FOURIER ANALYS

ÖV. F/KF

1995

SIDOR: 46

PRIS: ~~15:-~~ 25:-

Fourieranalys, övning 1

Trigonometriska fourierserier

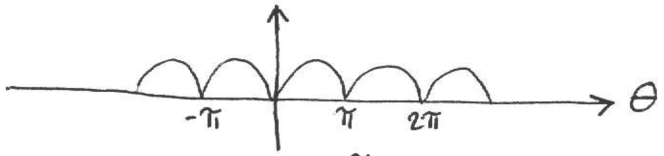
f periodisk med period T

har fourierserie $\dfrac{a_0}{2} + \sum_{1}^{\infty} a_n \cos \dfrac{2\pi}{T} nt + b_n \sin \dfrac{2\pi}{T} nt$

där $a_n = \dfrac{2}{T} \displaystyle\int_a^{a+T} f(t) \cos \dfrac{2\pi}{T} nt$

2.1 Verifiera i tabell 1 (2π-periodiska)

8: $f(\theta) = |\sin\theta| \sim \dfrac{2}{\pi} - \dfrac{4}{\pi} \sum_{1}^{\infty} \dfrac{\cos 2n\theta}{4n^2 - 1}$



f jämn $\Rightarrow b_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta = 0$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta = \frac{2}{\pi} \int_0^{\pi} f(\theta) \cos n\theta \, d\theta = \frac{2}{\pi} \int_0 \sin\theta \cos n\theta \, d\theta =$$

$$\left[\text{ty } \sin\alpha \cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)\right] = \frac{2}{\pi} \int_0^{\pi} \frac{1}{2}\Big(\sin\big((n+1)\theta\big) - \sin\big((n-1)\theta\big)\Big) d\theta =$$

$$= \frac{1}{\pi}\left[-\frac{\cos((n+1)\theta)}{n+1} + \frac{\cos((n-1)\theta)}{n-1}\right]_0^{\pi} = \frac{1}{\pi}\left(-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right) =$$

$$= \frac{1}{\pi}\left(1 - (-1)^{n+1}\right)\left(\frac{1}{n+1} - \frac{1}{n-1}\right) = \frac{1}{\pi}\left(1 - (-1)^{n+1}\right)\frac{(-2)}{n^2-1} = \begin{cases} \dfrac{-4}{\pi} \dfrac{1}{n^2-1}, & n \text{ jämn} \\ 0, & n \text{ udda} \end{cases}$$

$$\therefore f(\theta) = -\frac{2}{\pi} - \frac{4}{\pi} \sum_{1}^{\infty} \frac{\cos 2k\theta}{4k^2-1}$$

16: $f(\theta) = \theta^2 \quad (-\pi < \theta < \pi)$ [se bild (16) sida 29 i Folland]

f jämn $\Rightarrow b_n = 0$

Partiell Integration

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(\theta) \cos n\theta \, d\theta = \frac{2}{\pi} \int_0^{\pi} \theta^2 \cos n\theta \, d\theta = [P.I] = \frac{2}{\pi}\left(\left[\frac{1}{n}\theta^2 \sin n\theta\right]_0^{\pi} - \right.$$

$$\left. - \frac{2}{n} \int_0^{\pi} \theta \sin n\theta \, d\theta\right) = \frac{2}{\pi}\left(0 + \left[\frac{2}{n^2}\theta \cos n\theta\right]_0^{\pi} - \int_0^{\pi} \frac{2}{n^2} \cos n\theta \, d\theta\right) =$$

$$= \frac{2}{\pi}\left(\frac{2\pi}{n^2}(-1)^n - 0 - \left[\frac{2}{n^3} \sin n\theta\right]_0^{\pi}\right) = \frac{4}{n^2}(-1)^n, \quad n \neq 0$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \theta^2 \cdot 1 \, d\theta = \frac{2}{\pi}\,\frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

$$\therefore\ f(\theta) \sim \frac{\pi^2}{3} - 4\sum_1^{\infty} \frac{(-1)^n}{n^2}\cos n\theta$$

<u>18</u>: $f(\theta) = e^{b\theta} \quad (-\pi < \theta < \pi) \sim \dfrac{\sinh b\pi}{\pi} \displaystyle\sum_{-\infty}^{\infty} \frac{(-1)^n}{b-in} e^{in\theta}$

repetition: $\left[f(\theta) \sim \sum_{-\infty}^{\infty} C_n e^{in\theta}, \quad C_n = \frac{1}{2\pi}\int_a^{a+2\pi} f(\theta) e^{-in\theta} d\theta \right]$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{b\theta} e^{-in\theta} d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(b-in)\theta} d\theta = \frac{1}{2\pi}\left[\frac{e^{(b-in)\theta}}{b-in}\right]_{-\pi}^{\pi}$$

$$\frac{1}{2\pi}\left(\frac{e^{b\pi}e^{-in\pi} - e^{-b\pi}e^{in\pi}}{b-in}\right) = \frac{1}{2\pi}(-1)^n \frac{e^{b\pi} - e^{-b\pi}}{b-in} = \frac{\sinh b\pi}{\pi(b-in)}(-1)^n \Rightarrow$$

$$\Rightarrow f(\theta) \sim \frac{\sinh b\pi}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^{-n}}{b-in} e^{in\theta}$$

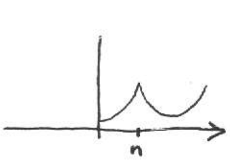
Sats: f periodisk & styckvis deriverbar
$$\sum_{-\infty}^{\infty} C_n e^{in\theta} = \frac{f(\theta^-) + f(\theta^+)}{2}$$

<u>2.2</u> <u>4</u>: $\displaystyle\sum_1^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

använd (16) (förrförra uppgiften): $f(\theta) = \theta^2, \ -\pi < \theta < \pi$

$$\sim \frac{\pi^2}{3} + 4\sum_1^{\infty} \frac{(-1)^n}{n^2}\cos n\theta$$

Lösning: Tag $\theta = \pi \Rightarrow \displaystyle\sum_{-\infty}^{\infty} C_n e^{in\theta} = \frac{\pi^2}{3} + 4\sum_1^{\infty} \frac{(-1)^n \overbrace{\cos n\pi}^{(-1)^n}}{n^2} = \pi^2$



$$\sum_1^{\infty} \frac{1}{n^2} = \frac{1}{4}\left(\pi^2 - \frac{\pi^2}{3}\right) = \frac{\pi^2}{6}$$

Tag $\theta = 0 \Rightarrow \dfrac{\pi^2}{3} + 4\displaystyle\sum \frac{(-1)^n \cdot 1}{n^2} = 0$

$$\therefore\ \sum \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

2.2 **6:** $\sum_{1}^{\infty} \frac{(-1)^n}{n^2+b^2} = \frac{\pi}{2b} \underbrace{\operatorname{csch}}_{\frac{1}{\sinh}} b\pi - \frac{1}{2b^2} = \frac{\pi}{2b}\frac{1}{\sinh b\pi} - \frac{1}{2b^2}$

förrförra uppgiften (18): $e^{b\theta} \sim \frac{\sinh b\pi}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n e^{in\theta}}{b-in}$

$-\pi < \theta < \pi$

tänk på att $(b-in)(b+in) = n^2+b^2$

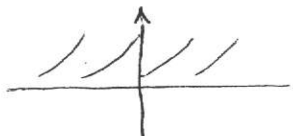
Tag $\theta = 0 \Rightarrow 1 = \frac{\sinh b\pi}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n \cdot 1}{b-in} =$

$$= \frac{\sinh b\pi}{\pi}\left(\frac{1}{b} + \sum_{1}^{\infty} \frac{(-1)^n}{b-in} + \frac{(-1)^{-n}}{b+in}\right) = \frac{\sinh b\pi}{\pi}\left(\frac{1}{b} + \sum_{1} (-1)^n \left(\frac{2b}{b^2+n^2}\right)\right) \Rightarrow$$

$$\Rightarrow \sum_{1}^{\infty} \frac{(-1)^n}{b^2+n^2} = \frac{\frac{\pi}{\sinh b\pi} - \frac{1}{b}}{2b} = \frac{\pi}{2b}\frac{1}{\sinh b\pi} - \frac{1}{2b^2}$$

