MATEMATIK
Chalmers tekniska högskola

Hjälpmedel: Beta, Standard math. tabels, typgodkänd räknedosa
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TMA132 Fourieranalys F2/Kf2, 5 poäng

OBS! Ange namn, personnummer samt linje och inskrivningsår.

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1. Ett linjärt tidsinvariant kausalt system har stegsvaret $f(t) = e^{-t}\theta(t)$, (dvs. $f(t)$ är utsignalen då insignalen är $\theta(t)$).

   a) Beräkna utsignalen då insignalen är $t\theta(t)$.

   b) En sinusformad insignal ger en utsignal, vars amplitud är hälften av insignalens. Beräkna vinkelfrekvensen.

2. Bestäm det polynom $P(x)$ av högst andra graden som minimerar

$$\int_{-\infty}^{\infty} [|x| - P(x)]^2 \, e^{-x^2} dx.$$

3. a) Lös Laplaces differentialekvation

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < 2, \quad y > 0, \\ u(0,y) = u_x(2,y) = 0, & \lim_{y\to\infty} u(x,y) = 0, \\ u(x,0) = \begin{cases} 0, & 0 < x < 1, \\ 1, & 1 < x < 2. \end{cases} \end{cases}$$

   b) Ge någon fysikalisk tolkning av problemet i uppgift (a).

4. Bestäm en lösning till problemet,

$$\begin{cases} u_t = u_{xx} + u_x, & -\infty < x < \infty, \quad t > 0, \\ u(x,0) = x^2 e^{-x^2}, & -\infty < x < \infty. \end{cases}$$

   Ledning: Fouriertransformera i x-led.

5. Låt f vara en funktion i $L^1(\mathbb{R})$. Definiera $F(x) = \sum_{k=-\infty}^{\infty} f(x + 2k\pi)$.

   a) Visa att F är 2π periodisk.

   b) Härled ett samband mellan F:s Fourierkoefficienter och f:s Fouriertransform.

   c) Bevisa (under lämpliga förutsättningar) *Poissons Summationsformel:*

$$\sum_{k=-\infty}^{\infty} f(2k\pi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

6. Bestäm temperaturen $u = u(r,t)$ i klotet $r = \sqrt{x^2 + y^2 + z^2} < 1$, då

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial u}{\partial r}), & 0 < r < 1, \quad t > 0, \\ u(1,t) + u_r(1,t) = 0, & u(r,0) = f(r), \quad u \text{ begränsad.} \end{cases}$$

7. Formulera och bevisa samplingsteoremet.

8. $J_n(x)$ är Besselfunktion an orning n. Visa *genererande funktionsfomel:*

$$\sum_{-\infty}^{\infty} J_n(x) z^n = e^{\frac{x}{2}(z - \frac{1}{z})}, \quad \forall z \neq 0, \quad \forall x.$$

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Lösningar, TMA 132. Fourieranalys F2/Kf2, 2004-01-17

1. a) $\theta(t) \overset{S}{\curvearrowright} e^{-t}\theta(t) \xrightarrow{L\text{-transf.}} \frac{1}{s+1} = \frac{1}{s} H(s) \Rightarrow H(s) = \frac{s}{s+1}$

så $t\theta(t) \overset{S}{\curvearrowright} y(t)$. Eftersom $t\theta(t) \overset{L}{\supset} \frac{1}{s^2}$ får

$Y(s) = \frac{1}{s^2} H(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$. Dvs

$\underline{\underline{y(t) = (1-e^{-t})\theta(t)}}$

b) $\sin(\omega t) \overset{S}{\curvearrowright} A(\omega)\sin(\omega t + \varphi) = \frac{1}{2}\sin(\omega t + \varphi) \Rightarrow$

$|\hat{H}(\omega)| = |H(i\omega)| = \left|\frac{i\omega}{i\omega+1}\right| = \frac{1}{2} \Longrightarrow$

$\frac{|\omega|}{\sqrt{1+\omega^2}} = \frac{1}{2} \Rightarrow \omega^2 + 1 = 4\omega^2 \Rightarrow \underline{\underline{\omega = \frac{1}{\sqrt{3}}}}$.

