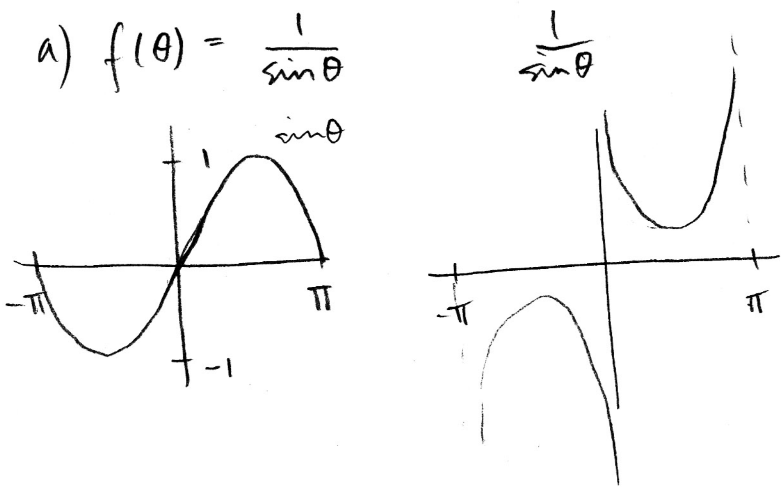
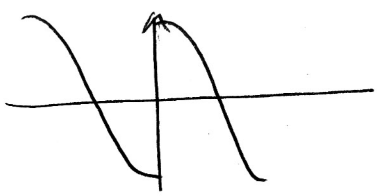
$\boxed{2.2.1}$ $[-\pi, \pi]$

a) $f(\theta) = \frac{1}{\sin\theta}$



ej kontin i $\pi, 0, -\pi$
ej styckvis glatt/kontin

d) $f(\theta) = \begin{cases} \cos\theta & 0 < \theta \leq \pi \\ -\cos\theta & -\pi < \theta \leq 0 \end{cases}$



ej kontin i 0
styckvis kontin
— " — glatt

$\boxed{2.2.4}$ Visa $\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$

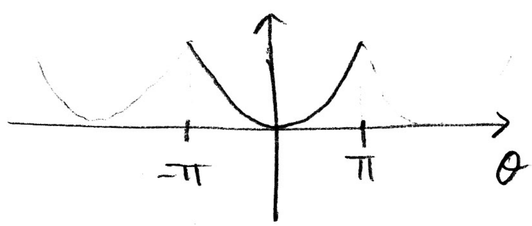
$$\sum_1^\infty \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

m.h.a #16:

$f(\theta) = \theta^2; \quad -\pi < \theta < \pi$

har Fourierserie

$$\frac{\pi^2}{3} + 4\sum_1^\infty \frac{(-1)^n}{n^2}\cos n\theta$$



f kontin, styckvis glatt
F-serien konv. punktvis

$$\theta^2 = \frac{\pi^2}{3} + 4\sum_1^\infty \frac{(-1)^n}{n^2}\cos n\theta$$

$\cos n\pi = (-1)^n$ så välj $\theta = \pi$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_1^\infty \frac{1}{n^2} \Rightarrow \frac{2\pi^2}{12} = \sum_1^\infty \frac{1}{n^2}$$

6

$\theta = 0$ ger $\cos n\theta = 1 \quad \forall n$

$$0 = \frac{\pi^2}{3} + 4\sum_1^\infty \frac{(-1)^n}{n^2}$$

$$\sum_1^\infty \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

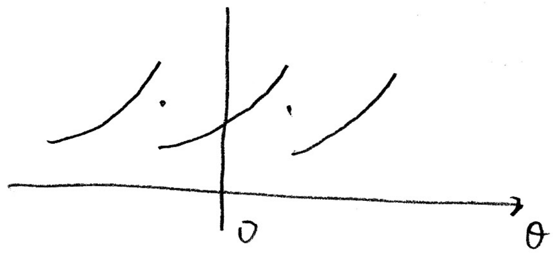
$\boxed{2.2.6}$ Visa att $(b>0)$

$$\sum_1^\infty \frac{(-1)^n}{n^2+b^2} = \frac{\pi}{2b}\operatorname{csch}(b\pi) - \frac{1}{2b^2}$$

m.h.a #18

$f(\theta) = e^{b\theta} \qquad -\pi < \theta < \pi$

har F-serie $\dfrac{\sinh(b\pi)}{\pi}\displaystyle\sum_{-\infty}^{\infty}\frac{(-1)^n}{b-in}e^{in\theta}$



$\theta = 0$ ger:

$$1 = \frac{\sinh(b\pi)}{\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n}{b-in}$$

$$\sum_{-\infty}^{\infty}\frac{(-1)^n}{b-in} = \sum_{-\infty}^{\infty}\frac{(-1)^n(b+in)}{b^2+n^2} = 2b\sum_{n=1}^{\infty}\frac{(-1)^n}{b^2+n^2} + \frac{1}{b}$$

$$\frac{(-1)^n b}{b^2+n^2} + \underbrace{\frac{(-1)^n in}{b^2+n^2}}_{\text{udda i } n}$$

alltså:

$$1 = \frac{\sinh(b\pi)}{\pi}\left(2b\sum_1^\infty\frac{(-1)^n}{b^2+n^2} + \frac{1}{b}\right) \dots$$

Rö Fourier 080127 On Lv1

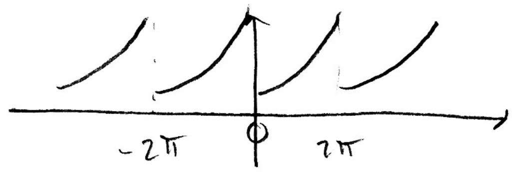
2.2.7 Visa att

$$\sum_1^\infty \frac{1}{n^2+b^2} = \frac{\pi}{2b}\coth(b\pi) - \frac{1}{2b^2}$$

mha #19:

$f(\theta) = e^{b\theta}$ $\quad 0 \le \theta \le 2\pi$ har F-serie

$$\frac{e^{2\pi b}-1}{2\pi}\sum_{-\infty}^{\infty}\frac{e^{in\theta}}{b-in}$$



$$\theta = 0: \quad \frac{e^{b2\pi}+1}{2} = \frac{e^{2\pi b}-1}{2\pi}\sum_{-\infty}^{\infty}\underbrace{\frac{e^{in\theta=0}}{b-in}}_{\frac{1}{b^2+n^2}(b+in)} \text{ jämn, udda} =$$

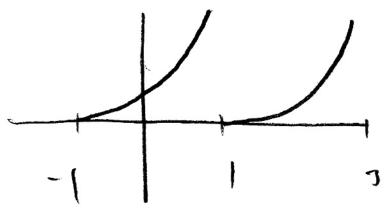
$$= \frac{e^{2\pi b}-1}{2\pi}\left(2b\sum_1^\infty \frac{1}{b^2+n^2} + \frac{1}{b}\right)$$

$$\Rightarrow \quad 2b\sum_1^\infty \frac{1}{b^2+n^2} + \frac{1}{b} = \pi\,\frac{e^{2\pi b}+1}{e^{2\pi}-1} = \pi\,\frac{(e^{\pi b}+e^{-\pi b})/2}{(e^{\pi b}-e^{-\pi b})/2} =$$

$$= \pi\,\frac{\cosh \pi b}{\sinh \pi b} = \pi \coth \pi b$$

EÖ1 $f(x)$ är 2-periodisk

$f(x) = (x+1)^2$ för $-1 < x < 1$



a) Utveckla f i en F-serie

b) Finn en 2-periodisk lösn till

$$2y'' - y' - y = f(x)$$

a) $f(x) \sim \sum_{-\infty}^{\infty} c_n e^{in\pi x/l}$ $\quad f$ 2l-per.

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-in\frac{\pi x}{l}}\, dx$$

$l=1$

$$c_n = \frac{1}{2}\int_{-1}^{1} (x+1)^2 e^{-in\pi x}\, dx =$$

$n \neq 0$

$$= \frac{1}{2}\left(\left[(x+1)^2 \frac{e^{-in\pi x}}{-in\pi}\right]_{-1}^{1} - \int_{-1}^{1} \frac{2(x+1)\, e^{-in\pi x}}{-in\pi}\, dx\right) =$$

$$= \frac{1}{2}\left(\left(4\frac{e^{-in\pi}}{-in\pi}\right) - \left[\frac{2(x+1)\, e^{-in\pi x}}{(-in\pi)^2}\right]_{-1}^{1} + \int_{-1}^{1} \frac{2e^{-in\pi x}}{(-in\pi)^2}\, dx\right) =$$

$$= \frac{1}{\cancel{2}}\left(\frac{\cancel{4}^{2} i e^{-in\pi}}{n\pi} + \frac{\cancel{4}^{2} e^{-in\pi}}{n^2\pi^2} + \left[\frac{\cancel{2} e^{-in\pi x}}{(-in\pi)^3}\right]_{-1}^{1}\right) =$$

$$= \frac{2i\overbrace{e^{-in\pi}}^{(-1)^n}}{n\pi} + \frac{2\overbrace{e^{-in\pi}}^{(-1)^n}}{n^2\pi^2} + \underbrace{\left(\frac{e^{-in\pi}}{in^3\pi^3} - \frac{e^{in\pi}}{in^3\pi^3}\right)}_{=0} =$$

$$= 2(-1)^n\left(\frac{i}{n\pi} + \frac{1}{n^2\pi^2}\right) = 2(-1)^n \frac{n\pi + i}{n^2\pi^2}$$

$$c_0 = \frac{1}{2}\int_{-1}^{1} (x+1)^2\, dx = \frac{1}{2}\left[\frac{1}{3}(x+1)^3\right]_{-1}^{1} = \frac{8}{6} = \frac{4}{3}$$

$$f(x) \sim = \frac{4}{3} + \frac{2}{\pi^2}\sum_{\substack{-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n (n\pi i + 1)}{n^2}\, e^{in\pi x}$$

b) y 2-periodisk

$$y = \sum_{-\infty}^{\infty} d_n e^{in\pi x}$$

$$y' = \sum_{-\infty}^{\infty} in\pi d_n e^{in\pi x}$$

$$y'' = \sum_{-\infty}^{\infty} (in\pi)^2 d_n e^{in\pi x}$$

$$2y'' - y' - y = \sum_{-\infty}^{\infty} d_n(-2(n\pi)^2 - in\pi - 1) e^{in\pi x}$$