7: $\sum_{1}^{\infty} \frac{1}{n^2+b^2} = \frac{\pi}{2b} \coth b\pi - \frac{1}{2b^2}$

(19): $f(\theta) = e^{b\theta}$, $0 < \theta < 2\pi \sim \frac{e^{2\pi b}-1}{2\pi} \sum_{-\infty}^{\infty} \frac{e^{in\theta}}{b-in}$



0:an språngpkt

Tag $\theta = 0 \Rightarrow \frac{f(0^-)+f(0^+)}{2} = \frac{e^{2\pi b}+1}{2} =$

$$= \frac{e^{2\pi b}-1}{2\pi} \sum_{-\infty}^{\infty} \frac{1}{b-in} = \frac{e^{2\pi b}-1}{2\pi}\left(\frac{1}{b} - \sum_{1}\left(\frac{1}{b-in} + \frac{1}{b+in}\right)\right) = \frac{e^{2\pi b}-1}{2\pi}\left(\frac{1}{b} + \sum_{1} \frac{2b}{b^2+n^2}\right) \Rightarrow$$

$$\Rightarrow \sum_{1}^{\infty} \frac{1}{n^2+b^2} = \left(\pi \frac{e^{2\pi b}+1}{e^{2\pi b}-1} - \frac{1}{b}\right)\cdot\frac{1}{2b} = \frac{\pi}{2b}\frac{e^{\pi b}+e^{-\pi b}}{e^{\pi b}-e^{-\pi b}} - \frac{1}{2b^2} = \frac{\pi}{2b}\coth \pi b - \frac{1}{2b^2}$$

Alt: Tag $\theta = \pi$ i (18)

2.3 Fourierserier kan deriveras & integreras termvis under lämpliga villkor

krävs försiktighet med konstanttermerna

2: (16) i tabell 1: $\theta^2 = \frac{\pi^2}{3} + 4\sum_{1}^{\infty} \frac{(-1)^n}{n^2} \cos n\theta$

a/ Visa: $\theta^3 - \pi^2\theta = 12 \sum_{1}^{\infty} \frac{(-1)^n}{n^3} \sin n\theta$

Lösning av a/: $\theta^2 - \frac{\pi^2}{3} = 4\sum_1^\infty \frac{(-1)^n}{n^2}\cos n\theta$

integrera: $\frac{\theta^3}{3} - \frac{\pi^2}{3}\theta = C_0 + 4\sum_1^\infty \frac{(-1)^n}{n^3}\sin n\theta$

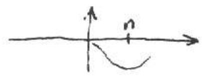
(VL) har medelvärde 0 över $[-\pi,\pi] \Rightarrow C_0 = 0$

$\theta^3 - \pi^2\theta = 12\sum \frac{(-1)^n}{n^3}\sin n\theta$

b/ $\theta^4 - 2\pi^2\theta^2 = 48\sum\frac{(-1)^{n+1}}{n^2}\cos n\theta - \frac{7\pi^4}{15}$

Lösning av b/: integrera resultatet från a/:

$\frac{\theta^4}{4} - \frac{\pi^2\theta^2}{2} = C_0 + 12\sum\frac{(-1)^{n+1}}{n^4}\cos n\theta$



$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\theta^4}{4} - \frac{\pi^2}{2}\theta^2\right)d\theta = \frac{1}{\pi}\int_0^{\pi}\left(\frac{\theta^4}{4} - \frac{\pi^2}{2}\theta^2\right)d\theta = \frac{1}{\pi}\left(\frac{\pi^5}{20} - \frac{\pi^5}{6}\right) = -\pi^4\,\frac{\cancel{14}^{7}}{\cancel{120}_{60}} \Rightarrow$

$\Rightarrow \theta^4 - 2\pi^2\theta^2 = -\frac{7\pi^4}{15} + 48\sum_1^\infty\frac{(-1)^{n+1}}{n^4}\cos n\theta$

c/ $\sum_1^\infty \frac{1}{n^4} = \frac{\pi^4}{90}$

Lösning av c/: Tag $\theta = \pi \Rightarrow \pi^4 - 2\pi^4 - \frac{7\pi}{15} + 48\sum\frac{(-1)^{n+1}(-1)^n}{n^4} \Rightarrow$

$\Rightarrow \sum_1^\infty \frac{1}{n^4} = \frac{1}{48}\left(\pi^4 - \frac{7\pi^4}{15}\right) = \frac{1}{\cancel{48}_6}\,\frac{\cancel{8}^1}{15}\,\pi^4 = \frac{\pi^4}{90}$

övning 2

Fouriertransformen

$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$ skrivs också $\mathcal{F}(f(t))(\omega)$

$f(t) = \frac{1}{2\pi}\int \hat{f}(\omega)e^{i\omega t}\,d\omega$

räkneregler:

$\mathcal{F}(f(t-a)) = e^{-ia\omega}\hat{f}(\omega)$

$\mathcal{F}(e^{iat}f(t)) = \hat{f}(\omega - a)$

$(f')^{\wedge}(\omega) = i\omega\hat{f}(\omega)$

$\mathcal{F}(xf(x)) = i(\hat{f})'(\omega)$

$\mathcal{F}(f(t)g(t)) = \frac{1}{2\pi}(\hat{f} * \hat{g})(\omega)$

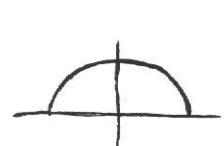
$\mathcal{F}((f*g)(t)) = \hat{f}(\omega)\hat{g}(\omega)$ där $(f*g)(t) = \int_{-\infty}^{\infty} f(t-s)g(s)\,ds = \int_{-\infty}^{\infty} f(s)g(t-s)\,ds$

Extrauppgifter (löst papper)

1. Fouriertransformera 4 st fkner

a/ $\frac{t}{(t^2+a^2)^2} = f(t)$

Metod 1: $\int_{-\infty}^{\infty} \frac{t}{(t^2+a^2)^2}e^{-i\omega t}\,dt$, Residuekalkyl



Metod II: Utgå från kända transformer

$$\mathcal{F}\left(\frac{1}{t^2+a^2}\right)=\frac{\pi}{a}e^{-a|\omega|}$$

derivera $\Rightarrow \mathcal{F}\left(-\frac{2t}{(t^2+a^2)^2}\right)=i\omega\frac{\pi}{a}e^{-a|\omega|}$

$\therefore \mathcal{F}\frac{t}{(t^2+a^2)^2}=\frac{i\pi}{2a}\omega e^{-a|\omega|}$

b/ $\frac{1}{(t^2+a^2)^2}$ Väljer metod 2 igen

$$\mathcal{F}\frac{1}{(t^2+a^2)^2}=\mathcal{F}\left(\frac{1}{t^2+a^2}\cdot\frac{1}{t^2+a^2}\right)=\frac{1}{2\pi}\frac{\pi}{a}e^{-a|\omega|}*\frac{\pi}{a}e^{-a|\omega|}=\frac{\pi}{2a^2}\int_{-\infty}^{\infty}e^{-a|u|}e^{-a|\omega-u|}du=$$

$$=\frac{\pi}{2a^2}\left(\int_{-\infty}^{\omega}e^{-a|u|}e^{-a(\omega-u)}du+\int_{\omega}^{\infty}e^{-a|u|}e^{-a(u-\omega)}du\right)$$

$$\omega>0:\ \frac{\pi}{2a^2}\left(\int_{-\infty}^{0}e^{2au}e^{-a\omega}du+\int_{0}^{\omega}\cancel{e^{-au}}\,\cancel{e^{au}}\,e^{-a\omega}du+\int_{\omega}^{\infty}e^{-2au}e^{a\omega}du\right)=$$

$$=\frac{\pi}{2a^2}\left(\frac{1}{2a}e^{-a\omega}+\omega e^{-a\omega}+\frac{-e^{-2a\omega}}{-2a}e^{a\omega}\right)=\frac{\pi}{2a^2}\left(\frac{1}{a}+\omega\right)e^{-a\omega}=\frac{\pi}{2a^3}(1+a\omega)e^{-a\omega}$$

$$\omega<0:\ \hat{f}(-\omega)=\frac{\pi}{2a^2}\int_{-\infty}^{\infty}e^{-a|u|}e^{-a\underbrace{|-\omega-u|}_{|\omega+u|}}du=[\text{Byt } u \text{ mot } -u]=\hat{f}(\omega)$$

