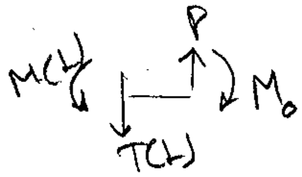
Givet w fås $M = -EIw''$,
$T = M' = (-EIw'')'$

Modellproblem



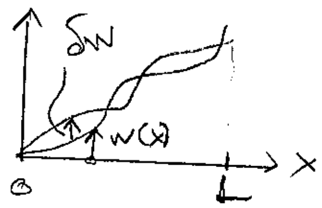
$$\begin{cases} \frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right] = q & 0<x<L \\ w(0)=0 \\ w'(0)=0 \end{cases} \quad \text{väsentliga r.v.}$$

$$w''(L) = \frac{-M_0}{EI}, \quad w'''(L) = \frac{-P}{EI} \quad \text{naturliga r.v.}$$



$\curvearrowright$: $\underline{M_0} = M(L) = \underline{-EIw''(L)}$

$\uparrow$: $\underline{P} = T(L) = \underline{-EIw'''(L)}$



Virtuella arbetets princip (VAP)

$\delta W_y = \delta W_i$ p.g.a. δw $\delta w(x)$ en nästan godt. störning från jämviktsläget

$\delta w(0)=0, \ \delta w'(0)=0$

$$\underbrace{\int_0^L \delta w \frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right]dx}_{} = \int_0^L \delta w \cdot q\,dx$$

Partialintegrera: $\left[\delta w \underbrace{\frac{d}{dx}\left[EI\frac{d^2w}{dx^2}\right]}_{-T}\right]_0^L -$

$$-\int_0^L \frac{d\delta w}{dx}\cdot\frac{d}{dx}\left[EI\frac{d^2w}{dx^2}\right]dx =$$

$$= T(0)\,\delta w(0) - T(L)\,\delta w(L) - \left[\frac{d\delta w}{dx} \overbrace{EI \frac{d^2 w}{dx^2}}^{-M} \right]_0^L +$$

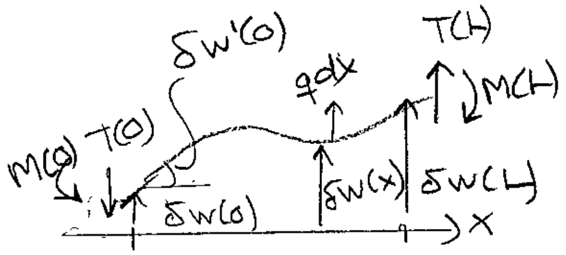
$$+ \int_0^L \frac{d^2 \delta w}{dx^2} EI \frac{d^2 w}{dx^2} dx =$$

$$= \int_0^L \delta w'' EI w'' dx = \int_0^L \delta w\, q\, dx + \underbrace{T(L)}_{=P} \delta w(L) - T(0) \underbrace{\delta w(0)}_{=0} -$$

$$- \underbrace{M(L)}_{=M_0} \delta w'(L) + M(0) \underbrace{\delta w'(0)}_{=0}$$

VAP

$$\Rightarrow \boxed{\int_0^L \delta w'' EI w'' dx = \int_0^L \delta w\, q\, dx + P\delta w(L) - M_0 \delta w''(L)}$$



Potentiell energi: $U = W_i +$ yttre lastens potential

$\bar{w} = \bar{w}(x)$, $\bar{w}(0) = 0$, $\bar{w}'(0) = 0$

$$U(\bar{w}) = \int_0^L \underbrace{\frac{1}{2} EI (\bar{w}'')^2}_{\frac{1 \cdot M^2}{2EI}} dx - \int_0^L \bar{w} q\, dx - P\bar{w}(L) + M_0 \bar{w}'(L)$$

$(M = -EIw'')$

principen om pot. energins minimum:

$$U(w) < U(\bar{w}) \quad \forall \bar{w} \neq w$$

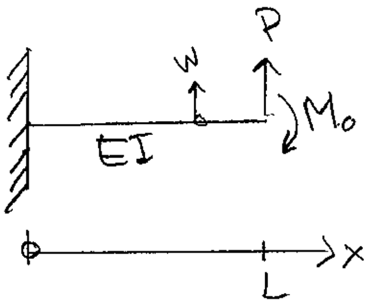
Bevis: Välj ett $\overline{w}$ och sätt δw så att $\overline{w} = w + \delta w$

$$\underline{U(\overline{w})} = U(w+\delta w) = \underbrace{\frac{1}{2}\int_0^L EI(w'')^2 dx - \int_0^L wq\,dx - Pw(L) + M_0 w'(L)}_{U(w)} +$$

$$+ \frac{1}{2}\int_0^L EI(\delta w'')^2 dx + \underbrace{0}_{\text{enl. VAP}}$$

$$0 = \int_0^L EI\,\delta w''\, w'' dx - \int_0^L \delta w\, q\, dx - P\delta w(L) + M_0 \delta w'(L)$$

$$= U(w) + \underbrace{\frac{1}{2}\int_0^L EI(\delta w'')^2 dx}_{>0} > \underline{U(w)}$$



Exempel:

modellproblemet med $q=0$ och EI konstant

Approximera: $w \approx w_a = C_0 + C_1 x + C_2 x^2$

$w_a(0) = 0 \Rightarrow C_0 = 0$, $w_a'(0) = 0 \Rightarrow C_1 = 0$

$w_a = C_2 x^2$, $w_a' = 2C_2 x$, $w_a'' = 2C_2$

$$U(w_a) = \frac{EI}{2}\int_0^L (2C_2)^2 dx - PC_2L^2 + M_0 2C_2 L =$$

$$= 2EIL\,C_2^2 - PL^2 C_2 + 2M_0 L C_2$$

$$\frac{\partial U}{\partial C_2} = 0 \implies 4EIL\,C_2 - PL^2 + 2M_0L = 0, \quad C_2 = \frac{PL}{4EI} - \frac{M_0}{2EI}$$