Entydighet ger:

$d_0(-1) = \frac{4}{3}$ så $d_0 = -\frac{4}{3}$

$$d_n = -\frac{2(-1)^n (n\pi i + 1)}{\pi^2 n^2 (2(n\pi)^2 + in\pi + 1)} \qquad n \neq 0$$

Rö Fourier 080130 OnLv2

$\boxed{E013}$ Lös diff/integral-ekv

$$u'(t) + 2u(t) + \int_{-\infty}^{t} e^{-2(t-\tau)} u(\tau)\, d\tau = \delta(t)$$

$$u'(t) + 2u(t) + \int_{-\infty}^{\infty} e^{-2(t-\tau)} \theta(t-\tau) u(\tau)\, d\tau = \delta(t)$$

$$u'(t) + 2u(t) + (\theta(\cdot)e^{-2(\cdot)} * u)(t) = \delta(t)$$

Fouriertransformera:

$$i\omega \hat{u}(\omega) + 2\hat{u}(\omega) + \widehat{\theta e^{-2(\cdot)}}(\omega)\cdot \hat{u}(\omega) = 1$$

$$\widehat{\theta(t) e^{-2t}}(\omega) = \int_{-\infty}^{\infty} \theta(t) e^{-2t} e^{-i\omega t}\, dt = \int_{0}^{\infty} e^{-t(2+i\omega)}\, dt =$$

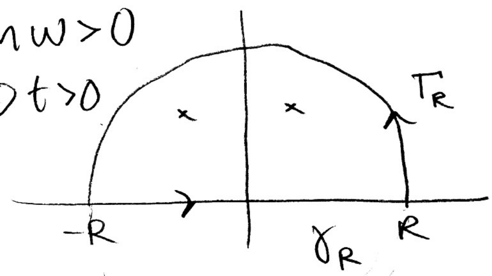
$$= \left[\frac{e^{-t(2+i\omega)}}{-(2+i\omega)}\right]_0^{\infty} = +\frac{1}{2+i\omega}$$

$$\hat{u}(\omega)\left(i\omega + 2 + \frac{1}{2+i\omega}\right) = 1$$

$$\hat{u}(\omega) = \frac{2+i\omega}{(2+i\omega)^2+1}$$

$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} \underbrace{\frac{2+i\omega}{1+(2+i\omega)^2}}_{f(\omega)}\, d\omega$$

$\operatorname{Im}\omega > 0$
$\Rightarrow t > 0$



Poler: $1 + (2+i\omega)^2 = 0 \Rightarrow \omega = \pm 1 + 2i$

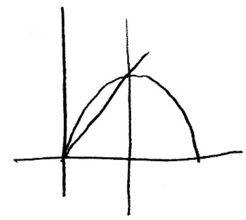
$$\int_{-R}^{R} f(\omega)\, d\omega + \int_{T_R} f(\omega)\, d\omega = 2\pi i \left(\operatorname{Res}(f, 1+2i) + \operatorname{Res}(f, -1+2i)\right)$$

$$\operatorname{Res}_{-1+2i} f = \lim_{\omega \to -1+2i} (\omega - (-1+2i)) f(\omega) = -i\, \frac{e^{-2t-it}}{2}$$

$$\operatorname{Res}_{1+2i} f = -i\, \frac{e^{-2t+it}}{2}$$

$$\frac{1}{2\pi}\int_{-R}^{R} f(w)\,dw + \frac{1}{2\pi}\int_{T_R} f(w)\,dw = i(-i)\left(e^{-2t}\left(\frac{e^{-it}+e^{it}}{2}\right)\right) = e^{-2t}\cos t$$

$$\left|\int_{T_R} e^{iwt}\frac{2+iw}{1+(2+iw)^2}\,dw\right| = \begin{bmatrix} w = Re^{i\theta} & 0\le\theta\le\pi \\ dw = Rie^{i\theta}d\theta & \end{bmatrix} \le$$

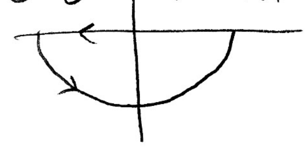


$$\le \frac{C}{R}\int_0^{\pi}\left|e^{it(Re^{i\theta})}Rie^{i\theta}\right|d\theta = C\int_0^{\pi}\left|e^{itR(\cos\theta+i\sin\theta)}\right|d\theta =$$

$$= C\int_0^{\pi} e^{-tR\sin\theta}\,d\theta = 2C\int_0^{\pi/2} e^{-tR\sin\theta}\,d\theta \le 2C\int_0^{\pi/2} e^{-tR\frac{2\theta}{\pi}}\,d\theta =$$

$$= 2C\left[\frac{e^{-tR\frac{2\theta}{\pi}}}{\frac{2tR}{\pi}}\right]_0^{\pi/2} = \frac{\pi C}{tR}\left(1-e^{-tR}\right) \xrightarrow{R\to\infty} 0$$

$t<0 \Rightarrow u(t)=0$



$u(t) = e^{-2t}\cos t \qquad t>0$

så: $u(t) = e^{-2t}\cos t\,\theta(t)$

$\boxed{7.2.13}$ a) Visa $\displaystyle\int_{-\infty}^{\infty}\frac{\sin(at)}{t}\frac{\sin(bt)}{t}\,dt = \pi\min(a,b)$

Plancherel: $\langle f,g\rangle = \frac{1}{2\pi}\langle \hat{f},\hat{g}\rangle$

$f(t) = \frac{\sin at}{t}, \quad g(t) = \frac{\sin bt}{t}$

sid 223:

$$\widehat{\frac{\sin at}{t}}(w) = \pi\chi_a(w) = \begin{cases}\pi & |w|<a \\ 0 & |w|>a\end{cases}$$

Rö Fourier 080130 on Lv2.

$$I = \langle f, g \rangle = \frac{1}{2\pi} \langle \hat{f}, \hat{g} \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi \chi_a(w) \pi \chi_b(w)\, dw =$$

$$= \frac{\pi}{2} \int_{-\min(a,b)}^{+\min(a,b)} 1\, dw = \pi \min(a,b)$$

b) Visa att $I = \int_{-\infty}^{\infty} \underbrace{\frac{t}{t^2+a^2}}_{f_a(t)} \cdot \frac{t}{t^2+b^2}\, dt = \frac{\pi}{a+b}$

$$I = \langle f_a, f_b \rangle, \quad \widehat{\frac{t}{t^2+a^2}}(w) = i \widehat{\frac{1}{t^2+a^2}}'(w) = \qquad \left[\widehat{tf} = i\hat{f}'(w) \right]$$

$$= i \frac{d}{dw}\left(\frac{\pi}{a} e^{-a|w|} \right) = -i\pi \operatorname{sgn}(w)\, e^{-a|w|}$$

$$\operatorname{sgn} w = \begin{cases} 1 & w > 0 \\ -1 & w < 0 \end{cases}$$

$$I = \langle f_a, f_b \rangle = \frac{1}{2\pi} \langle \hat{f}_a, \hat{f}_b \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-i\pi \operatorname{sgn} w\, e^{-a|w|}) \cdot$$

$$\cdot \overline{(-i\pi \operatorname{sgn} w\, e^{-b|w|})}\, dw = \frac{\pi}{2} \int_{-\infty}^{\infty} e^{-(a+b)|w|}\, dw = \pi \int_0^{\infty} e^{-(a+b)} \quad dw$$

$$= \pi \left[\frac{e^{-(a+b)w}}{-(a+b)} \right]_0^{\infty} = \frac{\pi}{a+b}$$

$\boxed{7.2.1}$ $f(x) = e^{-ax^2/2}$ $\quad a > 0$

Beräkna $\hat{f}(\xi)$

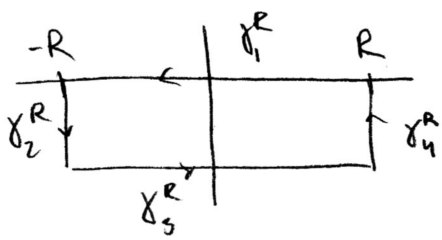
$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x - ax^2/2}\, dx$$

kvadr.-kompl.