$\therefore \hat{f}$ jämn dvs $\hat{f}(\omega)=\frac{\pi}{2a^3}(1+a|\omega|)e^{-a|\omega|}$

c/ $$\frac{t}{\underbrace{(t^2+1)}_{(t-i)(t+i)}(t^2+2t+5)}=\frac{At+B}{\underbrace{t^2+1}_{(t-i)(t+i)}}+\frac{Ct+D}{t^2+2t+5}$$ Multiplicera med $t-i$, låt $t\to i$ (HPL – handpåläggningsmetoden)

$$\frac{i}{2i(4+2i)}=\frac{Ai+B}{2i}+0\Leftrightarrow Ai+B=\frac{i}{4+2i}=\frac{i(4-2i)}{20}=\frac{2+4i}{20}=\frac{1+2i}{10}\Rightarrow\begin{cases}A=2/10\\B=1/10\end{cases}$$

Tag $t=0$: $0=B+\frac{D}{5}\Rightarrow D=-5/10$

Multiplicera med t, låt $t\to\infty$: $0=A+C\Rightarrow C=-2/10$

$$f(t)=\frac{1}{10}\left(\frac{2t+1}{t^2+1}-\underbrace{\frac{2t+5}{t^2+2t+5}}\right)\quad \frac{2(t+1)+3}{(t+1)^2+2^2}$$

$$\mathcal{F}\left(\frac{1}{t^2+a^2}\right)=\frac{\pi}{a}e^{-a|\omega|}\Rightarrow\mathcal{F}\left(\frac{t}{t^2+a^2}\right)=i\left(\frac{\pi}{a}e^{-a|\omega|}\right)'=\frac{i\pi}{a}(-a)\operatorname{sgn}\omega e^{-a(\omega)}=$$

$=-i\pi\operatorname{sgn}\omega\, e^{-a|\omega|}$, där $\operatorname{sgn}\omega=\begin{cases}1,&\omega>0\\-1,&\omega<0\end{cases}$

$$\mathcal{F}\left(\frac{1}{(t+1)^2+a^2}\right)=e^{i\omega}\frac{\pi}{a}e^{-a|\omega|}\Rightarrow\mathcal{F}\left(\frac{t+1}{(t+1)^2+a^2}\right)=e^{i\omega}\left(-i\pi\operatorname{sgn}\omega\, e^{-a|\omega|}\right)$$

$$\hat{f}(\omega)=\frac{1}{10}\left(2\left(-i\pi\operatorname{sgn}\omega\, e^{-1|\omega|}+\frac{\pi}{1}e^{-1|\omega|}\right)-\left(2e^{i\omega}(-\pi i\operatorname{sgn}\omega\, e^{-2|\omega|})+3e^{i\omega}\frac{\pi}{2}e^{-2|\omega|}\right)\right)=$$

$$= \frac{\pi}{10}\left(1 - 2i\,\mathrm{sgn}\,\omega\ e^{-|\omega|}\right) - \frac{\pi}{10}\left(\frac{3}{2} - 2i\,\mathrm{sgn}\,\omega\right)e^{i\omega}\, e^{-2|\omega|}$$

d/ $$e^{-a|t|}\sin bt = e^{-a|t|}\,\frac{e^{ibt}-e^{-ibt}}{2i} = \frac{1}{2i}\left(e^{ibt}e^{-a|t|} - e^{-ibt}e^{-a|t|}\right)$$

$$\mathcal{F}(f(t)) = g(\omega) \Rightarrow \mathcal{F}(g(t)) = 2\pi f(-\omega)$$

Vet $$\mathcal{F}\left(\frac{1}{t^2+a^2}\right) = \frac{\pi}{a}e^{-a|\omega|} \Rightarrow \mathcal{F}\left(\frac{\pi}{a}e^{-a|t|}\right) = 2\pi\frac{1}{(-\omega)^2+a^2}$$

$$\mathcal{F}\left(e^{-a|t|}\right) = 2a\frac{1}{\omega^2+a^2}$$

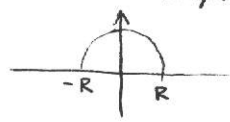
$$\mathcal{F}\left(e^{-a|t|}\sin bt\right) = \frac{1}{\cancel{8}i}\left(\frac{\cancel{8}a}{(\omega-b)^2+a^2} - \frac{8a}{(\omega+b)^2+a^2}\right) = \frac{4iab\omega}{(\omega^2-2b\omega+a^2+b^2)(\omega^2+2b\omega+a^2+b^2)}$$

---

2. $$\hat{f}(\omega) = \frac{\omega}{1+\omega^4}$$

a/ $$\int_{-\infty}^{\infty} t\,f(t)\,dt = [t\,f(t)]^\wedge(0) = i(\hat{f})'(0) = i\frac{d}{d\omega}\left(\frac{\omega}{1+\omega^4}\right)\Bigg|_{\omega=0} = i\left(\frac{1}{1+\omega^4} - \frac{\omega\,4\omega^3}{(1+\omega^4)^2}\right)\Bigg|_{\omega=0} = i$$

b/ $$f'(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(f')^\wedge(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} i\omega\hat{f}(\omega)\,d\omega = \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{\omega^2}{1+\omega^4}d\omega$$ lös den med residuekalkyl



$$\frac{i}{2\pi}2\pi i\sum_{\mathrm{Im}\, z>0}\mathrm{Res}\,\frac{z^2}{1+z^4} = -\left(\underset{z=e^{i\pi/4}}{\mathrm{Res}} + \underset{e^{i3\pi/4}}{\mathrm{Res}}\ \frac{z^2}{1+z^4}\right) =$$

$$= \left[\underset{z=a}{\mathrm{Res}}\ \frac{f(z)}{g(z)} = \frac{f(a)}{g'(a)}\right] = -\frac{z^2}{4z^3}\Bigg|_{z=e^{i\pi/4}} + \Bigg|_{z=e^{i3\pi/4}} = -\frac{1}{4}\left(\underbrace{e^{-i\pi/4}}_{\frac{1-i}{\sqrt{2}}} + \underbrace{e^{-i3\pi/4}}_{\frac{-1-i}{\sqrt{2}}}\right) = \frac{i}{2\sqrt{2}}$$

förresten... borde tagit reda på att $\in L^1$ först;
en något bättre början:

$$(f')^\wedge = i\omega\hat{f}(\omega) = \frac{i\omega^2}{1+\omega^2} \in L^1$$

$$f'(0) = \int\frac{i\omega^2}{1+\omega^4}d\omega = \ldots$$

Plancherels formel:

$$\langle\hat{f},\hat{g}\rangle = 2\pi\langle f,g\rangle$$

$$\|\hat{f}\|_{L^2}^2 = 2\pi\|f\|_{L^2}^2$$

L^2-norm, betyder ofta energin som finns i signalen

---

4. $$f(t) = \int_0^1\sqrt{\omega}\,e^{\omega^2}\cos\omega t\,d\omega$$

$$\int_{-\infty}^{\infty}|f'(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|(f')^\wedge(\omega)|^2d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}|i\omega\hat{f}(\omega)|^2d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^2|\hat{f}(\omega)|^2d\omega$$