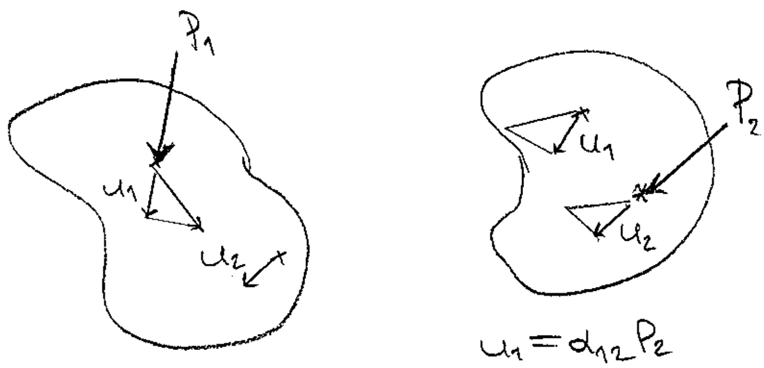
$$w_a = C_2x^2 = \frac{PL^3}{4EI}\left(\frac{x}{L}\right)^2 - \frac{M_0L^2}{2EI}\left(\frac{x}{L}\right)^2$$

F.s. sid. 8: $w(x) = \frac{PL^3}{6EI}\left(3\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3\right) - \frac{M_0L^2}{2EI}\left(\frac{x}{L}\right)^2$

[Avsn. 15.4.3: $w_a = C_2x^2 + C_3x^3$]

Maxwells reciprocitetssats

$u_1 = \alpha_{11}P_1$
$u_2 = \alpha_{21}P_1$



$u_1 = \alpha_{12}P_2$
$u_2 = \alpha_{22}P_2$

sats:

$$\boxed{\alpha_{ij} = \alpha_{ji}}$$

W_i oberoende av lasthistorien
$\Rightarrow W_y = W_i$ oberoende av lasthistorien

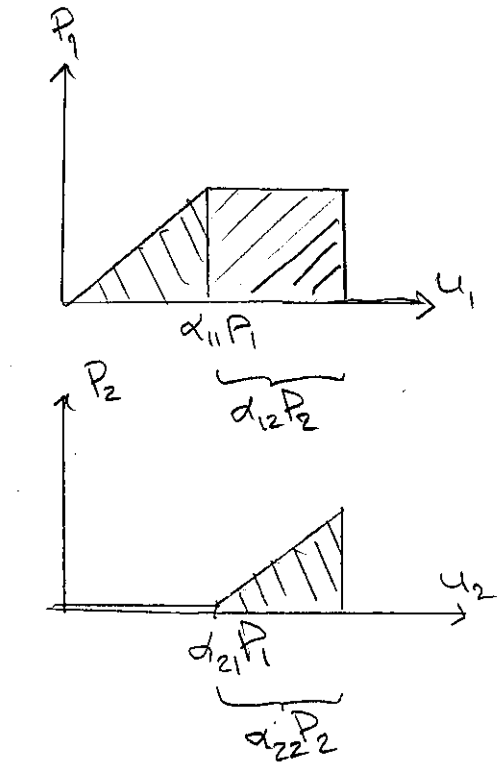
Fall 1: Låt $P_2 = 0$ och $0 \to P_1$ (långsamt)

$$W_y^{(1)} = \frac{1}{2} P_1 u_1 = \frac{1}{2} \alpha_{11} P_1^2$$

Håll P_1 konstant och $0 \to P_2$

$$W_y^{(2)} = \frac{1}{2} P_2 \alpha_{22} P_2 + P_1 \alpha_{12} P_2 =$$
$$= \alpha_{12} P_1 P_2 + \frac{1}{2} \alpha_{22} P_2^2$$

$$W_y^{fall1} = \frac{1}{2} \alpha_{11} P_1^2 + \alpha_{12} P_1 P_2 + \frac{1}{2} \alpha_{22} P_2^2$$



Fall 2: $P_1 = 0$, $0 \to P_2$

$$W_y^{(3)} = \frac{1}{2} \alpha_{22} P_2^2$$

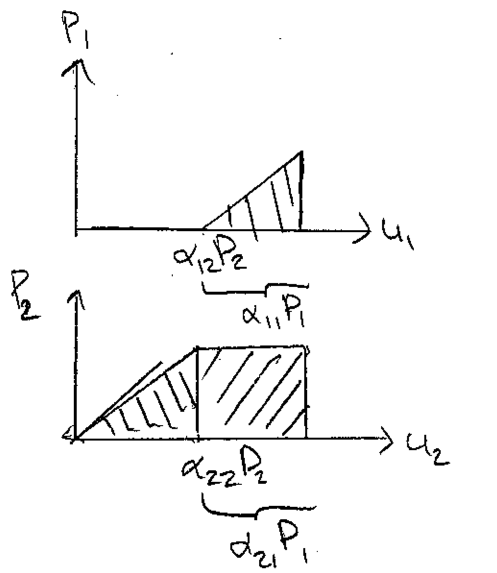
P_2 konstant och $0 \to P_1$

$$W_y^{(4)} = \frac{1}{2} \alpha_{11} P_1^2 + P_2 \alpha_{21} P_1$$

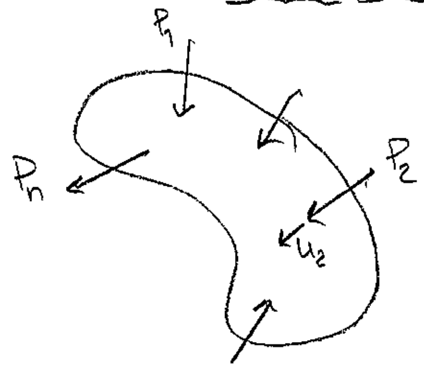
$$W_y^{fall2} = \underline{\frac{1}{2} \alpha_{11} P_1^2} + \alpha_{21} P_1 P_2 + \underline{\frac{1}{2} \alpha_{22} P_2^2}$$