$$-\frac{ax^2}{2} - ix\xi = -\frac{a}{2}\left(x^2 + x \frac{2i\xi}{a} \right) = -\frac{a}{2}\left(\left(x + \frac{i\xi}{a} \right)^2 + \left(\frac{\xi}{a} \right)^2 \right)$$

$g(x) = e^{\wedge}$

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$



Vill ha

$$\lim_{R\to\infty} -\int_{\gamma_1^R} g(x)\,dx$$, g holomorf (x komplex var.)

$$\Rightarrow \int_{\gamma_1^R} + \int_{\gamma_2^R} + \int_{\gamma_3^R} + \int_{\gamma_4^R} = 0$$

$$\int_{\gamma_3^R} g(x)\,dx = \left\{ \begin{array}{l} s = x + \frac{i\xi}{a} \\ ds = dx \\ -R < s < R \end{array} \right\} = \int_{-R}^{R} e^{-\frac{a}{2}\left(s^2 + \left(\frac{\xi}{a}\right)^2\right)}\,dx =$$

$$= e^{-\frac{\xi^2}{2a}} \int_{-R}^{R} e^{-\frac{a}{2}s^2}\,ds = \left\{ \begin{array}{l} t = s\sqrt{\frac{a}{2}} \\ dt = \sqrt{\frac{a}{2}}\cdot ds \end{array} \right\} = e^{-\frac{\xi^2}{2a}} \int_{-R\sqrt{\frac{a}{2}}}^{R\sqrt{\frac{a}{2}}} e^{-t^2}\cdot \sqrt{\frac{2}{a}}\,dt$$

$$\xrightarrow{R\to\infty} \sqrt{\frac{2\pi}{a}}\, e^{-\frac{\xi^2}{2a}}$$

$$\left| \int_{\gamma_2^R} g(x)\,dx \right| = \left\{ \begin{array}{l} x = -R - it \\ dx = -i\,dt \end{array} \quad 0 \le t \le \frac{\xi}{a} \right\} = \left| \int_0^{\xi/a} e^{i\xi(R+it) - a\frac{(R+it)^2}{2}}\, i\,dt \right| \le$$

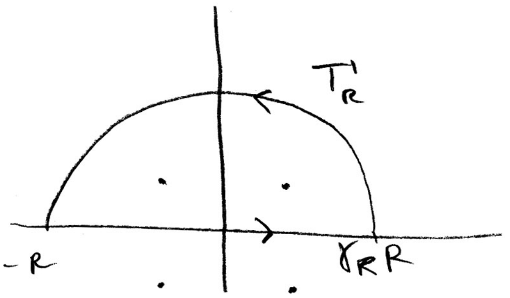
$$\le \int_0^{\xi/a} \left| e^{i\xi(R+it) - a\frac{(R+it)^2}{2}} \right| dt = \int_0^{\xi/a} e^{-t\xi - aR^2/2 + at^2/2}\,dt =$$

$$= e^{-a\frac{R^2}{2}} \int_0^{\xi/a} e^{-t\xi + at^2/2}\,dt \xrightarrow{R\to\infty} 0$$ pss med $\int_{\gamma_4^R}$

$\boxed{7.2.9}$ Visa att

$$\mathcal{F}\left(\frac{1}{1+x^4}\right) = \frac{\pi}{\sqrt{2}} e^{-|\xi|/\sqrt{2}}\left(\cos\left(\frac{\xi}{\sqrt{2}}\right) + \sin\left(\frac{|\xi|}{\sqrt{2}}\right)\right)$$

$$\| \quad \int_{-\infty}^{\infty} \frac{e^{-ix\xi}}{1+x^4}\,dx$$



$1+x^4=0$ då $x = \frac{1}{\sqrt{2}}(\pm 1 \pm i)$

$\mathrm{Re}(-i\xi x) < 0$ om $\xi < 0$

$$\int_{\gamma_R} g(x)\,dx + \int_{\Gamma_R} g(x)\,dx = 2\pi i\left(\mathrm{Res}\left(g, \tfrac{1}{\sqrt{2}}(-1+i)\right) + \mathrm{Res}\left(g, \tfrac{1}{\sqrt{2}}(+1+i)\right)\right)$$

$$\mathrm{Res}\left(\frac{e^{-i\xi x}}{1+x^4}, \frac{1}{\sqrt{2}}(-1+i)\right) = \frac{e^{-i\xi x}}{4x^3} = -\frac{x e^{-i\xi x}}{4} = \frac{(1-i)\, e^{\frac{i\xi(1-i)}{\sqrt{2}}}}{\sqrt{2}\cdot 4}$$

$$\mathrm{Res}\left(\frac{e^{-ix\xi}}{1+x^4}, \frac{1}{\sqrt{2}}(1+i)\right) = -\frac{(1+i)\, e^{-\frac{i\xi(1+i)}{\sqrt{2}}}}{\sqrt{2}\cdot 4}$$

$$HL \overset{\xi<0}{=} \frac{\pi}{\sqrt{2}} e^{-|\xi|/\sqrt{2}}\left(\cos(\xi/\sqrt{2}) + \sin(|\xi|/\sqrt{2})\right)$$

$$x^4 = -1 \Rightarrow x^3 = -\frac{1}{x}$$

$$\cos(\theta) = \{\text{integrering}\} = 1 - \frac{2}{\pi}\theta + \frac{4}{\pi}\sum_1^\infty \frac{\sin 2n\theta}{2n(4n^2-1)}$$

$$\cos(\theta) = \{\text{derivering}\} = \frac{4}{\pi}\sum_1^\infty \frac{\sin 2n\theta}{2n} + \frac{4}{\pi}\sum_1^\infty \frac{\sin 2n\theta}{2n(4n^2-1)}$$

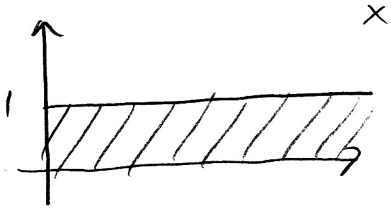
$$f(\theta) = \pi - \theta \sim 2\sum_1^\infty \frac{\sin n\theta}{n} \qquad 0<\theta<2\pi$$

$$\frac{1}{\pi}\cdot f(2\pi) = 1 - \frac{2}{\pi}\theta \sim \frac{4}{\pi}\sum_1^\infty \frac{\sin 2n\theta}{2n} \qquad 0<\theta<\pi$$

$\boxed{7.4.6}$ Lös Laplace ekv

$u''_{xx} + u''_{yy} = 0$ i det semioändliga bandet

$x>0, \quad 0<y<1$



$u'_x(0,y) = 0$

$u'_y(x,0) = 0$

$u(x,1) = e^{-x}$

Variabelsep.: ansätt:

$$u(x,y) = X(x)\,Y(y)$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\xi^2$$

$$X''(x) + \xi^2 X(x) = 0$$

$$Y''(y) - \xi^2 Y(y) = 0$$

$$\Rightarrow X(x) = A(\xi)\cos\xi x + B(\xi)\sin\xi x$$

$$Y(y) = C(\xi)\cosh\xi y + D(\xi)\sinh\xi y$$

$$0 = X'(0) = \xi B(\xi) \Rightarrow B(\xi) = 0$$

$$0 = Y'(0) = \xi D(\xi) \Rightarrow D(\xi) = 0$$

Låt $E(\xi) = A(\xi) \cdot C(\xi)$

$$u(x,y) = \int_0^\infty E(\xi) \cosh \xi y \, \cancel{\cos} \xi x \, d\xi$$

$E(\xi)$ bestäms av $u(x,1) = e^{-x}$

$$e^{-x} = u(x,1) = \int_0^\infty E(\xi) \cosh \xi \cos \xi x \, d\xi =$$

$$\mathcal{F}_c(e^{-x})(\xi) = \frac{1}{1+\xi^2}$$

Invers:

$$e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{1}{1+\xi^2} \cos \xi x \, d\xi$$

$$E(\xi) \cosh(\xi) = \frac{2}{\pi} \frac{1}{1+\xi^2}$$

$$\Rightarrow u(x,y) = \frac{2}{\pi} \int_0^\infty \frac{\cosh(\xi x) \cos(\xi x)}{(1+\xi^2)(\cosh(\xi x))} \, d\xi$$