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2. Använd Hermite polynomen $H_n(x)$. Skriv

$P(x) = a_0 H_0(x) + a_1 H_1(x) + a_2 H_2(x)$

$\int_{-\infty}^{\infty} \left[|x| - \sum_0^2 a_n H_n(x)\right]^2 e^{-x^2} dx$ blir minimal

precis då $a_n = c_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} |x| H_n(x) e^{-x^2} dx$

$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2$

$c_0 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} |x| e^{-x^2} dx = \frac{2}{\sqrt{\pi}} \int_0^{\infty} x e^{-x^2} dx = \frac{1}{\sqrt{\pi}} \left[-e^{-x^2}\right]_0^{\infty} = \frac{1}{\sqrt{\pi}}$

$c_1 = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} |x| \cdot 2x \cdot e^{-x^2} dx = 0$ (udda integrand)

$c_2 = \frac{1}{8\sqrt{\pi}} \int_{-\infty}^{\infty} |x|(4x^2-2) e^{-x^2} = \frac{1}{\sqrt{\pi}} \int_0^{\infty} x^3 e^{-x^2} dx - \frac{c_0}{4} =$

$= \frac{1}{\sqrt{\pi}} \left[\cancel{-\frac{x^2}{2} e^{-x^2}}\right]_0^{\infty} + \frac{1}{\sqrt{\pi}} \int_0^{\infty} x e^{-x^2} dx = \frac{c_0}{2} - \frac{c_0}{4} = \frac{1}{4\sqrt{\pi}}$

$P(x) = \frac{1}{\sqrt{\pi}} \cdot 1 + \frac{1}{4\sqrt{\pi}}(4x^2 - 2) = \underline{\underline{\frac{1}{\sqrt{\pi}}\left(x^2 + \frac{1}{2}\right)}}$.

3. a) (DE): $u_{xx} + u_{yy} = 0 \qquad 0 \le x \le 2, \quad 0 \le y < \infty$

(RV1): $u(0,y) = 0$ (RV3): $u(x,0) = f(x) = \begin{cases} 0, & 0<x<1 \\ 1, & 1<x<2 \end{cases}$

(RV2): $u'_x(2,y) = 0$ (RV4): $\lim\limits_{y\to\infty} u(x,y) = 0$.

Delproblem: Finn lösningar $u(x,y) = X(x)Y(y)$ till DE + RV1 + RV2, som inte är $\equiv 0$.

$$X''(x)Y(y) + X(x)Y''(y) = 0; \quad \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda.$$

Egenvärdesproblem: $X'' = \lambda X; \quad X(0) = X'(2) = 0.$

Egenlösningar: $X_n(x) = \sin\left(n+\frac{1}{2}\right)\frac{\pi x}{2}; \quad n = 0,1,2,\ldots$

Egenvärden $\lambda_n(x) = -\left[\left(n+\frac{1}{2}\right)\frac{\pi}{2}\right]^2 = -\alpha_n^2.$

$$Y_n''(y) - \left[\left(n+\frac{1}{2}\right)\frac{\pi}{2}\right]^2 Y_n(y) = 0,$$

$$Y_n(y) = A_n e^{\alpha_n y} + B_n e^{-\alpha_n y}, \qquad \alpha_n = \left(n+\frac{1}{2}\right)\frac{\pi}{2}$$

$X_n(x)Y_n(y)$ klarar delproblemet.

Superposition:

$$u(x,y) = \sum_{n=0}^{\infty}\left[A_n e^{\alpha_n y} + B_n e^{-\alpha_n y}\right]\sin \alpha_n x$$

uppfyller DE + RV1 + RV2.

RV4 uppfyllt om $A_n = 0, \ \forall n$.

$u(x,y) = \sum_{n=0}^{\infty} B_n e^{-\alpha_n y}\sin(\alpha_n x)$ klarar DE+RV1+RV2+RV4.