$$W_y^{fall1} = W_y^{fall2}$$

$$\Longrightarrow \alpha_{21} = \alpha_{12}$$



Castiglianos 2:a sats



$$W_y = \frac{1}{2}\sum_{i=1}^{n} P_i u_i$$

$$u_i = \sum_{j=1}^{n} \alpha_{ij} P_j$$

$$W_y = \frac{1}{2}\sum_i \sum_j \alpha_{ij} P_i P_j = W_i$$

Sats: $\boxed{\frac{\partial W_i}{\partial P_k} = u_k}$

$$\frac{\partial W_i}{\partial P_k} = \frac{1}{2}\frac{\partial}{\partial P_k}\sum_i \sum_j \alpha_{ij} P_i P_j = \frac{\partial}{\partial P_k}\left[\frac{1}{2}\sum_i \left(\alpha_{i1} P_i P_1 + \dots + \alpha_{ik} P_i P_k + \dots + \alpha_{in} P_i P_n\right)\right]$$

$$= \frac{\partial}{\partial P_k}\frac{1}{2}\Big[\left(\alpha_{11} P_1^2 + \alpha_{21} P_2 P_1 + \dots + \alpha_{k1} P_k P_1 + \dots + \alpha_{n1} P_n P_1\right) +$$

$$\left(\alpha_{12} P_1 P_2 + \dots + \alpha_{k2} P_k P_2 + \dots + \alpha_{n2} P_n P_2\right) +$$

$$\vdots$$

$$\left(\alpha_{1k} P_k P_1 + \alpha_{2k} P_k P_2 + \dots + \alpha_{kk} P_k^2 + \dots + \alpha_{nk} P_n P_k\right) +$$

$$\vdots$$

$$\left(\alpha_{1n} P_1 P_n + \dots + \alpha_{kn} P_k P_n + \dots + \alpha_{nn} P_n^2\right)\Big] =$$

$$= \frac{1}{2}\left(\alpha_{k1} P_1 + \alpha_{k2} P_2 + \dots + \alpha_{kk} P_k + \dots + \alpha_{kn} P_n\right) +$$

$$\frac{1}{2}\left(\alpha_{1k} P_1 + \alpha_{2k} P_2 + \dots + \alpha_{kk} P_k + \dots + \alpha_{nk} P_n\right) = \{\alpha_{ij} = \alpha_{ji}\}$$

$$= \sum_{i=1}^{n} \alpha_{ki} P_i = \boxed{u_k = \frac{\partial W_i}{\partial P_k}}$$

/end ons LV5

<table><tr><td>2011-05-09
Måndag Lv 6</td></tr></table>

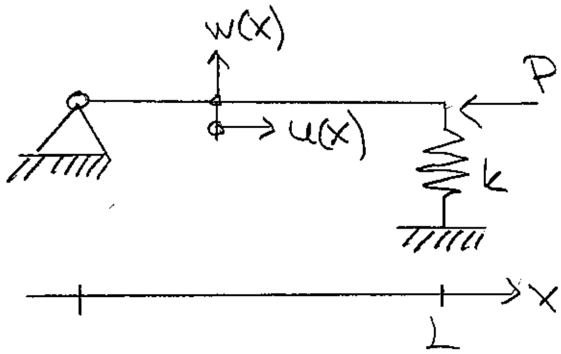
<u>Spänningsproblem:</u>
Sambandet mellan kraft och förskjutning är entydigt och svarar mot <u>stabila jämviktslägen</u>.

liten ändring i kraft, ger liten ändring i respons

<u>Stabilitetsproblem:</u> sambandet kraft/förskjutning är inte entydigt, liten ändring i kraft ger stor förändring i respons.

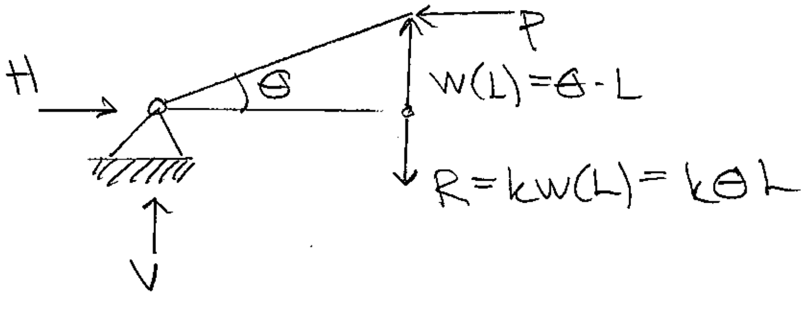
- kantring/vippning
- buckling
- <u>böjknäckning</u>

Analys av stabilitet kräver att jämvikt ställs upp i det förskjutna läget.