$$f(t)=\int_0^1\sqrt{\omega}\,e^{\omega^2}\frac{e^{i\omega t}+e^{-i\omega t}}{2}\,d\omega=\int_0^1\sqrt{\omega}\,\frac{e^{\omega^2}e^{i\omega}}{2}\,d\omega+\int_{-1}^{0}\sqrt{\omega}\,\frac{e^{\omega^2}e^{i\omega t}}{2}\,d\omega=$$

$$=\int_{-1}^{1}\frac{\sqrt{|\omega|}\,e^{\omega^2}}{2}e^{i\omega t}\,d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\pi\sqrt{|\omega|}\,e^{\omega^2}\underbrace{\chi_{[-1,1]}}_{=\begin{cases}1, \text{ PÅ } [-1,1]\\ 0, \text{ ANNARS}\end{cases}}e^{i\omega t}$$

$$\hat{f}(\omega)=\pi\sqrt{|\omega|}\,e^{\omega^2}\chi_{[-1,1]}\in L^1\cap L^2$$

$$\int_{-\infty}^{\infty}|f'(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^2|\hat{f}(\omega)|^2d\omega=\frac{1}{2\pi}\int_{-1}^{1}\omega^2\pi^2|\omega|e^{2\omega^2}d\omega\overset{\text{JÄMN fkn}}{=}\frac{\pi}{2}2\int_0^1\omega^3e^{2\omega^2}d\omega=$$

$$=\begin{bmatrix}u=\omega^2\\ du=2\omega d\omega\end{bmatrix}=\pi\int_0^1 ue^{2u}\frac{du}{2}=[\text{PARTIELL INTEGRATION}]=\frac{\pi}{2}\left[u\frac{e^{2u}}{2}-\frac{e^{2u}}{4}\right]_0^1=$$

$$=\frac{\pi}{2}\left(\frac{e^2}{2}-\frac{e^2}{4}-\left(0-\frac{1}{4}\right)\right)=\frac{\pi}{8}(e^2+1)$$

ÖVNING 3

extra övn

EÖ5 $u'(t)+2u(t)+e^{-2t}\int_{-\infty}^{t}e^{2\tau}u(\tau)d\tau=\delta(t)$

$(f*g)(t)=\int_{-\infty}^{\infty}f(t-\tau)g(\tau)d\tau$

$$e^{-2t}\int_{-\infty}^{t}e^{2\tau}u(\tau)d\tau=\int_{-\infty}^{t}e^{-2(t-\tau)}u(\tau)d\tau=\int_{-\infty}^{\infty}\underbrace{e^{-2(t-\tau)}\theta(t-\tau)}_{f(t-\tau)}u(\tau)d\tau$$

Fouriertransformera ekv $\qquad \theta(t)=\begin{cases}1, & t>0\\ 0, & t<0\end{cases}$

$\mathcal{F}(f*g)=\hat{f}\cdot\hat{g}$

$i\omega\hat{u}(\omega)+2\hat{u}(\omega)+\left(e^{-2t}\theta(t)\right)^{\wedge}\hat{u}(\omega)=1$

$$\mathcal{F}\left(e^{-2t}\theta(t)\right)=\int_{-\infty}^{\infty}e^{-2t}\theta(t)e^{-i\omega t}dt=\int_0^{\infty}e^{-(2+i\omega)t}dt=\left[-\frac{e^{-(2+i\omega)t}}{2+i\omega}\right]_0^{\infty}=\frac{1}{2+i\omega}$$

$$\left(i\omega+2+\frac{1}{2+i\omega}\right)\hat{u}(\omega)=1$$

$$\hat{u}(\omega)=\frac{1}{2+i\omega+\frac{1}{2+i\omega}}=\frac{2+i\omega}{(2+i\omega)^2+1}$$

$$\frac{y}{y^2+1}=\frac{y}{(y+i)(y-i)}=\frac{A}{y+i}+\frac{B}{y-i}\qquad A=\frac{-i}{-2i}=\frac{1}{2},\quad B=\frac{1}{2}$$

$$\hat{u}(\omega)=\frac{1}{2}\left(\frac{1}{2+i\omega+i}+\frac{1}{2+i\omega-i}\right)$$

$\mathcal{F}\left(e^{iat}f(t)\right)=\hat{f}(\omega-a)$

$$u(t)=\frac{1}{2}\left(e^{-2t}\theta(t)e^{it}+e^{-2t}\theta(t)e^{-it}\right)=\theta(t)e^{-2t}\cos t$$

EÖ7 $\quad \xrightarrow{x(t)}\boxed{L}\xrightarrow{y(t)}$

$L(e^{i\omega t})=\hat{h}(\omega)e^{i\omega t}$

$\hat{y}(\omega)=\hat{h}(\omega)\hat{x}(\omega)$

$y(t)=(h*x)(t)=\int_{-\infty}^{\infty}h(t-\tau)x(\tau)d\tau$

beteckningar:

Vi/Holmåker	Folland	
h	H	impulssvar
$\hat{h}$	h	överföringsfkn

Vet: $x(t)=\frac{1}{1+t^2}$ ger $y(t)=\frac{t}{(4+t^2)^2}$

$\mathcal{F}\left(\frac{1}{t^2+a^2}\right)=\frac{\pi}{a}e^{-a|\omega|}$

$\mathcal{F}\left(\frac{t}{(t^2+a^2)^2}\right)=-\frac{\pi}{2a}i\omega e^{-a|\omega|}$ (från övning 2)

$\hat{y}(\omega)=\hat{h}(\omega)\hat{x}(\omega)$ ger $-\frac{\pi}{4}i\omega e^{-2|\omega|}=h(\omega)\cdot\pi e^{-|\omega|}$

$\hat{h}(\omega)=-\frac{i}{4}\omega e^{-|\omega|}=\frac{1}{2\pi}\left(-\frac{\pi\omega}{2\cdot 1}e^{-|\omega|}\right)$ inverstransformera

impulssvaret:

$h(t)=\frac{1}{2\pi}\frac{t}{(t^2+1)^2}$

svaret på cos ωt:

$x(t)=\cos\omega t \Rightarrow y(t)=?$

$x(t)=\frac{1}{2}\left(e^{i\omega t}+e^{-i\omega t}\right)$

$y(t)=\frac{1}{2}\left(\hat{h}(\omega)e^{i\omega t}+\hat{h}(-\omega)e^{-i\omega t}\right)=\frac{1}{2}\left(-\frac{i}{4}\omega e^{-|\omega|}e^{i\omega t}-\frac{i}{4}\left(-\omega e^{-|\omega|}e^{-i\omega t}\right)\right)=$

$=-\frac{i}{4}\omega e^{-|\omega|}\underline{\left(e^{i\omega t}-e^{-i\omega t}\right)}_{2i}=\frac{1}{4}\omega e^{-|\omega|}\sin\omega t$

kausalt om $h(t)=0$ för $t<0$

Ej fallet här

stabilitet: $\int_{-\infty}^{\infty}|h(t)|\,dt<\infty$ Ja!