$\boxed{\text{EÖ 45}}$ Lös $(k>0)$:

$$u'_t = k u''_{xx}, \quad x \in \mathbb{R}, \ t > 0$$

$$u(x,0) = (1-2x^2)e^{-x^2}, \quad x \in \mathbb{R}$$

sök begr. lösn

Fouriertransformering:

$$\hat{u}'_t(\xi,t) = -k\xi^2 \hat{u}(\xi,t)$$

$$\hat{u}(\xi,0) = \mathcal{F}((1-2x^2)e^{-x^2})(\xi)$$

Tabell:

$$\mathcal{F}(e^{-x^2}) = \sqrt{\pi}\, e^{-\xi^2/4}$$

$$\mathcal{F}(xf) = i\hat{f}'(\xi)$$

$$\mathcal{F}(x^2e^{-x^2}) = i^2\mathcal{F}(e^{-x^2})''(\xi) = -\frac{d^2}{d\xi^2}\left(\sqrt{\pi}\, e^{-\xi^2/4}\right) =$$

$$= \frac{\sqrt{\pi}}{2}e^{-\xi^2/4} - \frac{\sqrt{\pi}}{4}\xi^2 e^{-\xi^2/4}$$

$$\mathcal{F}((1-2x^2)e^{-x^2}) = \sqrt{\pi}\, e^{-\xi^2/4} - \sqrt{\pi} e^{-\xi^2/4} + \frac{\sqrt{\pi}}{2}\xi^2 e^{-\xi^2/4} = \frac{\sqrt{\pi}}{2}\xi^2 e^{-\xi^2/4}$$

$$\hat{u}(\xi,t) = C(\xi)e^{-tk\xi^2}$$

$$C(\xi) = \hat{u}(\xi,0) = \frac{\sqrt{\pi}}{2}\xi^2 e^{-\xi^2/4}$$

$$\hat{u}(\xi,t) = \frac{\sqrt{\pi}}{2}\xi^2 e^{-\xi^2(\frac{1}{4}+kt)}$$

$$u(x,t) = \frac{\sqrt{\pi}}{2}i^2\frac{d^2}{dx^2}\mathcal{F}^{-1}\left(e^{-\xi^2\underbrace{(\frac{1}{4}+kt)}_{=b}}\right)$$

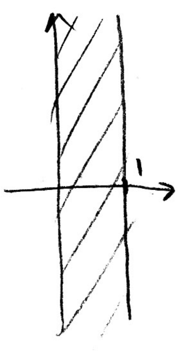
$$\mathcal{F}^{-1}(e^{-b\xi^2}) = e^{-x^2/4b}\cdot\frac{1}{\sqrt{4\pi b}}$$

$$u(x,t) = \frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{4\pi b}}\frac{d^2}{dx^2}\left(e^{-x^2/4b}\right) = \dots = \frac{1}{16b^{5/2}}e^{-x^2/4b}(2b-x^2) =$$

$$= \frac{1+4kt-2x^2}{(1+4kt)^{5/2}}e^{-x^2/(4kt+1)}$$

$\boxed{\text{EÖ 46}}$ Lös probl

$$\begin{cases} u''_{xx} + u''_{yy} = x & 0<x<1,\ -\infty<y<\infty \\ u'_x(0,y) = 0 \\ u(1,y) = ye^{-|y|} \end{cases}$$



$u(x,y) = v(x,y) + S(x)$

Bestäm $S(x)$ så att:

$$v''_{xx} + v''_{yy} = u''_{xx} + u''_{yy} - S''(x) = x - S''(x) \overset{!}{=} 0 \Rightarrow S''(x) = x$$

$$v'_x(0,y) = u'_x(0,y) - S'(0) = -S'(0) \overset{!}{=} 0 \Rightarrow S'(0) = 0$$

$$v(1,y) = u(1,y) - S(1) = ye^{-|y|} - S(1) \overset{!}{=} ye^{-|y|} \Rightarrow S(1) = 0$$

$$S'(x) = \frac{x^2}{2} + A \qquad 0 = S'(0) = A$$

$$\Rightarrow S(x) = \frac{x^3}{6} + B \qquad 0 = S(1) \Rightarrow B = -\frac{1}{6}$$

$$S(x) = \frac{1}{6}(x^3 - 1)$$

$$\begin{cases} v''_{xx} + v''_{yy} = 0 \\ v'_x(0,y) = 0 \\ v(1,y) = ye^{-|y|} \end{cases} \qquad \begin{matrix} 0 < x < 1 \\ -\infty < y < \infty \end{matrix}$$

FT:

(1) $\hat{v}''_{xx}(x,\eta) - \eta^2 \hat{v}(x,\eta) = 0$

(2) $\hat{v}'_x(0,\eta) = 0$

(3) $\hat{v}(1,\eta) = \mathcal{F}(ye^{-|y|})$, $\quad \mathcal{F}(e^{-|y|}) = \dfrac{2}{1+\eta^2}$

$$\Rightarrow \mathcal{F}(ye^{-|y|}) = i \frac{d}{d\eta}\left(\frac{2}{1+\eta^2}\right) = -\frac{4i\eta}{(1+\eta^2)^2}$$

$$\hat{v}(x,\eta) = A(\eta)\cosh(x\eta) + B(\eta)\sinh(\eta x)$$

$(2) \Rightarrow B(\eta) = 0$

$(3) \Rightarrow -\dfrac{4i\eta}{(1+\eta^2)^2} = \hat{v}(1,\eta) = A(\eta)\cosh(\eta)$

$$\Rightarrow A(\eta) = -\frac{4i\eta}{(1+\eta^2)^2\cosh(\eta)}$$

Inverstransform:

$$v(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{-\frac{4i\eta\cosh(\eta x)}{(1+\eta^2)^2\cosh(\eta)}}_{\text{udda i } \eta} \underbrace{e^{i\eta y}}_{\cos(\eta y)+i\sin(\eta y)} d\eta =$$

$$= \frac{4}{\pi}\int_0^{\infty} \frac{\eta\cosh(\eta x)\sin(\eta y)}{(1+\eta^2)^2\cosh(\eta)} d\eta$$

$\boxed{\text{EÖ47}}$ $f \in L^2(\mathbb{R})$

1) Sök lösn till

$$\begin{cases} u''_{xx} + u''_{yy} = 0 & x \in \mathbb{R},\ 0<y<a \\ u(x,0) = 0 \\ u(x,a) = f(x) \end{cases}$$

2) Visa att $\int_{-\infty}^{\infty} |u(x,y)|^2 dx \leq \int_{-\infty}^{\infty} |f(x)|^2 dx$

$$\begin{cases} -\xi^2\hat{u}(\xi,y) + \hat{u}''_{yy}(\xi,y) = 0 \\ \hat{u}(\xi,0) = 0 \\ \hat{u}(\xi,a) = \hat{f}(x) \end{cases}$$

$$\hat{u}(\xi,y) = C_1(\xi)\sinh(\xi y) + C_2(\xi)\cosh(\xi y)$$

$$0 = \hat{u}(\xi,0) = C_2(\xi)$$

$$\hat{f}(\xi) = \hat{u}(\xi,a) = C_1(\xi)\sinh(\xi a)$$

$$C_1(\xi) = \frac{\hat{f}(\xi)}{\sinh(\xi a)}$$

Inverstransf.

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sinh(\xi y)}{\sinh(\xi a)} \hat{f}(\xi) e^{i\xi x} \, d\xi$$

$$\mathcal{F}\left(\frac{\sinh(a_1 x)}{\sinh(bx)}\right)(\xi) \overset{\text{BETA}}{=} \frac{\pi \sin\left(\pi \frac{a_1}{b}\right)}{b\cosh\left(\xi \frac{a_1}{b}\right) + b\cos\left(\pi \frac{a_1}{b}\right)} \qquad 0 < a < b$$

sätt $a_1 = y$
$b = a$ symmetriregeln

$$\mathcal{F}\left(\frac{1}{2a} \frac{\sin\left(\pi \frac{y}{a}\right)}{\cosh\left(\frac{\pi x}{a}\right) + \cos\left(\frac{\pi y}{a}\right)}\right) = \frac{\sinh(\xi y)}{\sinh(\xi a)}$$

$$\int u(x)\hat{v}(x)\,dx = \int \hat{u}(x) v(x)\,dx$$

$$u(x,y) = \frac{1}{2a}\int_{-\infty}^{\infty} \frac{\sin\frac{\pi y}{a}}{\cosh\left(\frac{\pi}{a}(x-t)\right) + \cos\left(\frac{\pi y}{a}\right)} f(t)\,dt$$