$$u(x) = \frac{-Px}{EA}$$

$$w(x) \equiv 0$$



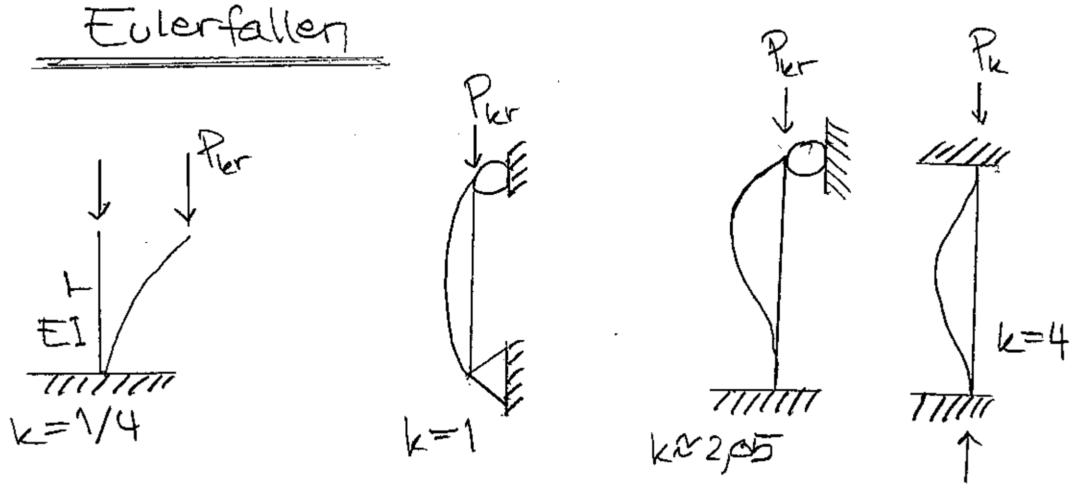
$$\theta \ll 1$$

$\curvearrowright$: $R \cdot L - Pw(L) = 0$

$k\theta L^2 - P\theta L = 0$, $\theta(kL - P) = 0$

$\theta \neq 0$ är möjligt om $P = \underline{kL = P_{kr}}$

Eulerfallen



$$P_{kr} = k \frac{\pi^2 EI}{L^2}$$

Differentialekvationen för den axialbelastade balken

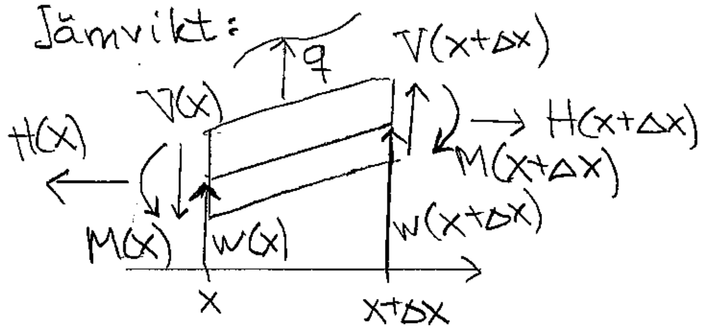
- Andra ordningens teori:
  jämvikt i utböjt läge, med inverkan av axiallast

- förskjutningar och deformationer antas fortfarande vara små

1) Konstitutivt samband: $M = EI\kappa$ (Hooke)

2) Kinematiskt samband: $\kappa = \dfrac{-w''}{(1+(w')^2)^{3/2}} \approx -w''$

3) Jämvikt:



$\rightarrow$: $H(x+\Delta x) - H(x) = 0 \Rightarrow \dfrac{dH}{dx} = 0$ (H konst.)

$\uparrow$: $V(x+\Delta x) - V(x) + q\cdot\Delta x = 0$

$\times \dfrac{1}{\Delta x}$, $\Delta x \to 0 \Rightarrow \dfrac{dV}{dx} = -q$

$\overset{\frown}{x+\Delta x}$: $M(x+\Delta x) - M(x) + q\Delta x \dfrac{\Delta x}{2} - V(x)\Delta x + H(w(x+\Delta x) - w(x)) = 0$

$\times \dfrac{1}{\Delta x}$, $\Delta x \to 0 \Rightarrow \dfrac{dM}{dx} = V - H\dfrac{dw}{dx}$

Kom ihåg detta!

$$\frac{d^2M}{dx^2} = \frac{dV}{dx} - H\frac{d^2w}{dx^2} = \underline{\underline{-q - H\frac{d^2w}{dx^2}}}$$

4) Sammanställ:

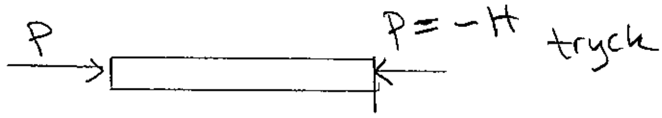
$$q = -\frac{d^2M}{dx^2} - H\frac{d^2w}{dx^2} = \left\{ M = EIK = -EI\frac{d^2w}{dx^2} \right\} =$$

$$= \boxed{\frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right] - H\frac{d^2w}{dx^2} = q}$$

Stabilitetsproblemet:

$q \equiv 0$ och bara homogena randvillkor ger $w \equiv 0$ trivialt. Hitta värden på H som medger $w \neq 0$.

1) $H > 0$ (drag) ger ingen lösning
$\left(w = w_h = A + Bx + C\cosh(nx) + D\sinh(nx)\right)$

2) 

Låt EI vara konstant, $H = -P$; $q \equiv 0 \Rightarrow$

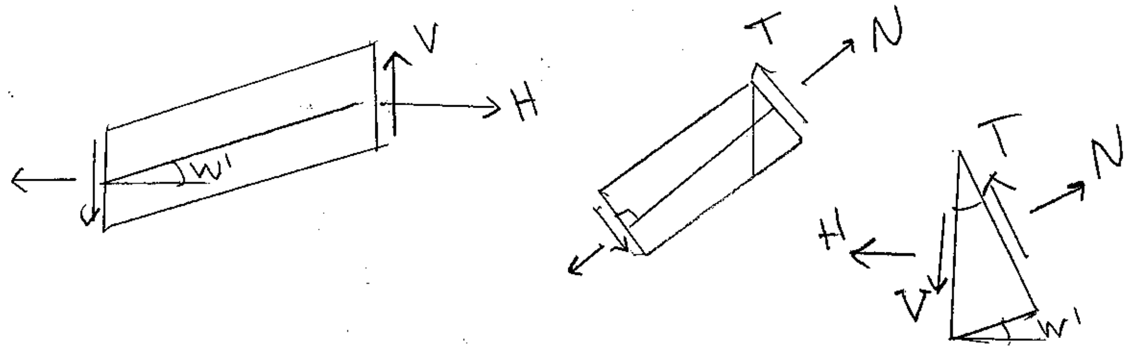
$$w^{IV} + \frac{P}{EI}w'' = 0 \qquad n^2 = \frac{P}{EI}$$