<u>Eö9</u> $x(t)=\frac{1}{4+t^2}$ ger $y(t)=e^{-2t^2}$

$\hat{x}(\omega)=\frac{\pi}{2}e^{-2|\omega|}$

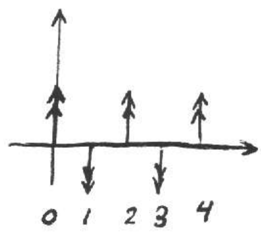
$\hat{y}(\omega)=\sqrt{\frac{\pi}{2}}e^{-\omega^2/2\cdot 4}=\sqrt{\frac{\pi}{2}}e^{-\omega^2/8}$

$\hat{y}(\omega)=\hat{h}(\omega)\hat{x}(\omega)$

$\hat{h}(\omega)=\frac{\hat{y}(\omega)}{\hat{x}(\omega)}=\sqrt{\frac{2}{\pi}}e^{-\omega^2/8+2|\omega|}$

$x(t)=\sum_{n=-\infty}^{\infty}\left(2\delta(t-2n)-\delta(t-2n-1)\right)$

$y(t)=?$



Vet att $\begin{cases}\delta(t)\rightsquigarrow h(t)\\ \delta(t-a)\rightsquigarrow h(t-a)\end{cases}$

$y(t)=\sum_{-\infty}^{\infty}\left(2h(t-2n)-h(t-2n-1)\right)$

2-periodisk funktion ⇒ kan utvecklas i Fourierserie

$$y(t)=\sum c_m e^{im\pi t}$$

$$c_m=\frac{1}{2}\int_0^2 y(t)e^{-im\pi t}\,dt=\frac{1}{2}\int_0^2\sum_{-\infty}^{\infty}\big(2h(t-2n)-h(t-2n-1)\big)e^{-im\pi t}\,dt=$$

$$=\frac{1}{2}\sum_{-\infty}^{\infty}\big(2h(t-2n)-h(t-2n-1)\big)e^{im\pi t}\,dt=\frac{1}{2}\sum_{-\infty}^{\infty}\int_{2n}^{2n+2}2h(t)\underbrace{e^{im\pi(t+2n)}}_{e^{im\pi}}-$$

$$-\int_{2n+1}^{2n+3}h(t)\underbrace{e^{-im\pi(t+2n+1)}}_{e^{-im\pi}(-1)^m}\,dt=\frac{1}{2}\left(\int_{-\infty}^{\infty}2h(t)e^{-im\pi t}\,dt-(-1)^m\int_{-\infty}^{\infty}h(t)e^{-im\pi t}\,dt\right)=\hat{h}(m\pi)-$$

$$-\frac{1}{2}(-1)^m\hat{h}(m\pi)$$

Enklare lösn: (Vi kan inte visa att den är korrekt, men det är den, det går att visa)

Utveckla $x(t)$ i Fourierserie

$$x(t)=\sum d_m e^{im\pi t}$$

$$d_m=\frac{1}{2}\int_{-1/2}^{3/2}x(t)e^{im\pi t}\,dt=\frac{1}{2}\int_{-1/2}^{3/2}\left(\sum 2\delta(t-2n)-\delta(t-2n-1)\right)$$

↑ Väljer de här gränserna för att slippa få en impuls på någon av ändpunkterna

$$e^{-im\pi t}\,dt=\frac{1}{2}\int_{-1/2}^{3/2}2\delta(t)-\delta(t-1)\,e^{-im\pi t}\,dt=\frac{1}{2}\left(2e^{-im\pi 0}-e^{-im\pi\cdot 1}\right)=$$

$$=\frac{1}{2}\left(2-(-1)^m\right)=1-\frac{1}{2}(-1)^m$$

$$x(t)=\sum_{-\infty}^{\infty}\left(1-\frac{1}{2}(-1)^m\right)e^{im\pi t}$$

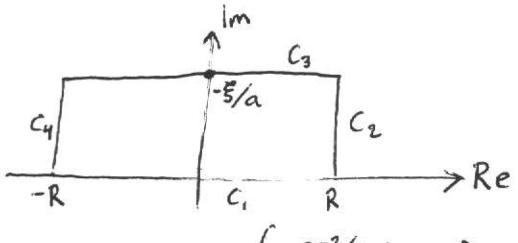
Vet: $e^{i\omega t}\rightsquigarrow\hat{h}(\omega)e^{i\omega t}$

$$y(t)=\sum_{-\infty}^{\infty}\left(1-\frac{1}{2}(-1)^m\right)\hat{h}(m\pi)e^{i\omega t}=\sqrt{\frac{2}{\pi}}\sum_{-\infty}^{\infty}\left(1-\frac{1}{2}(-1)^m\right)e^{-m^2\pi^2/8+2\pi|m|}e^{i\omega t})$$

7.2:1 $\mathcal{F}\left(e^{-ax^2/2}\right)=\sqrt{\frac{2\pi}{a}}\,e^{-\xi^2/2a}$

$$\mathcal{F}\left(e^{-ax^2/2}\right)=\int_{-\infty}^{\infty}e^{-ax^2/2}e^{-i\xi x}\,dx=\text{ det av fouriertransf}$$

$$=\int_{-\infty}^{\infty}e^{-a\left(x+\frac{i\xi}{a}\right)^2/2-\xi^2/2a}\,dx=e^{-\xi^2/2a}\underbrace{\int_{-\infty}^{\infty}e^{-a(x+i\xi/a)^2/2}\,dx}_{=I}$$



Cauchys sats: $\int_{C_1+C_2+C_3+C_4}e^{-az^2/2}\,dz=0$ $\qquad\int_{C_1}=\int_{-R}^{R}e^{-ax^2/2}\,dx$

$$\int_{C_3} = -\int_{-R}^{R} e^{-a(x+i\xi/a)^2/2}\,dx$$

$$z = x + i\xi/a$$
$$dz = dx$$

$$\left|\int_{C_2}\right| = \left[z = R+iy\right] = \left|\int_0^{\xi/a} e^{-a(R+iy)^2/2}\, i\,dy\right| = \left|\int_0^{\xi/a} e^{-aR^2/2} e^{ay^2/2} e^{i(\;)}\, i\,dy\right|$$

$$\leq \left|\frac{\xi}{a}\right| e^{-aR^2/2} e^{\xi^2/2a} \longrightarrow 0 \text{ då } R\to\infty$$

PSS $\left|\int_{C_4}\right| \to 0$, då $R\to\infty$

Låt $R\to\infty$:

$$\int_{-\infty}^{\infty} e^{-ax^2/2}\,dx + 0 - \int_{-\infty}^{\infty} e^{-a(x+i\xi/a)^2/2}\,dx$$

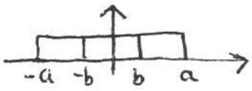
$$I = \int_{-\infty}^{\infty} e^{-ax^2/2}\,dx = \begin{bmatrix} x = \sqrt{\frac{2}{a}}\,y \\ dx = \sqrt{\frac{2}{a}}\,dy \end{bmatrix} = \sqrt{\frac{2}{a}}\int_{-\infty}^{\infty} e^{-y^2}\,dy = \sqrt{\frac{2\pi}{a}}$$

$$\therefore\; \mathcal{F}\left(e^{-ax^2/2}\right) = \sqrt{\frac{2\pi}{a}}\, e^{-\xi^2/2a}$$

Plancherel: $2\pi\langle f,g\rangle = \langle \hat{f},\hat{g}\rangle$ $\qquad \mathcal{F}\left(\frac{\sin at}{t}\right) = \chi_{[-a,a]} = \begin{cases} 1 & \text{på } [-a,a] \\ 0 & \text{annars}\end{cases}$