2) $$\int_{-\infty}^{\infty} |u(x,y)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{u}(\xi,y)|^2 d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\left|\frac{\sinh(\xi y)}{\sinh(\xi a)}\right|^2}_{\leq 1} |\hat{f}(\xi)|^2 d\xi \leq$$

$$\leq \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$\boxed{\text{EÖ25}}$

$$\begin{cases} u''_{xx} + u''_{yy} = y & 0<x<2,\ 0<y<1 \\ u(x,0) = 0 \\ u(x,1) = 0 \\ u(0,y) = y - y^3 \\ u(2,y) = 0 \end{cases}$$

$$w(x,y) = u(x,y) - v(y)$$

$$w''_{xx} + w''_{yy} = u''_{xx} + u''_{yy} - v''(y) = y - v''(y) \overset{!}{=} 0$$

$$w(x,0) = u(x,0) - v(0) = -v(0) \overset{!}{=} 0$$

$$w(x,1) = u(x,1) - v(1) = -v(1) \overset{!}{=} 0$$

$$w(0,y) = u(0,y) - v(y) = y - y^3 - v(y) = \tfrac{7}{6}(y - y^3)$$

$$w(2,y) = u(2,y) - v(y) = -v(y) = \tfrac{1}{6}(y^3 - y)$$

$$\begin{cases} v''(y) = y \\ v(0) = 0 \\ v(1) = 0 \end{cases} \Rightarrow \quad v(y) = \frac{y^3}{6} + by + c$$

$$0 = v(0) = c$$

$$0 = v(1) = \tfrac{1}{6} + b \Rightarrow b = -\tfrac{1}{6}$$

$$v(y) = \tfrac{1}{6}(y^3 - y)$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \nu^2 \quad (>0 \text{ kolla själva andra fallen})$$

$$\begin{cases} Y''(y) + \nu^2 Y = 0 \\ Y(0) = 0 \\ Y(1) = 0 \end{cases}$$

så $\nu = n\pi, \quad n = 1, 2, 3, \ldots$

$$Y_n(y) = A_n \sin(n\pi y)$$

$$X_n''(x) + n^2\pi^2 X_n(x) = 0$$

$$X_n(x) = c_n \cosh(n\pi x) + d_n \sinh(n\pi x)$$

$$w(x,y) = \sum_1^\infty \left(c_n \cosh(n\pi x) + d_n \sinh(n\pi x)\right) \sin(n\pi y)$$

$$\frac{7}{6}(y - y^3) = w(0,y) = \sum_1^\infty c_n \sin(n\pi y)$$

$$\Rightarrow c_n = 2\int_0^1 \frac{7}{6}(y - y^3)\sin(n\pi y)\,dy =$$

$$= \cdots = \frac{14(-1)^n}{(n\pi)^3}$$

$$\frac{1}{6}(y - y^3) = w(2,y) = \sum_1^\infty \underbrace{\left(c_n \cosh(n\pi \cdot 2) + d_n \sinh(n\pi \cdot 2)\right)}_{k_n} \sin n\pi y$$

$$k_n = 2\int_0^1 \frac{1}{6}(y - y^3)\sin(n\pi y)\,dy = \frac{2(-1)^n}{(n\pi)^3}$$

$$\Rightarrow d_n = \frac{k_n - c_n \cosh(2\pi n)}{\sin(2\pi n)} = \frac{2(-1)^{n+1}(1 + 7\cosh(2\pi n))}{(n\pi)^3 \sinh(2\pi n)}$$

$\boxed{\text{EÖ28}}$ Lös problemet:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = t \sin x & 0 < x < 1,\ t > 0 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = \sin(2\pi x) \end{cases}$$

Homogent problem:

$$\begin{cases} u_t' - u_{xx}'' = 0 \\ u(0,t) = u(1,t) = 0 \end{cases}$$

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T} = -(n\pi)^2$$

$$X_n = \sin(n\pi x)$$

Ansätt:

$$u(x,t) = \sum_1^\infty \beta_n(t) \sin(n\pi x)$$

$$\frac{\partial u}{\partial t} = \sum_1^\infty \beta_n'(t) \sin(n\pi x)$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_1^\infty -(n\pi)^2 \beta_n(t) \sin(n\pi x)$$

$$\sin x = \sum_1^\infty c_n \underbrace{\sin(n\pi x)}_{\varphi_n}$$

$$\sum_1^\infty \left(\beta_n'(t) + (n\pi)^2 \beta_n(t) - t c_n\right) \sin(n\pi x) = 0$$

Så:

$$\beta_n'(t) + (n\pi)^2 \beta_n(t) - t c_n = 0, \quad n = 1, 2, \ldots$$

$$c_n = \frac{\langle \sin x, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} = \frac{\int_0^1 \sin x \sin(n\pi x)\, dx}{\int_0^1 \sin^2(n\pi x)\, dx} = \frac{2(-1)^{n+1} n\pi \sin(1)}{(n\pi)^2 - 1}$$

β_n: Homogen lösn: $\beta_n^h(t) = A_n e^{-(n\pi)^2 t}$

Part-lösn. ansätt: $\beta_n^p(t) = a_n t + b_n$

$$a_n + (n\pi)^2(a_n t + b_n) - c_n t = 0 \quad \Rightarrow \quad \begin{cases} a_n = \dfrac{c_n}{(n\pi)^2} \\ b_n = -\dfrac{a_n}{(n\pi)^2} = -\dfrac{c_n}{(n\pi)^4} \end{cases}$$

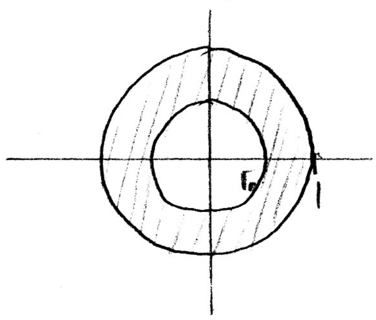
$$u(x,t) = \sum_1^\infty \left(A_n e^{-(n\pi)^2 t} + \frac{c_n t}{(n\pi)^2} - \frac{c_n}{(n\pi)^4}\right)\sin(n\pi x)$$

$$\sin(2\pi x) = u(x,0) = \sum_1^\infty \left(A_n - \frac{c_n}{(n\pi)^4}\right)\sin(n\pi x)$$

$$\begin{cases} A_n = \dfrac{c_n}{(n\pi)^4} & \text{för } n \neq 2 \\ A_2 = 1 + \dfrac{c_2}{(2\pi)^4} \end{cases}$$

$\boxed{4.4.5}$

$$(\Delta u = \nabla^2 u =) \begin{cases} u''_{rr} + \frac{1}{r}u'_r + \frac{1}{r^2}u''_{\theta\theta} = 0 \\ u'_r(r_0, \theta) = 0 \\ u(1,\theta) = f(\theta) \\ f(\theta) = f(\theta + 2\pi) \end{cases}$$



$$u(r,\theta) = R(r)\Theta(\theta)$$

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

Dela med $\frac{1}{r^2}R\Theta$:

$$\frac{r^2 R''(r)}{R(r)} + \frac{r R'(r)}{R(r)} = -\frac{\Theta''}{\Theta} = A$$

$$\Theta'' - A\Theta = 0$$

$$r^2 R''(r) + r R'(r) - A R(r) = 0$$

$\Theta'' = -A\Theta$

$\underline{A<0}$: $A = -\mu^2$, $\mu > 0$

$\Theta(\theta) = Ae^{\mu\theta} + Be^{-\mu\theta}$

aldrig 2π-per.

$\Rightarrow A = B = 0$

$\underline{A=0}$:

$\Theta = a\theta + b$

endast 2π-per om $\Theta = b$:

$\underline{A>0}$: $A = \nu^2$, $\nu \geq 0$

$\Theta(\theta) = Ae^{i\nu\theta} + Be^{-i\nu\theta}$

$\Theta(\theta+2\pi) = Ae^{i\nu\theta}\underbrace{e^{i\nu 2\pi}}_{=1} + Be^{-i\nu\theta}\underbrace{e^{-i\nu 2\pi}}_{=1} \Rightarrow \nu = \text{heltal}$

$\Theta(\theta) = A_n e^{in\theta}$, $n \in \mathbb{Z}$

$r^2R'' + rR'(r) - n^2R(r) = 0$

$R_n(r) = a_n r^n + b_n r^{-n}$, $n \neq 0$

$R_0(r) = a_0 + b_0 \ln r$

$$u(r,\theta) = \sum_{n\neq 0}(a_n r^n + b_n r^{-n})A_n e^{in\theta} + (a_0 + b_0 \ln r)A_0 =$$

$$= \sum_{n\neq 0}(a_n' r^n + b_n' r^{-n})e^{in\theta} + (a_0' + b_0' \ln r)$$

$$0 = u_r'(r_0,\theta) = \sum_{n\neq 0} (a_n' n r_0^{n-1} + b_n'(-n) r_0^{-n-1})\, e^{in\theta} + \frac{b_0'}{r_0}$$

$$\Rightarrow b_0' = 0$$

$$b_n' = a_n' r_0^{2n}$$

$$u(r,\theta) = a_0' + \sum_{n\neq 0} a_n' (r^n + r_0^{2n} r^{-n})\, e^{in\theta}$$

$$f(\theta) = u(1,\theta) = a_0' + \sum_{n\neq 0} a_n' (1 + r_0^{2n})\, e^{in\theta}$$

$$c_0 = a_0' = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$c_n = a_n'(1 + r_0^{2n}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in\theta}\, d\theta$$

$$f(\theta) = \sum c_n e^{-in\theta}$$

$$u(r,\theta) = \sum_{-\infty}^{\infty} c_n \left(\frac{r^n + r_0^{2n} r^{-n}}{1 + r_0^{2n}}\right) e^{in\theta}$$

b) $f(\theta) = 1 + 2\sin\theta = 1 + \frac{e^{i\theta}}{i} - \frac{e^{-i\theta}}{i}$

$c_n = 0$ om $n \neq -1, 0, 1$

ying koeff:

$c_{-1} = i$

$c_0 = 1$

$c_1 = -i$

$$u(r,\theta) = i\left(\frac{r^{-1} + r_0^{-2} r}{1 + r_0^{-2}}\right) e^{-i\theta} + 1 - i\left(\frac{r + r_0^2 r^{-1}}{1 + r_0^2}\right) e^{i\theta} =$$

$$= 1 + \frac{r^2 + r_0^2}{r(1 + r_0^2)} 2\sin\theta$$

Rö Fourier 080220 On Lv5

$\boxed{4.4.6}$ Låt D vara enhetsskivan

$$\{(r,\theta):\ 0\le r\le 1,\ 0\le\theta\le 2\pi\}$$

Låt $P(r,\theta)$ vara Poissonkärnan och låt $u(r,\theta)$ vara lösn till Dirichlet-probl. $\nabla^2 u = 0$ i D, $u(1,\theta)=f(\theta)$

a) Visa att

$$u(0,\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

b) Visa att $P(r,\theta) > 0$ och

$$\int_{-\pi}^{\pi} P(r,\theta)\,d\theta = 2\pi, \quad \text{då } r<1$$

c) använd b) för att visa:
Om $f(\theta)\le M\ \forall\theta$, då gäller att $u(r,\theta)\le M\ \forall\theta,\ 0\le r\le 1$

a)
$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)}}_{P(r,\,\theta-\varphi)} f(\varphi)\,d\varphi$$

$$u(0,\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{1}f(\varphi)\,d\varphi = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\,d\varphi$$

b)
$$P(r,\theta) = \frac{1-r^2}{1+r^2+2r\cos(\theta)} = \frac{(1+r)(1-r)}{(1-r)^2+2r(1-\cos\theta)} > 0$$

$$P(r,\theta) = \sum_{-\infty}^{\infty} r^{|n|}e^{in\theta} \qquad r<1$$

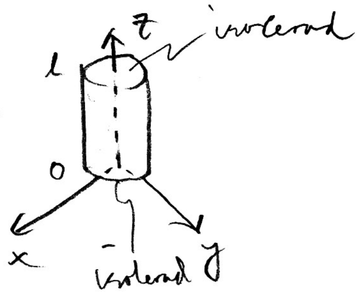
$$\int_{-\pi}^{\pi} P(r,\theta)\,d\theta = \sum_{-\infty}^{\infty} r^{|n|}\int_{-\pi}^{\pi} e^{in\theta}\,d\theta = 2\pi$$

c) Antag $f(\theta) \leq M$

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{P(r,\theta-\varphi)}_{\geq 0} \underbrace{f(\varphi)}_{\leq M} \, d\varphi \leq \frac{M}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-\varphi)\, d\varphi = M$$

Rö Fourier 080227 On Lv6

$\boxed{\text{EÖ 33}}$



$u'_t = \Delta u$

$u = u(r, \varphi, z, t)$

$u(b, \varphi, z, t) + 2u'_r(b, \varphi, z, t) = 0$

$u(r, \varphi, z, 0) = r^2$

$u'_z(r, \varphi, \overset{0}{l}, t) = 0$

u ej ber. av φ

u ej ber. av z

$\Downarrow$

$u = u(r, t)$

$$\begin{cases} u'_t = u''_{rr} + \frac{1}{r}u'_r + \cancel{\frac{1}{r^2}u''_{\varphi\varphi} + u''_{zz}} \\ u(b,t) + 2u'_r(b,t) = 0 \\ u(r,0) = r^2 \\ u \text{ begr} \end{cases}$$

$u = R(r)\,T(t)$

$\Rightarrow \; T'(t)R(r) = R''(r)T(t) + \frac{1}{r}R'(r)T(t)$

$\Rightarrow \; \frac{T'}{T} = \frac{R'' + \frac{1}{r}R'}{R} = -\mu^2, \quad \mu > 0 \quad T = e^{-\mu^2 t}$

$r^2R'' + rR' + \mu^2 r^2 R = 0$

$R(b) + 2R'(b) = 0$

$$\left[\begin{array}{l} x^2 f''(x) + x f'(x) + (\mu^2 x^2 - \nu^2) f(x) = 0 \\ f(x) = A J_\nu(\mu x) + B Y_\nu(\mu x) \end{array}\right]$$

$R(r) = AJ_0(\mu r) + \underbrace{BY_0(\mu r)}_{\text{ej begr.}}$

$\Rightarrow B = 0$

$R(r) = J_0(\mu r)$

$$J_0(\mu b) + 2\mu J_0'(\mu b) = 0$$

$$\frac{b}{2} J_0(\mu b) + \mu b J_0'(\mu b) = 0$$

Välj $0 < \mu_1 < \mu_2 < \ldots$ s.a $\{\mu_k\}_1^\infty$ är de pos. nollst

till $\frac{b}{2} J_0(x) + x J_0'(x)$

Enl sats 5.3 b) är då $\{J_0(\mu_k b r/b)\}_{k=1}^\infty$ bas

för $L^2_w]0,b[$ och $\|J_0(\mu_k r)\|_w^2 = \frac{b^2((\mu_k b)^2 + (\frac{b}{2})^2)}{(\mu_k b)^2} J_0(\mu_k b)^2$

$$u(r,t) = \sum_1^\infty c_k e^{-\mu_k^2 t} J_0(\mu_k r)$$

$$u(r,0) = \sum_1^\infty c_k J_0(\mu_k r) = r^2$$

$$\Rightarrow c_k = \frac{\langle r^2, J_0(\mu_k r)\rangle_w}{\|J_0(\mu_k r)\|_w^2}$$

$$\langle r^2, J_0(\mu_k r)\rangle_w = \int_0^b r^3 J_0(\mu_k r)\,dr = \begin{bmatrix} s = \mu_k r & \\ ds = \mu_k dr & 0 \xrightarrow{s} \mu_k b \end{bmatrix} =$$

$$= \int_0^{\mu_k b} \frac{1}{\mu_k^3} s^3 J_0(s) \frac{1}{\mu_k} ds = \frac{1}{\mu_k^4} \int_0^{\mu_k b} s^3 J_0(s)\,ds = \begin{bmatrix} \frac{d}{dx}(x^\nu J_\nu(x)) = x^\nu J_{\nu-1}(x) \\ (5.14) \end{bmatrix}$$

$$= \frac{1}{\mu_k^4} \int_0^{\mu_k b} s^2 (s J_1(s))' ds = \frac{1}{\mu_k^4}\left(\left[s^2 s J_1(s)\right]_0^{\mu_k b} - 2\int_0^{\mu_k b} (s^2 J_2)' ds \right) =$$

$$= \frac{1}{\mu_k^4}\left((\mu_k b)^3 J_1(\mu_k b) - 2(\mu_k b)^2 J_2(\mu_k b) \right) =$$

$$= \frac{b^3}{\mu_k} J_1(\mu_k b) - 2\frac{b^2}{\mu_k^2} J_2(\mu_k b)$$

$\boxed{E\bar{O}34}$ a) Bestäm en begr. lösn på formen

$$u(r,t) = v(r)\, e^{i\omega t}$$

till:
$$\begin{cases} u''_{tt} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{n^2}{r^2}u & 0<r<a \\ & n \geq 0,\ n \in \mathbb{Z} \\ u(a,t) = e^{i\omega t} \end{cases}$$

För vilka ω finns sådan lösn?

Ekv:
$$-\omega^2 v(r)\, e^{i\omega t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)e^{i\omega t} - \frac{n^2}{r^2}v(r)\, e^{i\omega t}$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \left(\omega^2 - \frac{n^2}{r^2}\right)v(r) = 0$$

$$\Rightarrow v_n(r) = c_n J_n(\omega r) + d_n Y_n(\omega r)$$

v begr $\Rightarrow d_n = 0$ (Y_n ej begr i 0)

$v_n(r) = c_n J_n(\omega r)$

$u(r,t) = c_n J_n(\omega r)\, e^{i\omega t}$

$e^{i\omega t} = u(a,t) = c_n \cdot J_n(\omega a)\, e^{i\omega t}$

så $c_n = \frac{1}{J_n(\omega a)}$ <u>om</u> $J_n(\omega a) \neq 0$

$$u(r,t) = \frac{J_n(\omega r)}{J_n(\omega a)}\, e^{i\omega t}$$

b) Låt ω vara s.a. $J_n(\omega a) \neq 0$. Lös nu

(i) $\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial r}\right) - \frac{n^2}{r^2}u$

(ii) $u(a,t) = \sin \omega t$, u begr. (iii)

(iv) $u(r,0) = 0$, $u'_t(r,0) = 0$ (v)

"Imaginärdelen av a)":

$$v(r,t) = \frac{J_n(\omega r)}{J_n(\omega a)} \sin \omega t \quad \text{löser (i)} \to \text{(iii)}$$

$$y(r,t) = u(r,t) - \frac{J_n(\omega r)}{J_n(\omega a)} \sin(\omega t)$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial y}{\partial r}\right) - \frac{n^2}{r^2} y(r,t)$$

$$y(a,t) = 0$$

$$y(r,0) = 0$$

$$y_t'(r,0) = -\omega \frac{J_n(\omega r)}{J_n(\omega a)}$$

Var.-sep.: $y(r,t) = R(r)T(t)$

$$\frac{T''(t)}{T(t)} = \frac{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - \frac{n^2}{r^2}R(r)}{R(r)} = \underbrace{-\lambda^2}_{\text{måste vara negativt enl. T-ekv.}}$$

$$T(t) = A\sin(\lambda t) + B\cos(\lambda t)$$

$$0 = T(0) = B$$

$$\Rightarrow T(t) = A\sin(\lambda t)$$

$$\begin{cases} R(r) = cJ_n(\lambda r) + \cancel{dY_n(\lambda r)} & \text{obegr.} \\ R(a) = 0 \end{cases}$$

$J_n(\lambda a) = 0$, Låt $0 < \lambda_1 < \lambda_2 < \ldots$ vara s.a.