Kar. ekv. $r^4 + n^2r^2 = 0$, $r_{1,2} = 0$, $r^2 + n^2 = 0$
$r_{3,4} = \pm in$

$$\Rightarrow w = (A + Bx)e^{0 \cdot x} + C_1e^{-inx} + C_2e^{inx} =$$

$$= A + Bx + C\cos(nx) + D\sin(nx) \qquad (8-66)$$

A, B, C och D ur randvillkor.
Vissa r.v. kräver att vi håller isär $V(x)$ och $T(x)$ (tvärkraft)



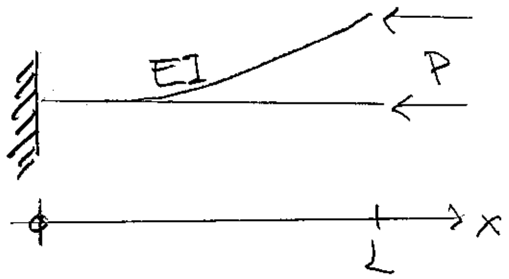
$\leftarrow$: $H - N\cos w' + T\sin w' = 0$

$w' \ll 1 \Rightarrow H = N - Tw'$, T måttlig $\Rightarrow \underline{\underline{N \approx H}}$

$\nwarrow$: $T - V\cos w' + H\sin w' = 0$, $\underline{T = V - Hw'}$ (8-59)

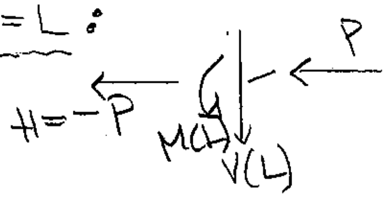
$$\left.\begin{array}{l} M = EI\kappa = -EIw'', \quad \dfrac{dM}{dx} = -EIw''' \\ \text{mom. jämvikt: } \dfrac{dM}{dx} = V - Hw' = T \end{array}\right\} \Rightarrow T = -EIw'''$$

Exempel (Euler 1)



$$\begin{cases} w^{IV} + n^2 w'' = 0 \\ \quad 0 < x < L, \; n = \sqrt{\frac{P}{EI}} \\ w(0) = 0 \\ w'(0) = 0 \\ w''(L) = 0 \\ w'''(L) + n^2 w'(L) = 0 \end{cases}$$

$x = L$:



$\curvearrowleft$: $M(L) = 0$

$M = -EIw'' \Rightarrow w''(L) = 0$

$\downarrow$: $V(L) = 0$, $\quad V = T + Hw' = \underbrace{-EIw''' - Pw' = 0}_{\text{för } x = L}$

Trivial lösning: $w \equiv 0$

$$w = A + Bx + C\cos(nx) + D\sin(nx)$$
$$w' = B - Cn\sin(nx) + Dn\cos(nx)$$
$$w'' = -Cn^2\cos(nx) - Dn^2\sin(nx)$$
$$w''' = Cn^3\sin(nx) - Dn^3\cos(nx)$$

$$w(0) = 0 \Rightarrow A + C = 0 \qquad (1)$$
$$w'(0) = 0 \Rightarrow B + Dn = 0 \qquad (2)$$

$$w''(L) = 0 \Rightarrow C\cos(nL) + D\sin(nL) = 0 \qquad (3)$$
$$w'''(L) + n^2 w'(L) = 0 \Rightarrow Cn^3\cancel{\sin}(nL) -$$
$$- Dn^3\cancel{\cos}(nL) + Bn^2 - Cn^3\cancel{\sin}(nL) +$$
$$+ Dn^3\cancel{\cos}(nL) \Rightarrow B = 0 \qquad (4)$$

$$\underset{\sim}{\Lambda}(n)\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \underset{\sim}{0}, \quad \underbrace{\det \Lambda = 0}_{\text{knäckekvationen}} \Rightarrow n$$

$$\underset{\sim}{\Lambda}(n) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & n \\ 0 & 0 & \cos(nL) & \sin(nL) \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(2) & (4) ger $B = D = 0$

$$\left.\begin{array}{ll} (1) & A + C = 0 \\ (3b) & C\cos(nL) = 0 \end{array}\right\} C = 0 \Rightarrow A = 0 \Rightarrow w \equiv 0$$

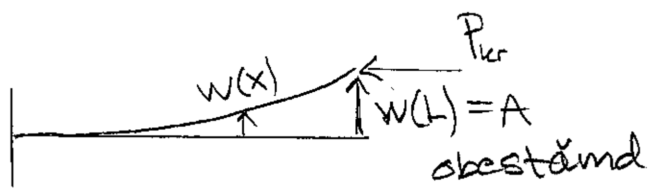
måste ha $\cos(nL) = 0$ för icke-trivial lösning $\Rightarrow nL = \frac{\pi}{2} + m\pi \quad m = 0, 1, 2, \ldots$

Lägsta (positiva) roten ger P_{kr}:

$nL = \frac{\pi}{2}$, $n^2 = \frac{\pi^2}{4L^2} = \frac{P_{kr}}{EI}$, $\underline{\underline{P_{kr} = \frac{\pi^2 EI}{4L^2}}}$

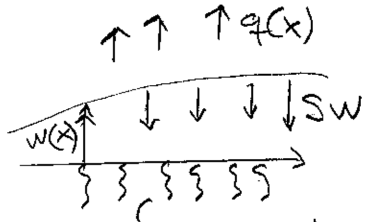
$$w(x) = A + C\cos nx = \left\{\begin{array}{l} C = -A \\ n = \frac{\pi}{2L} \end{array}\right\} = A\left(1 - \cos\frac{\pi x}{2L}\right)$$

$w(L) = A$



Balk på elastiskt underlag

2011-05-10
Tisdag Lv6



Winklerbädd med styvhet S $\left(\frac{\text{kraft}}{\text{längd}}\right)$

$$EIw^{IV} = q - Sw$$

$$w^{IV} + \frac{S}{EI} w = q$$

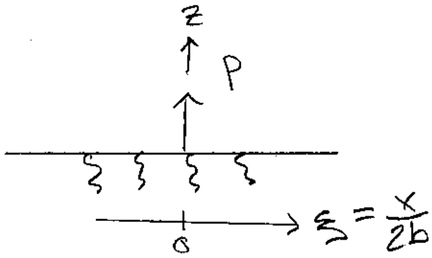
$$b = \left(\frac{EI}{4S}\right)^{1/4}, \quad \xi = \frac{x}{2b}$$

$$\frac{dw}{dx} = \frac{dw}{d\xi} \cdot \frac{d\xi}{dx} = \frac{1}{2b} \frac{dw}{d\xi}$$

$$\frac{1}{16b^4} \frac{d^4w}{d\xi} + \frac{S}{EI} w = 0, \quad \frac{d^4w}{d\xi^4} + 4w = 0$$

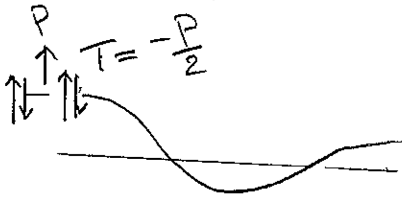
Kar. ekv. har rötterna $r_{1,2,3,4} = \pm(1 \pm i)$

$$w = e^{\xi}(A\cos\xi + B\sin\xi) + e^{-\xi}(C\cos\xi + D\sin\xi)$$



$x \geq 0$ $\quad w \to 0$ då $\xi \to \infty$

$\Rightarrow A = B = 0$



$$\frac{dw}{dx} = \frac{1}{2b}\frac{dw}{d\xi} = 0 \text{ för } x = 0$$

$\Rightarrow C = D$

$$w = e^{-\xi}(C\cos\xi + D\sin\xi)$$

$$\frac{dw}{d\xi} = -e^{-\xi}\left(C\cos\xi + D\sin\xi\right) + e^{-\xi}\left(-C\sin\xi + D\cos\xi\right) =$$

$$= e^{-\xi}\left((D-C)\cos\xi - (C+D)\sin\xi\right)$$

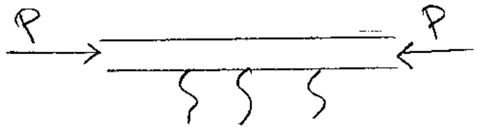
$$\left.\frac{dw}{d\xi}\right|_{\xi=0} = D-C = 0$$

$$T = -EI\frac{d^2w}{dx^2} = \frac{-P}{2} \text{ för } x=0 \Rightarrow C = \frac{Pb^3}{EI}$$

$$\frac{d^3w}{dx^3} = \frac{1}{8b^3}\frac{d^3w}{d\xi^3}$$

$$w = e^{-\frac{x}{2b}}\frac{Pb^3}{EI}\left(\cos\frac{x}{2b} + \sin\frac{x}{2b}\right)$$

Inverkan av axialkraft: $EIw^{IV} + Pw'' + Sw = 0$



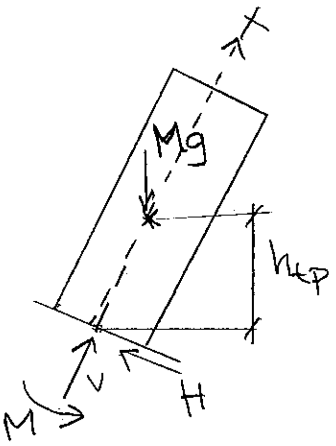
$$w^{IV} + n^2 w'' + \frac{S}{EI}w = 0$$

$$n^2 = \frac{P}{EI}$$

$$\frac{d^4w}{d\xi^4} + 4b^2n^2\frac{d^2w}{d\xi^2} + 4w = 0 \qquad \text{etc.}$$

$$r = \pm\left(-2b^2n^2 \pm \sqrt{4b^4n^4 - 4}\right)^{1/2}$$

/end tis LV6



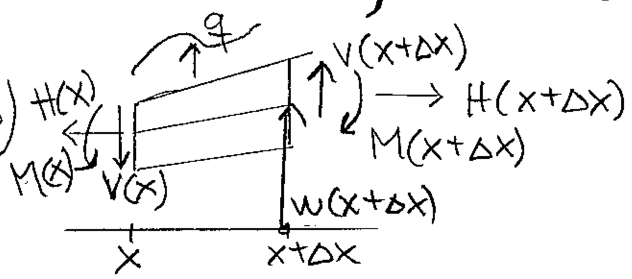
$$\sigma = \frac{N}{A} + \frac{Mz}{I}$$

2011-05-11
Onsdag LV 6

inlämn. 4

Diff.ekv. för axialbelastad balk

1) $\kappa = -w''$ 2) $M = EI\kappa$

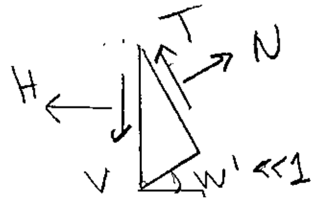
3)


$$\uparrow : \frac{dV}{dx} = -q$$

$$\curvearrowright : \frac{dM}{dx} = V - Hw'$$

$$q = -\frac{dV}{dx} = -\frac{d^2M}{dx^2} - H\frac{d^2w}{dx^2} = \frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right] - H\frac{d^2w}{dx^2}$$