13a $\displaystyle\int_{-\infty}^{\infty} \frac{\sin at}{t^2}\sin bt\,dt = \int_{-\infty}^{\infty} \frac{\sin at}{t}\,\overline{\frac{\sin bt}{t}}\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi_{[-a,a]}\,\overline{\chi_{[-b,b]}}\,d\omega =$

$$= \frac{1}{2\pi}\int_{-\min(a,b)}^{\min(a,b)} 1\,d\omega = \frac{1}{2}\min(a,b)$$



b $\displaystyle\int_{-\infty}^{\infty} \frac{t^2\,dt}{(t^2+a^2)(t^2+b^2)} = \int_{-\infty}^{\infty} \frac{t}{(t^2+a^2)}\,\overline{\frac{t}{t^2+b^2}}\,dt = \frac{1}{2\pi}\int i\pi\,\mathrm{sgn}\,\omega\, e^{-a|\omega|}\,\overline{i\pi\,\mathrm{sgn}\,\omega\, e^{-b|\omega|}}\,d\omega$

$$= \frac{\pi}{2}\int_{-\infty}^{\infty} e^{-(a+b)|\omega|}\,d\omega = \frac{\pi}{\cancel{2}}\,\cancel{2}\int_0^{\infty} e^{-(a+b)\omega}\,d\omega$$

ty jämn fkn

$$\mathcal{F}\left(\frac{1}{t^2+a^2}\right) = \frac{\pi}{a}\,e^{-a|\omega|}$$

$$\mathcal{F}\left(\frac{t}{t^2+a^2}\right) = i\left(\frac{\pi}{a}\,e^{-a|\omega|}\right)' =$$

$$= i\,\frac{\pi}{\cancel{a}}\,\cancel{a}\;\mathrm{sgn}\,\omega\; e^{-a|\omega|}$$

7.3 Vågekv $u_{tt} = c^2 u_{xx}$

$$(BV): \quad u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

Fouriertransformera i x-led

$\hat{u}_{tt}(\xi) = c^2 (i\xi)^2 \hat{u}(\xi)$

$\hat{u}_{tt}(\xi) + (c\xi)^2 \hat{u}(\xi) = 0$

Fixera ξ. Då fås en diff ekv m konst koeff

Kar ekv: $r^2 + (c\xi)^2 = 0 \Rightarrow r = \pm i c\xi$

$\Rightarrow \hat{u}(\xi, t) = A(\xi) \cos c\xi t + B(\xi) \sin c\xi t$

Fouriertransformera (BV)

$\hat{u}(\xi, 0) = \hat{f}(\xi)$

$\hat{u}_t(\xi, 0) = \hat{g}(\xi)$

$\Rightarrow A(\xi) = \hat{f}(\xi)$

$c\xi B(\xi) = \hat{g}(\xi) \Rightarrow B(\xi) = (c\xi)^{-1} \hat{g}(\xi)$

$\therefore \hat{u}(\xi, t) = \hat{f}(\xi) \cos c\xi t + \hat{g}(\xi)(c\xi)^{-1} \sin c\xi t$

ÖVNING 4

Det kallas för d'Alamberts formel $G(x+ct) - G(x-ct)$

7.3:3 $u_{tt} = c^2 u_{xx}$

$u(x,0) = f(x)$

$u_t(x,0) = g(x)$

a/ $\hat{u}(\xi, t) = \hat{f}(\xi) \cos ct\xi + \hat{g}(\xi)(c\xi)^{-1} \sin ct\xi$

b/ $\hat{u}(\xi, t) = \hat{f}(\xi)\left(\frac{e^{it\xi} + e^{-ict\xi}}{2}\right) + \frac{1}{2c} \hat{g}(\xi) \frac{2 \sin \overbrace{ct}^{=a} \xi}{\xi}$

ty $(\chi_{[-a,a]})^\wedge = \frac{2 \sin a\xi}{\xi}$

$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} +$

$+ \frac{1}{2c} g * \chi_{[-ct,ct]}(x) =$

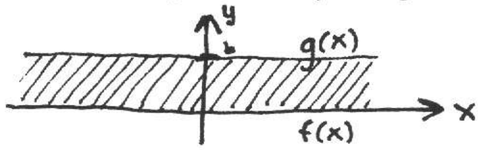
$= \frac{1}{2}\left(f(x+ct) + f(x-ct) + \frac{1}{c} \int_{-\infty}^{\infty} g(y) \underbrace{\chi_{[-ct,ct]}(x-y)}_{\substack{=1 \text{ om } -ct \leq x-y \leq ct \\ \Leftrightarrow x-ct \leq y \leq x-ct}} dy\right) = \frac{1}{2}(f(x+ct) + f(x-ct)) +$

$+ \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)\, dy$

7.3:4 $u_{xx} + u_{yy} = 0 \quad (\Delta u = 0), \quad x \in \mathbb{R}, \; 0 < y < b$

$u(x,0) = f(x)$

$u(x,b) = g(x)$



Fouriertransformera i x-led

$(i\xi)^2 \hat{u} + \hat{u}_{yy} = 0$

$\hat{u}_{yy}(\xi,y) - \xi^2 \hat{u}(\xi,y) = 0$

kar ekv: $r^2 - \xi^2 = 0 \Rightarrow r = \pm \xi$

$$\hat{u}(\xi,y) = \underbrace{A(\xi)e^{\xi y} + B(\xi)e^{-\xi y}}_{C(\xi)\sinh \xi y + D(\xi)\cosh \xi y}$$

Fouriertransformera randvillkoren

$\hat{u}(\xi,0) = \hat{f}(\xi)$

$\hat{u}(\xi,b) = \hat{g}(\xi)$

Lös ekv med $f = 0$

$\hat{u}(\xi,0) = A(\xi) + B(\xi) = 0 \Rightarrow B(\xi) = -A(\xi)$

$\hat{u}(\xi,b) = A(\xi)(e^{\xi b} - e^{-\xi b}) = \hat{g}(\xi)$

$$A(\xi) = \frac{\hat{g}(\xi)}{2\left(\frac{e^{\xi b} - e^{-\xi b}}{2}\right)} = \frac{1}{2}\frac{\hat{g}(\xi)}{\sinh \xi b}$$

$$\Rightarrow \hat{u}(\xi,y) = \frac{1}{2}\frac{\hat{g}(\xi)}{\sinh \xi b}\left(e^{\xi y} - e^{-\xi y}\right) = \hat{g}(\xi)\frac{\sinh \xi y}{\sinh \xi b}$$

$$7.2{:}10 \Rightarrow \mathcal{F}\left(\frac{\sinh ax}{\sinh \pi x}\right) = \frac{\sin a}{\cosh \xi + \cos a}, \quad 0 < a < \pi$$

$$\mathcal{F}\, f(\delta(x)) = \frac{1}{\delta}\hat{f}\left(\frac{\xi}{\delta}\right) \quad \left(=\left(\hat{f}\right)_\delta(x)\right)$$

Tag $\delta = b/\pi$

$$\mathcal{F}\left(\frac{\sinh \frac{ab}{\pi} x}{\sinh bx}\right) = \frac{\pi}{b}\frac{\sin a}{\cosh \frac{\pi\xi}{b} + \cos a}$$

Välj $a = y\pi/b$

$$\mathcal{F}\left(\frac{\sinh yx}{\sinh bx}\right) = \frac{\pi}{b}\sin\frac{\pi y}{b} \Big/ \left(\cosh\frac{\pi\xi}{b} + \cos\frac{\pi y}{b}\right)$$

$\hat{f} = g, \qquad \hat{g} = 2\pi f(-\xi)$

dvs $f(x) \overset{\mathcal{F}}{\to} g(\xi), \quad g(x) \overset{\mathcal{F}}{\to} 2\pi f(-\xi)$

$$\mathcal{F}\left(\frac{\pi}{2b} \frac{\sin \pi y/b}{\cosh \frac{\pi x}{b} + \cos \frac{\pi y}{b}}\right) = 2\pi \frac{\cancel{\ } \sinh(\cancel{\ }\xi y)}{\cancel{\ } \sinh(\cancel{\ }\xi b)}$$

Lösn till ekv med $f=0$ är alltså

$$u_1(x,y) = g(x) \underset{\text{map } x}{*} \frac{1}{2b} \frac{\sin \pi y/b}{\cosh \frac{\pi x}{b} + \cos \frac{\pi y}{b}} = \frac{1}{2b} \int_{-\infty}^{\infty} \frac{\sin \pi y/b \; g(t) \; dt}{\cosh \frac{\pi(x-t)}{b} + \cos \frac{\pi y}{b}}$$