$J_n(\lambda_k a) = 0$

Sats 5.3 ger att $\{J_n(\lambda_k r)\}_1^\infty$ bas

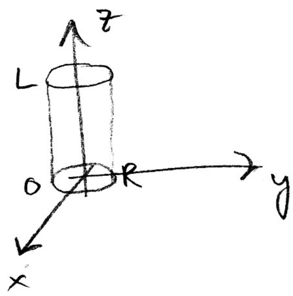
$$y(r,t) = \sum_1^\infty A_k \sin(\lambda_k t)\, J_n(\lambda_k r)$$

$$y_t'(r,0) = \sum_1^\infty A_k \lambda_k J_n(\lambda_k r) = -\omega \frac{J_n(\omega r)}{J_n(\omega a)}$$

$$\Rightarrow \quad A_k \lambda_k = \frac{\langle -\omega \frac{J_n(\omega r)}{J_n(\omega a)},\ J_n(\lambda_k r)\rangle_w}{\| J_n(\lambda_k r)\|_w^2}$$

$$\Rightarrow \quad A_k = -\frac{\omega}{J_n(\omega a)} \frac{\int_0^a J_n(\omega r) J_n(\lambda_k r)\, r\, dr}{\lambda_k \frac{a^2}{2} J_{n+1}(\lambda_k a)^2}$$

$\boxed{\text{EÖ35}}$ Lös $\Delta u = 0$ i cylindern:



$$u(r,\varphi,0) = u(r,\varphi,L) = 0$$

$$u(R,\varphi,z) = \sin\left(\frac{\pi z}{L}\right)\left(1-\cos\left(\frac{\pi z}{L}\right)\right)$$

$$u = u(r,z)$$

$$\Delta u = u''_{rr} + \frac{1}{r}u'_r + \cancel{\frac{1}{r^2}u''_{\varphi\varphi}} + u''_{zz}$$

Var.-sep.: $u = R(r)Z(z)$

$$\Rightarrow \quad \frac{R''(r) + \frac{1}{r}R(r)}{R(r)} = -\frac{Z''(z)}{Z(z)} = \lambda^2$$

$$Z(z) = A\sin(\lambda z) + B\cos(\lambda z)$$

$$Z(0) = B = 0$$

$$Z(L) = A\sin(\lambda L) = 0$$

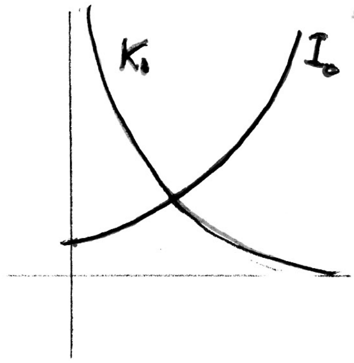
$$\lambda_n = \frac{n\pi}{L}, \quad n = 1,2,\ldots$$

$$Z_n(z) = \sin\left(\frac{n\pi z}{L}\right)$$

$$r^2 R_n''(r) + rR_n'(r) - \lambda_n^2 r^2 R(r) = 0$$

$$\left[\, x^2 f''(x) + x f'(x) + (\nu^2 - \lambda^2 x^2) f(x) = 0 \quad \text{Modifierad Bessel}\right.$$
$$\text{ger: } f(x) = A I_\nu(\lambda x) + B K_\nu(\lambda x)$$

$$R_n(r) = A I_0(\lambda_n r) + B K_0(\lambda_n r)$$



$$\overset{\text{se bild}}{\Longrightarrow} \quad B = 0$$

$$u(r,z) = \sum_{1}^{\infty} c_n I_0(\lambda_n r) \sin\left(\frac{n\pi z}{L}\right)$$

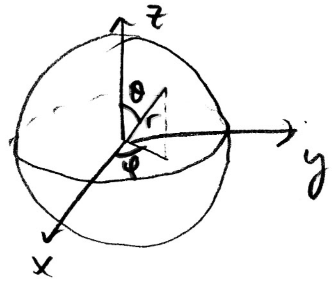
$$u(R,z) = \sum_{1}^{\infty} c_n I_0(\lambda_n R) \sin\left(\frac{n\pi z}{L}\right) = \sin\left(\frac{\pi z}{L}\right)\left(1 - \cos\left(\frac{\pi z}{L}\right)\right) =$$

$$= \sin\left(\frac{\pi z}{L}\right) - \frac{1}{2}\sin\left(\frac{2\pi z}{L}\right)$$

$$\Longrightarrow \begin{cases} c_n = 0, \quad n \neq 1,2 \\ c_1 = \dfrac{1}{I_0(\lambda_1 R)}, \quad c_2 = -\dfrac{1}{2 I_0(\lambda_2 R)} \end{cases}$$

$$\Longrightarrow u(r,z) = \frac{I_0\left(\frac{\pi r}{L}\right)}{I_0\left(\frac{\pi R}{L}\right)} \sin\left(\frac{\pi z}{L}\right) - \frac{1}{2} \frac{I_0\left(\frac{2\pi r}{L}\right)}{I_0\left(\frac{2\pi R}{L}\right)} \sin\left(\frac{2\pi z}{L}\right)$$

<u>G special</u> $u(r,\varphi,\theta,t)$



$$\begin{cases} u'_t = \Delta u & \text{i } B(0,1) \\ u(1,\varphi,\theta,t) = 0 \\ u(r,\varphi,\theta,0) = r\cos\varphi \\ u(r,\varphi+2\pi,\theta,t) = u(r,\varphi,\theta,t) \end{cases}$$

u begr.

Separera tidsvariabeln

$$u(r,\varphi,\theta,t) = \Psi(r,\varphi,\theta)\cdot T(t)$$

$$\Rightarrow \frac{T'}{T} = \frac{\Delta\Psi}{\Psi} = -\mu^2$$

$$T(t) = e^{-\mu^2 t}$$

$$0 = \Delta\Psi + \mu^2\Psi = \Psi''_{rr} + \frac{2}{r}\Psi'_r + \frac{1}{r^2}\underbrace{\left(\frac{1}{\sin\theta}(\Psi'_\theta\sin\theta)'_\theta + \frac{1}{\sin^2\theta}\Psi''_{\varphi\varphi}\right)}_{\Lambda\Psi} + \mu^2\Psi$$

$$= \Psi''_{rr} + \frac{2}{r}\Psi'_r + \frac{1}{r^2}\Lambda\Psi + \mu^2\Psi$$

$$\Psi(r,\varphi,\theta) = R(r)f(\varphi,\theta)$$

$$\frac{r^2R'' + 2rR' + \mu^2r^2R}{R} = -\frac{\Lambda f}{f} = \lambda$$

<u>Sats:</u> (s.180) $\lambda = n(n+1)$, $n = 0,1,2,\dots$

$$f_{nm}(\varphi,\theta) = e^{im\varphi}P_n^{|m|}(\cos\theta), \quad |m| \le n$$

utgör en ortogonal bas för $L_2(S, \sin\theta\, d\varphi\, d\theta)$ och

$$\|f_{nm}\|^2 = \frac{4\pi}{2n+1}\frac{(n+|m|)!}{(n-|m|)!} \qquad (\Lambda f_{nm} = -n(n+1)f_{nm})$$

$$r^2R'' + 2rR' + (\mu^2r^2 - n(n+1))R = 0$$

$$g(r) = r^{1/2}R(r)$$

$$\Rightarrow r^2g'' + rg' + (\mu^2r^2 - (n+\tfrac{1}{2})^2)g = 0$$

$$g(r) = A J_{n+\frac{1}{2}}(\mu r) + \underbrace{\cancel{B Y_{n+\frac{1}{2}}(\mu r)}}_{\text{obegr}}$$

$$\Rightarrow R(r) = r^{-1/2} J_{n+\frac{1}{2}}(\mu r)$$

$$R(1) = J_{n+\frac{1}{2}}(\mu) = 0$$

Låt $0 < \mu_1^n \leq \mu_2^n \leq \dots$ vara nollst till $J_{n+\frac{1}{2}}$

Då utgör (för varje n) $\{J_{n+\frac{1}{2}}(\mu_l^n r)\}_{l=1}^{\infty}$ en ortogonal bas i $L_r^2]0,1[= L_2(]0,1[, r\,dr)$ och $\|J_{n+\frac{1}{2}}(\mu_l^n r)\|_r^2 = \frac{1}{2} J_{n+\frac{3}{2}}(\mu_l^n)^2$

$\{r^{-1/2} J_{n+\frac{1}{2}}(\mu_l^n r)\}_{l=1}^{\infty}$ utgör bas i $L_2(]0,1[, r^2 dr)$