Med EI konstant fås $w^{IV} - \frac{H}{EI}w'' = \frac{q}{EI}$



$$\rightarrow : H \approx N$$

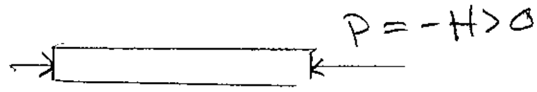
$$\nwarrow : T = V - Tw'$$

$$M = -EIw'' \Rightarrow \frac{dM}{dx} = \underline{-EIw''' = T}$$

$$\frac{dM}{dx} = V - Hw' = T$$

Lösningar

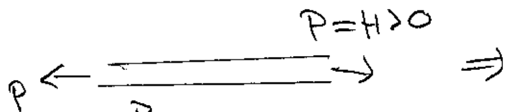
1) $H = 0$ eller inverkan av H försummas
$\Rightarrow$ 1:a ordningens teori. $w^{IV} = \frac{q}{EI}$

2) $H < 0$. Tryckt balk 

$w^{IV} + n^2 w'' = \frac{q}{EI}$, $n^2 = \frac{P}{EI}$

$w = w_p + w_h$. $w_h = A + Bx + C\cos(nx) + D\sin(nx)$

Specialfall: inga transversallaster.
Stabilitetsproblemet – hitta det lägsta positiva n som medger $w = w_h \neq 0$

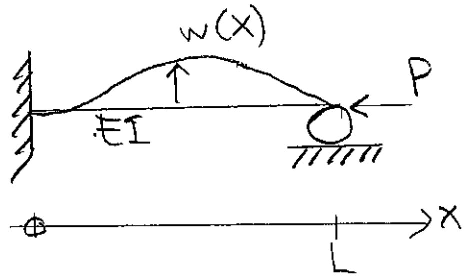
3) $H > 0$ dragen balk 

$\Rightarrow w^{IV} - n^2 w'' = \frac{q}{EI}$, $n^2 = \frac{P}{EI}$

Karakteristisk ekv. $r^4 - n^2 r^2 = 0$, $r_{1,2} = 0$
$r^2 - n^2 = 0$, $r_{3,4} = \pm n$

$$w_h = (A + Bx)e^{0 \cdot x} + C_1 e^{nx} + C_2 e^{-nx}$$
$$= A + Bx + C\cosh(nx) + D\sinh(nx)$$

Euler 3



$$\begin{cases} w^{IV} + n^2 w'' = 0 & \\ w(0) = 0 & (1) \\ w'(0) = 0 & (2) \\ w(L) = 0 & (3) \\ w''(L) = 0 & (4) \end{cases}$$

$M(L) = 0$, $M = -EIw''$

$$w = A + Bx + C\cos(nx) + D\sin(nx)$$
$$w' = B - Cn\sin(nx) + Dn\cos(nx)$$
$$w'' = -Cn^2\cos(nx) - Dn^2\sin(nx)$$

(1) ger $A + C = 0$, $A = -C = D\tan(nL)$

(2) $B + Dn = 0$, $B = -Dn$

(4) $C\cos(nL) + D\sin(nL) = 0 \Rightarrow C = -D\tan(nL)$

$$w = D\left(\tan(nL) - nx - \tan(nL)\cos(nx) + \sin(nx)\right)$$

(3) : $D\left(\tan(nL) - nL - \underbrace{\tan(nL)\cos(nL)}_{\sin(nL)} + \sin(nL)\right) = 0$

$D = 0 \Rightarrow w \equiv 0.$

Icke-triviala lösningar kräver $\underbrace{\frac{\tan(nL)}{nL} - 1 = 0}_{\text{knäckekv.}} = f(nL)$

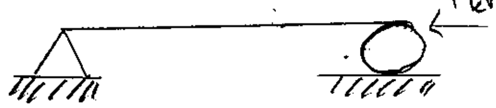
lägsta positiva rot ligger i intervallet $[\pi, 2\pi]$

Vi finner $nL = 4{,}49341$

$$(nL)^2 = \frac{P_{kr}}{EI}L^2 = (4{,}49\ldots)^2$$

$$P_{kr} \approx \frac{20{,}19\,EI}{L^2} \approx \frac{2{,}05\pi^2 EI}{L^2}$$

Euler 2

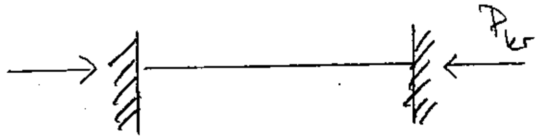


$P_{kr} = \frac{\pi^2 EI}{L^2}$

$(nL)^2 = \pi^2$

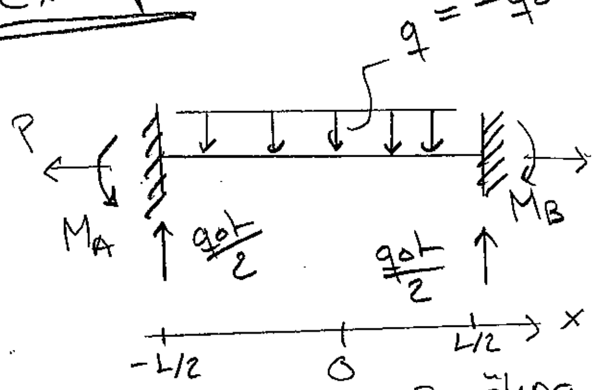
$nL = \pi$

Euler 4



$P_{kr} = \frac{4\pi^2 EI}{L^2}$, $nL = 2\pi$

Exempel



$P = \frac{4EI}{L^2}$; $nL = 2$

Beräkna M_A o M_B.