Lösn till ekv med $g=0$: u_2

$$u(x,0) = f(x) \qquad u_{xx} + u_{yy} = 0$$
$$u(x,b) = 0$$

byt y mot $b-y$

$$u(x,y) = v(x, b-y) \qquad \begin{cases} v_{xx} + v_{yy} = 0 \\ v(x,0) = 0 \\ v(x,b) = f(x) \end{cases} \Rightarrow v(x,y) = u_1(x,y) =$$

$$= \int_{-\infty}^{\infty} \frac{\sin \pi y/b \qquad dt}{\cosh \frac{\pi(x-t)}{b} + \cos \frac{\pi y}{b}}$$

$$u_2(x,y) = v(x, b-y) = \frac{1}{2b} \int_{-\infty}^{\infty} \frac{\sin \pi y/b \cdot f(t) \; dt}{\cosh \frac{\pi(x-t)}{b} - \cos \frac{\pi y}{b}}$$

<u>EÖ14</u> $\quad u_t = k u_{xx}, \quad x \in \mathbb{R}, \quad t > 0$

$$u(x,0) = (1-2x^2)\, e^{-x^2} = u_0(x)$$

Fouriertransformera i x-led

$\hat{u}_t = K(i\xi)^2 \hat{u}$

$\hat{u}_t = -K \xi^2 \hat{u}$

$\hat{u} = c(\xi) e^{-k\xi^2 t}$

Fouriertransformera (BV)

$\hat{u}(\xi,0) = \hat{u}_0(\xi) \Rightarrow \hat{u}(\xi,t) = \hat{u}_0(\xi) e^{-K\xi^2 t}$

$\hat{u}_0 = ? \quad \mathcal{F}(e^{-ax^2/2}) = \sqrt{\frac{2\pi}{a}}\, e^{-\xi^2/2a} \Rightarrow \mathcal{F}(e^{-x^2}) = \sqrt{\pi}\, e^{-\xi^2/4}$

$$\mathcal{F}(x^2e^{-x^2}) = \left(i\frac{d}{d\xi}\right)^2 \sqrt{\pi}\, e^{-\xi^2/4} = -\sqrt{\pi}\left(e^{-\xi^2/4}\right)'' = -\sqrt{\pi}\left(-\frac{\xi}{2}e^{-\xi^2/4}\right)' =$$

$$= -\sqrt{\pi}\left(-\frac{1}{2}+\frac{\xi^2}{4}\right)e^{-\xi^2/4}$$

$$\hat{u}_0(\xi) = \sqrt{\pi}\, e^{-\xi^2/4} + 2\sqrt{\pi}\left(-\frac{1}{2}+\frac{\xi^2}{4}\right)e^{-\xi^2/4} = \frac{\sqrt{\pi}}{2}\xi^2 e^{-\xi^2/4}$$

$$\hat{u}(\xi,t) = \frac{\sqrt{\pi}}{2}\xi^2 e^{-\xi^2/4-k\xi^2t} = \frac{\sqrt{\pi}}{2}\xi^2 e^{-(1+4kt)\xi^2/4}$$

Enligt ovan $\mathcal{F}\left((1-2x^2)e^{-x^2}\right) = \frac{\sqrt{\pi}}{2}\xi^2 e^{-\xi^2/4}$

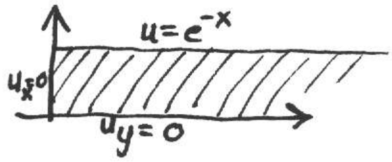
Skalning $\mathcal{F}\left(\frac{1}{\delta}f\left(\frac{x}{\delta}\right)\right) = \hat{f}(\delta\xi)$

$$\mathcal{F}\left(\frac{1}{\sqrt{1+4kt}}\left(1-2\frac{x^2}{1+4kt}\right)e^{-\frac{x^2}{1+4kt}}\right) = \frac{\sqrt{\pi}}{2}(1+4kt)\,\xi^2 e^{-(1+4kt)\xi^2/4}$$

$$\mathcal{F}\left(\frac{1+4kt-2x^2}{(1+4kt)^{5/2}}e^{-x^2/(1+4kt)}\right) = \frac{\sqrt{\pi}}{2}\xi^2 e^{-(1-4kt)\xi^2/4}$$

$$u(x,t) = \frac{1+4kt-2x^2}{(1+4kt)^{5/2}}e^{-x^2/1+4kt}$$

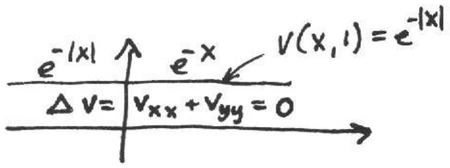
<u>7.4:6</u> $\Delta u = u_{xx}+u_{yy} = 0$



$$\mathcal{F}_c(f)(\xi) = \int_0^\infty f(x)\cos\xi x\,dx$$

för cos-transf

$$f(x) = \frac{2}{\pi}\int_0^\infty \mathcal{F}_c(f)(\xi)\cos\xi x\,dx$$



Fouriertransformera i x-led

$-\xi^2\hat{v}+\hat{v}_{yy}=0$ kar eku: $r^2-\xi^2=0$

$\hat{v}(\xi,y) = A(\xi)\cosh\xi y + B(\xi)\sinh\xi y$

$\hat{v}(\xi,0) = 0 \Rightarrow B(\xi)=0$

$\hat{v}(\xi,y) = A(\xi)\cosh\xi y$

$$\hat{v}(\xi,1) = \mathcal{F}(e^{-|x|}) = \int_{-\infty}^{\infty} e^{-|x|}e^{-i\xi x}\,dx = \int_{-\infty}^{0} e^{x-i\xi x}\,dx + \int_0^\infty e^{-x-i\xi x}\,dx =$$

$$= \left[\frac{e^{(1-i\xi)x}}{1-i\xi}\right]_{-\infty}^{0} + \left[-\frac{e^{-(1+i\xi)x}}{1+i\xi}\right]_0^\infty = \frac{1}{1-i\xi}+\frac{1}{1+i\xi} = \frac{2}{1+\xi^2}$$

Detta ger $A(\xi)\cosh\xi = \frac{2}{1+\xi^2} \Rightarrow A(\xi) = \frac{2}{(1+\xi^2)\cosh\xi}$

$$\hat{v}(\xi,y) = \frac{2\cosh \xi y}{(1+\xi^2)\cosh\xi}$$

$$v(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\cosh\xi y}{(1+\xi^2)\cosh\xi} e^{i\xi x}\, d\xi = \frac{2}{\pi}\int_0^{\infty} \frac{\cosh\xi y \cos\xi x}{(1+\xi^2)\cosh\xi}\, d\xi$$

$u(x,y) = v(x,y)$ för $x>0$

Egentligen ska man använda separabla variabler på den här uppgiften, men det kan vi inte än

---

Övning 5

3.3:1 Visa: $f_n \in L^2(a,b)$ och $f_n \to f$ i norm (dvs $\|f_n - f\| \to 0$)

$\Rightarrow \langle f_n, g\rangle \to \langle f, g\rangle \quad \forall g \in L^2(a,b)$

(svag konvergens)

$$|\langle f_n, g\rangle - \langle f,g\rangle| = |\langle f_n - f, g\rangle|$$

$$[\text{Cauchy-Schwartz}] \le \underbrace{\|f_n - f\|}_{\to 0}\cdot\|g\| \to 0$$

$\Rightarrow \langle f_n, g\rangle \to \langle f, g\rangle$

---

9 $\{\phi_n\}$ ON-bas för $L^2(a,b)$

Visa: $\langle f,g\rangle = \sum_1^\infty \langle f,\phi_n\rangle \overline{\langle g,\phi_n\rangle}$ om $f,g \in L^2(a,b)$