(RV3): $f(x) = \sum_{n=0}^{\infty} B_n \sin(\alpha_n x) = $ OG-serie i $(0,2)$

$$B_n = \frac{1}{M_n}\int_0^2 f(x)\sin(\alpha_n x)\,dx = \int_1^2 \sin(\alpha_n x)\,dx = \left[M_n = \int_0^2 \sin^2\alpha_n x\,dx = 1\right]$$

$$= \left[-\frac{\cos\left(n+\frac{1}{2}\right)\frac{\pi x}{2}}{\left(n+\frac{1}{2}\right)\frac{\pi}{2}}\right]_1^2 = \frac{\cos\left(n+\frac{1}{2}\right)\frac{\pi}{2}}{\left(n+\frac{1}{2}\right)\frac{\pi}{2}}.$$

Svar: $$u(x,y) = \sum_{n=0}^{\infty}\frac{\cos\alpha_n}{\alpha_n}e^{-\alpha_n y}\sin(\alpha_n x), \qquad \alpha_n = \left(n+\frac{1}{2}\right)\frac{\pi}{2}$$

b) Värmeledning i området $0<x<2, \ 0<y<\infty, \ -\infty<z<\infty$.

Begränsningsytan $x=0$ hålls vid temp $u=0$, ytan $x=2$ är isolerad, ytan $y=0$ vid temp. $f(x)$. Stationärt tillstånd. Inga inre värmekällor.

4. $\begin{cases} u_t = u_{xx} + u_x, & -\infty < x < \infty, \quad t > 0 \\ u(x,0) = x^2 e^{-x^2} = f(x). \end{cases}$

Fouriertransformera i x-led. $\mathcal{F}_x[u(x,t)] = \hat{u}(\xi, t)$.

$$\frac{\partial \hat{u}}{\partial t} = (i\xi)^2 \hat{u} + (i\xi)\hat{u} = (-\xi^2 + i\xi)\hat{u}$$

$$\hat{u}(\xi,t) = C(\xi)\, e^{(-\xi^2 + i\xi)t}$$

$$\hat{u}(\xi,0) = C(\xi) = \hat{f}(\xi)$$

$$e^{-x^2} \overset{\mathcal{F}}{\supset} \sqrt{\pi}\, e^{-\xi^2/4} \Rightarrow x e^{-x^2} \overset{\mathcal{F}}{\supset} i \frac{d}{d\xi} \sqrt{\pi}\, e^{-\xi^2/4} = -\frac{i\sqrt{\pi}}{2} \xi e^{-\xi^2/4}$$

$$f(x) = x^2 e^{-x^2} \overset{\mathcal{F}}{\supset} i \frac{d}{d\xi}\left(\frac{-i\sqrt{\pi}}{2} \xi e^{-\xi^2/4}\right) = \frac{\sqrt{\pi}}{2}\left(1 - \frac{\xi^2}{2}\right) e^{-\xi^2/4} = \hat{f}(\xi)$$

$$\hat{u}(\xi,t) = \frac{\sqrt{\pi}}{2}\left(1 - \frac{\xi^2}{2}\right) e^{-\xi^2/4}\, e^{-\xi^2 t} \cdot e^{i\xi t} =$$

$$= \frac{\sqrt{\pi}}{2}\left(1 - \frac{\xi^2}{2}\right) e^{-(t+\frac{1}{4})\xi^2}\, e^{i\xi t}$$