1) 1a ordningens teori: $M_A = M_B = \frac{q_0 L^2}{12}$

(F.S. sid 12)

<u>2a ordningens teori</u> $\quad w^{IV} - n^2 w'' = \frac{-q_0}{EI}$; $n^2 = \frac{P}{EI}$

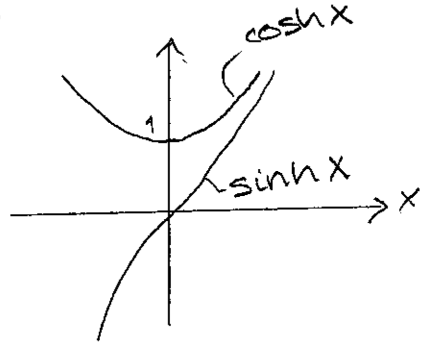
$$w = w_p + A + Bx + C\cosh(nx) + D\sinh(nx)$$

Ansätt $w_p = ax^2$, $w_p'' = 2a$, $w_p^{IV} = 0$

$$\Rightarrow -2n^2 a = \frac{-q_0}{EI}, \quad w_p = \frac{q_0 x^2}{2n^2 EI}$$

$$\therefore\ w(x) = \frac{q_0 x^2}{2n^2 EI} + A + Bx + C\cosh(nx) + D\sinh(nx)$$

Sym.: $w(x) = w(-x) \Rightarrow B = D = 0$



$w(\pm\frac{L}{2}) = 0 \Rightarrow$

$$\Rightarrow \frac{q_0 L^2}{8n^2 EI} + A + C\cosh\left(\frac{nL}{2}\right) = 0$$

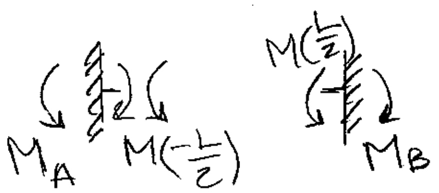
$$w' = \frac{q_0 x}{n^2 EI} + Cn\sinh(nx)$$

$$w'(\pm\tfrac{L}{2}) = 0 \Rightarrow \quad C = \frac{-q_0 L}{2n^3 EI \sinh\left(\frac{nL}{2}\right)}$$

$$\Rightarrow A = \frac{-q_0 L^2}{8n^2 EI} + \frac{q_0 L}{2n^3 EI} \cdot \frac{\cosh\left(\frac{nL}{2}\right)}{\sinh\left(\frac{nL}{2}\right)}$$

$$w'' = \frac{q_0}{n^2 EI} + Cn^2\cosh(nx)$$

$$M(x) = -EIw'' = \frac{-q_0}{n^2} + \frac{q_0 L\cosh(nx)}{2n\sinh\left(\frac{nL}{2}\right)}$$

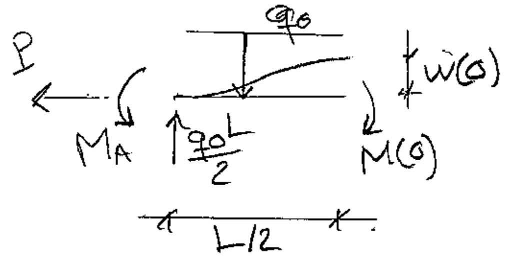


$$M_A = M_B = \frac{-q_0}{n^2} + q_0 L \frac{\cosh\left(\frac{nL}{2}\right)}{\sinh\left(\frac{nL}{2}\right)} = \frac{q_0 L^2}{12}\left(\frac{6}{nL}\frac{\cosh\left(\frac{nL}{2}\right)}{\sinh\left(\frac{nL}{2}\right)} - \frac{12}{(nL)^2}\right)$$

$$= f(nL)\cdot\frac{q_0 L^2}{12}$$ $f(nL)$ en Berryfunktion (sid 139)

$f(2) \approx 0{,}939.$ $\quad M_A = M_B = 0{,}94\,\frac{q_0 L^2}{12}$

$$M(0) = \frac{q_0 L}{2n\sinh\left(\frac{nL}{2}\right)} - \frac{q_0}{n^2}$$



$\underline{x=0}$: $M(0) = M_A + \frac{q_0 L}{2}\,\frac{L}{4} - \frac{q_0 L}{2}\cdot\frac{L}{2} - P\cdot w(0) =$

$$= \{ w(0) = A + C \} = \underbrace{\frac{-q_0}{n^2} + \frac{q_0 L \cosh\left(\frac{nL}{2}\right)}{2n\sinh\left(\frac{nL}{2}\right)}}_{M_A} -$$

$$-\frac{q_0 L^2}{8} + \underbrace{\frac{q_0 L^2}{8} - \frac{q_0 L}{2n}\,\frac{\cosh(nL/2)}{\sinh(nL/2)}}_{-P\cdot A} + \frac{q_0 L}{2n\sinh\left(\frac{nL}{2}\right)} =$$

$$= \frac{-q_0}{n^2} + \frac{q_0 L}{2n\sinh\left(\frac{nL}{2}\right)}$$

/end ons Lv6
end Hållf. teori

$\left[P = n^2 EI\right]$