Sätt $F_{lmn}(r,\varphi,\theta) = r^{-1/2} J_{n+\frac{1}{2}}(\mu_l^n r) e^{im\varphi} P_n^{|m|}(\cos\theta)$

Då utgör $\{F_{lmn}\}$ en ortogonal bas för $L_2(B(0,1), r^2 \sin\theta\, dr\, d\varphi\, d\theta)$

och:

$$\|F_{lmn}\|^2 = \frac{2\pi (n+|m|)!\, J_{n+\frac{3}{2}}(\mu_l^n)^2}{(2n+1)(n-|m|)!}$$

$$u(r,\varphi,\theta,t) = \sum_{l,m,n} c_{lmn} F_{lmn} e^{-(\mu_l^n)^2 t}$$

$$u(r,\varphi,\theta,0) = \sum_{l,m,n} c_{lmn} F_{lmn}(r,\varphi,\theta) = r\cos\varphi$$

$$c_{lmn} = \frac{\langle r\cos\varphi, F_{lmn}\rangle}{\|F_{lmn}\|^2}$$

$$\langle r\cos\varphi, F_{lmn}\rangle = \iiint_{B(0,1)} r\cos\varphi\, r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\mu_l^n r) e^{-im\varphi} P_n^{|m|}(\cos\theta)\, r^2 \sin\theta\, dr\, d\varphi\, d\theta =$$

$$= \int_0^1 r^{5/2} J_{n+\frac{1}{2}}(\mu_l^n r)\, dr \cdot \underbrace{\int_0^{2\pi} \cos\varphi\, e^{-im\varphi}\, d\varphi}_{\text{ger bidrag} \iff m=\pm 1} \cdot \underbrace{\int_0^{\pi} P_n^{|m|}(\cos\theta) \sin\theta\, d\theta}_{=\int_{-1}^{1} P_n^{|m|}(s)\, ds}$$

$\boxed{6.5.6}$ Utveckla $f(x) = e^{-bx}$, $b > \overset{(-1)}{0}$, $x > 0$

i en serie med Laguerre-polynom.

$\underline{\alpha > -1:}$

$$L_n^\alpha(x) = \frac{x^{-\alpha} e^{-x}}{n!} \frac{d^n}{dx^n}\left(x^{\alpha+n} e^{-x}\right)$$

$\underline{\text{Sats 6.15:}}$ $\{L_n^\alpha\}_{n=0}^\infty$ utgör fullständig ort. bas i $L_2(]0,\infty[, \underbrace{x^\alpha e^{-x}}_{w(x)} dx)$

$$\|L_n^\alpha\|_w^2 = \frac{\Gamma(n+\alpha+1)}{n!}$$

$\underline{\text{Sats 6.17:}}$ $x > 0$, $|z| < 1$

$$\sum_{n=0}^\infty L_n^\alpha(x) z^n = \frac{e^{-x\left(\frac{z}{1-z}\right)}}{(1-z)^{\alpha+1}}$$

Välj z s.a $\frac{z}{1-z} = b$, $1 - z = \frac{1}{1+b}$ => $z = \frac{b}{1+b}$

$$\sum_{n=0}^\infty L_n^\alpha(x) \frac{b^n}{(1+b)^n} = \frac{e^{-bx}}{\left(\frac{1}{1+b}\right)^{\alpha+1}}$$

$\boxed{\text{EÖ 38}}$ Bestäm det polynom av grad högst 2 som minimerar:

$$\int_0^\infty \left(e^{x/4} - p(x)\right)^2 x e^{-x} dx \qquad (*)$$

$\{L_n^{\alpha=1}\}_{n=0}^\infty$ utgör en bas i $L_2(]0,\infty[, x e^{-x} dx)$

$e^{x/4} = e^{-bx}$ => $b = -\frac{1}{4}$ => $z = \frac{b}{1+b} = -\frac{1}{3}$

$$e^{x/4} = \sum_0^\infty \frac{(-1/4)^n}{(3/4)^{n+2}} L_n^1(x) = \frac{16}{9} \sum_0^\infty \left(-\frac{1}{3}\right)^n L_n^1(x)$$

L_n polynom av grad n

Sats 3.8: Det polynom som minimerar (*) ges av:

$$p(x) = \frac{16}{9}\sum_0^2 \left(-\frac{1}{3}\right)^n L_n^1(x)$$

$$L_0^1(x) = 1, \quad L_1^1(x) = 2-x, \quad L_2^1(x) = \frac{1}{2}x^2 - 3x + 3$$

$$\Rightarrow p(x) = \frac{8}{81}(12+x^2)$$

$\boxed{6.4.6}$ Låt $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$

uttryck f i en serie av Hermite-polynom

Sats 6.11, cor. 6.12: $H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$, $n = 0, 1, 2, \ldots$

Utgör ortogonal bas på $L^2(\mathbb{R}, e^{-x^2}dx)$ och $\|H_n\|^2 = 2^n n! \sqrt{\pi}$

$$f(x) = \sum_0^\infty c_n H_n(x), \qquad c_n = \frac{\langle f(x), H_n(x)\rangle_{e^{-x^2}}}{\|H_n\|^2_{e^{-x^2}}}$$

$$\langle f, H_n\rangle_{e^{-x^2}} = \int_0^\infty 1 \cdot H_n(x) e^{-x^2} dx =$$

$$\left[6.34: \boxed{n \geq 1} \quad -\frac{d}{dx}\left(e^{-x^2}H_{n-1}(x)\right) = H_n(x)e^{-x^2} \right]$$

$$= \left[-e^{-x^2}H_{n-1}(x)\right]_0^\infty = H_{n-1}(0)$$

Övning 6.4.1: $H_{n-1}(x) = \sum_{j \leq \frac{n-1}{2}} \frac{(-1)^j (2x)^{n-1-2j}}{j!(n-1-2j)!}$ $\qquad j = \frac{n-1}{2}$

$$H_{n-1}(0) = \begin{cases} 0 & n \text{ jämnt} \\ \frac{(-1)^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)! \cdot 1} & n \text{ udda} \end{cases}$$

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$$\langle f, H_0 \rangle = \int_0^\infty 1 \cdot 1 \cdot e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$c_n = \begin{cases} 0 & n \text{ jämnt}, n \geq 1 \\ \frac{1}{2} & n = 0 \\ \dfrac{(-1)^{\frac{n-1}{2}}}{(\frac{n-1}{2})!\, 2^n n! \sqrt{\pi}} & n \text{ udda} \end{cases}$$

$\boxed{6.4.4}$ Utvidga $f(x) = x^{2m}$, m pos. heltal i en serie med Hermite-polynom.

$$f(x) = \sum_{n=0}^{\infty} c_n H_n(x), \quad c_n = \frac{\langle f, H_n \rangle_w}{\|H_n\|_w^2}, \quad w = e^{-x^2}$$

$$\langle f, H_n \rangle_w = \int_{-\infty}^{\infty} f(x) H_n(x) e^{-x^2} dx = \int_{-\infty}^{\infty} f(x) (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) e^{-x^2} dx =$$

$$= \{\text{part. int } n \text{ ggr}\} = \int_{-\infty}^{\infty} f^{(n)}(x) e^{-x^2} dx$$

$n > 2m \Rightarrow c_n = 0$

$\underline{n \leq 2m}$:

1) n udda: $f^{(n)}(x)$ monom av udda grad

$$\int_{-\infty}^{\infty} \frac{d^n}{dx^n}(x^{2m}) e^{-x^2} dx = 0, \quad \text{så } c_n = 0$$

2) n jämnt: $n = 2k$, $k \leq m$

$$\frac{d^{2k}}{dx^{2k}} x^{2m} = (2m)(2m-1) \ldots (2m-2k+1) x^{2m-2k} =$$

$$= \frac{(2m)!}{(2m-2k)!} x^{2m-2k}$$

$$\langle x^{2m}, H_{2k}(x) \rangle = \frac{(2m)!}{(2m-2k)!} \int_{-\infty}^{\infty} x^{2m-2k} e^{-x^2} dx =$$

$$= \frac{(2m)!}{(2m-2k)!} 2 \int_0^\infty x^{2m-2k} e^{-x^2} dx = \left[\begin{matrix} t = x^2 & \\ dt = 2x\,dx & 0 \xrightarrow{t} \infty \end{matrix}\right] =$$

$$= \frac{(2m)!}{(2m-2k)!} \underbrace{\frac{2}{2}\int_0^\infty t^{m-k-\frac{1}{2}} e^{-t}\, dt}_{=\Gamma(m-k-\frac{1}{2})} = \frac{(2m)!}{(2m-2k)!}\Gamma(m-k-\tfrac{1}{2})$$

$$c_{2k} = \frac{(2m)!\,\Gamma(m-k-\frac{1}{2})}{(2m-2k)!\,2^k(2k)!\sqrt{\pi}}$$

$$\Gamma(m-k-\tfrac{1}{2}) = \sqrt{\pi}\,\frac{(2m-2k)!\,2^{2k}}{2^{2m}(m-k)!} \Rightarrow c_{2k} = \frac{(2m)!}{2^{2m}(2k)!(m-k)!}$$