Låt $A = t + \frac{1}{4}$; $\quad \frac{1}{\sqrt{4\pi A}}\, e^{-x^2/4A} \overset{\mathcal{F}}{\supset} e^{-A\xi^2}$

$$-i \frac{d}{dx}\left(\frac{1}{\sqrt{4\pi A}}\, e^{-x^2/4A}\right) \overset{\mathcal{F}}{\supset} \xi e^{-A\xi^2}$$

$$\xi e^{-A\xi^2} \subset \frac{1}{\sqrt{4\pi A}} \frac{ix}{2A} e^{-x^2/4A}$$

$$\xi^2 e^{-A\xi^2} \subset -i \frac{d}{dx}\left(\frac{1}{\sqrt{4\pi A}} \frac{ix}{2A} e^{-x^2/4A}\right) = \frac{1}{2A\sqrt{4\pi A}}\left(1 - \frac{x^2}{2A}\right) e^{-x^2/4A}$$

$$\frac{\sqrt{\pi}}{2}\left(1 - \frac{\xi^2}{2}\right) e^{-A\xi^2} \overset{\mathcal{F}}{\subset} \frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{4\pi A}}\left[1 - \frac{1}{2} \cdot \frac{1}{2A}\left(1 - \frac{x^2}{2A}\right)\right] e^{-x^2/4A}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{4t+1}}\left[1 - \frac{1}{4t+1}\left(1 - \frac{2x^2}{4t+1}\right)\right] e^{-x^2/4t+1}$$

$$= \frac{2t(4t+1) + (x+t)^2}{(4t+1)^{5/2}}\, e^{-x^2/(4t+1)}$$

$$u(x,t) = \frac{2t(4t+1) + (x+t)^2}{(4t+1)^{5/2}}\, e^{-\frac{(x+t)^2}{4t+1}}.$$

5. a) $F(x) = \sum_{-\infty}^{\infty} f(x+2k\pi)$

$$F(x+2\pi) = \sum_{-\infty}^{\infty} f(x+2(k+1)\pi) = \sum_{-\infty}^{\infty} f(x+2k\pi) = F(x).$$

$\therefore$ F 2π-periodisk.

b) $$C_n(F) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)e^{-inx}\,dx = \frac{1}{2\pi}\sum_{-\infty}^{\infty}\int_{-\pi}^{\pi} f(x+2k\pi)e^{-inx}\,dx$$

$$= \sum_{-\infty}^{\infty} \frac{1}{2\pi}\int_{(2k-1)\pi}^{(2k+1)\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-inx} = \frac{\hat{f}(n)}{2\pi}.$$

c) $$\sum_{-\infty}^{\infty} f(2k\pi) = F(0) = \sum_{-\infty}^{\infty} C_n(F)\, e^{in0} = \frac{1}{2\pi}\sum_{-\infty}^{\infty}\hat{f}(n).$$

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6. $$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right); & 0<r<1;\ t>0, \\ u(r,t) \text{ begränsad} \\ u(1,t) + u_r(1,t) = 0 \\ u(r,0) = f(r). \end{cases}$$

Vi kan skriva $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru)$.

För $v = ru$ fås då ekvationerna:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial r^2} \\ v(0,t) = v_r(1,t) = 0, \quad v(r,0) = r f(r). \end{cases}$$

$v(r,t) = R(r)T(t)$ i de homogena ekvationer ger $\frac{T'}{T} = \frac{R''}{R} = -\lambda$,

$R'' + \lambda R$, $R(0) = R'(1) = 0 \Rightarrow R = R_n(r) = \sin(n+\tfrac{1}{2})\pi r$

$T' = -\lambda_n T \Rightarrow T = T_n(t) = C_n e^{-\lambda_n t}$. Ansatz

$v(r,t) = \sum_0^{\infty} C_n e^{-\lambda_n t}\sin(n+\tfrac{1}{2})\pi r \Rightarrow v(r,0) = \sum_0^{\infty} C_n \sin(n+\tfrac{1}{2})\pi r = r f(r)$.

$\{\sin(n+\tfrac{1}{2})\pi r\}_0^{\infty}$ OG-bas $\Rightarrow C_n = 2\int_0^1 r f(r)\sin(n+\tfrac{1}{2})\pi r\,dr$

Med denna C_n blir lösningen $u(r,t) = \frac{1}{r}\sum_{n=0}^{\infty} C_n e^{-(n+\frac{1}{2})^2\pi^2 t}\sin(n+\tfrac{1}{2})\pi r$

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