$f = \sum_1^\infty \langle f,\phi_n\rangle \phi_n \quad$ (jfr $x\cdot y = x_1\bar{y}_1 + x_2\bar{y}_2 + \ldots + x_n\bar{y}_n$)

$$\langle f,g\rangle = \langle \sum_1^\infty \langle f,\phi_n\rangle\phi_n, g\rangle = \sum_1^\infty \langle\langle f,\phi_n\rangle\phi_n, g\rangle =$$

$$= \sum_1^\infty \langle f,\phi_n\rangle\langle \phi_n, g\rangle = \sum_1^\infty \langle f,\phi_n\rangle \overline{\langle g,\phi_n\rangle}$$

$$\langle f,g\rangle = \int_a^b f\bar{g}\,dx$$

Parsevals formel (oändligdimensionell verson av pythagoras sats)

$$\|f\|^2 = \sum_1^\infty |\langle f, \phi_n \rangle|^2 \quad \text{om } \{\phi_n\} \text{ ON-bas}$$

$L^2(-\pi, \pi)$: $\{1, \cos nx, \sin nx\}$ ortogonala men inte normerade

$$f(x) = a_0\left(2 + \sum_1^\infty a_n \cos nx + b_n \sin nx\right)$$

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{\pi}{2}|a_0|^2 + \pi \sum_1^\infty |a_n|^2 + |b_n|^2$$

---

10 a) $\sum_1^\infty \frac{1}{n^4}$

Tabell 2.1: (16): $f(\theta) = \theta^2 \ (-\pi < \theta < \pi) = \frac{\pi^2}{3} + 4\sum_1^\infty \frac{(-1)^n}{n^2}\cos n\theta$

Parsevals formel $\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta = \frac{\pi}{2}\left(\frac{2\pi^2}{3}\right)^2 + \pi \sum_1^\infty \frac{16}{n^4}$

$$\int_{-\pi}^{\pi} \theta^4 d\theta = \frac{2\pi^5}{5} \Rightarrow \sum_1^\infty \frac{1}{n^4} = \frac{1}{16\pi}\left(\frac{2\pi^5}{5} - \frac{2\pi^5}{9}\right) = \frac{\pi^4}{\cancel{8}_2}\,\frac{\cancel{4}}{45} = \frac{\pi^4}{90}$$

b) $\sum_1^\infty \frac{1}{(2n-1)^6}$

Tab 2.1: $f(\theta) = \theta(\pi - |\theta|) \quad (-\pi < \theta < \pi)$

$$= \frac{8}{\pi}\sum \frac{\sin(2n-1)\theta}{(2n-1)^3}$$

Parsevals formel $\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta = \pi \cdot \sum \frac{64}{\pi^2}\,\frac{1}{(2n-1)^6}$

$$\int_{-\pi}^{\pi} (\theta\,\pi - |\theta|)^2 d\theta = 2\int_0^{\pi} \theta^2(\pi-\theta)^2 d\theta = \text{utveckla kvadraten integrera term för term} = \frac{\pi^5}{15} \Rightarrow$$

$$\Rightarrow \sum_1^\infty \frac{1}{(2n-1)^6} = \frac{\pi}{64}\,\frac{\pi^5}{15} = \frac{\pi^6}{960}$$

EÖ 13 Bestäm den lösn till $y''-y=0$ som minimerar

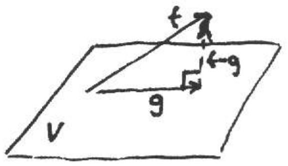
$$\int_{-1}^{1}\left(1+x-y(x)\right)^2 dx$$

Allmän lösn till $y''-y=0$: $y = Ae^{x}+Be^{-x} \overset{\text{eller}}{=} C\cosh x + D\sinh x$

Alt 1: Stoppa in allmänna lösningen i integralen och minimera den.

Alt 2: Känn igen L^2-normen i integralen

$\min \|1-x-y(x)\|^2_{L^2(-1,1)}$ då $y(x) = C\underbrace{\cosh x}_{=\varphi_1} + D\underbrace{\sinh x}_{=\varphi_2}$



min $\|f-g\|$ fås då
g = ortogonala projektionen av f på underrummet av alla lösningar till
V

Om $\{\phi_1, \phi_2\}$ är en ON-bas för V så ges proj av

$g = \langle f,\phi_1\rangle\phi_1 + \langle f,\phi_2\rangle\phi_2$

φ_1 och φ_2 är ortogonala $\Rightarrow$ behöver inte ortogonaliseras

$$\langle\varphi_1,\varphi_2\rangle = \int_{-1}^{1}\underbrace{\cosh x}_{\text{jämn}}\ \underbrace{\sinh x}_{\text{udda}}\,dx = 0$$

$$\|\varphi_1\|^2 = \int_{-1}^{1}(\cosh x)^2 dx = \int\frac{e^{2x}+2+e^{-2x}}{4}dx = \frac{1}{2}\sinh 2 + 1$$

PSS $\|\varphi_2\| = \ldots = \frac{1}{2}\sinh 2 - 1$

Lösningsrummet till $y''-y=0$ har alltså en ON-bas $\{\phi_1, \phi_2\}$

$$\phi_1(x) = \frac{\cosh x}{\sqrt{\frac{1}{2}\sinh 2 - 1}}, \quad \phi_2(x) = \frac{\sinh x}{\sqrt{\frac{1}{2}\sinh 2 + 1}}$$

Bästa y ges av projektionen

$y = \langle 1+x,\phi_1\rangle\phi_1 + \langle 1+x,\phi_2\rangle\phi_2$

$$y(x) = \langle 1+x, \cosh x\rangle \frac{\cosh x}{\frac{1}{2}\sinh 2 - 1} + \langle 1+x, \sinh x\rangle \frac{\sinh x}{\frac{1}{2}\sinh + 1}$$

$$\langle 1+x, \cosh x\rangle = \int_{-1}^{1}(1+x)\cosh x\,dx = [\text{jämn-udda-resonemang}] =$$
$$= 2\int_0^1 \cosh x\,dx = 2\left[\sinh x\right]_0^1 = 2\sinh 1$$

$$\langle 1+x, \sinh x\rangle = \int_{-1}^{1}(1+x)\sinh x\,dx = 2\int_0^1 x\sinh x\,dx = [P.I] =$$
$$= 2\left(\left[x\cosh x\right]_0^1 - \int_0^1 \cosh x\,dx\right) = 2\left(\cosh 1 - \sinh 1\right) = 2\left(\frac{e^1+e^{-1}}{2} -\right.$$
$$\left. - \frac{e^1 - e^{-1}}{2}\right) = 2e^{-1}$$

Alltså $y(x) = \dfrac{2\sinh 1}{\frac{1}{2}\sinh 2 + 1}\cosh x + \dfrac{2e^{-1}}{\frac{1}{2}\sinh 2 - 1}\sinh x$

<u>7.3:6</u> $f \in L^2(\mathbb{R})$ Visa min $\|f-g\|_{L^2}$ då $\hat{g} = 0$ utanför $[-\Omega, \Omega]$ fås för $g_0(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega}\hat{f}(\omega)e^{i\omega t}\,d\omega$

$$\|f-g\|^2 = [\text{Plancherels formel}] = \frac{1}{2\pi}\|\hat{f}-\hat{g}\|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\hat{f}(\omega) - \hat{g}(\omega)\right|^2 d\omega$$
$$= \frac{1}{2\pi}\int_{|\omega|\le\Omega}\left|\hat{f}(\omega) - \hat{g}(\omega)\right|^2 d\omega + \frac{1}{2\pi}\int_{|\omega|>\Omega}|\hat{f}(\omega) - \underbrace{\hat{g}(\omega)}_{0}|^2 d\omega \geq$$
$$\geq \frac{1}{2\pi}\int_{|\omega|>\Omega}|\hat{f}(\omega)|^2 d\omega \quad \text{med likhet dää } \hat{g}(\omega) = \hat{g}_0(\omega) = \begin{cases}\hat{f}(\omega) & \text{på } [-\Omega, \Omega] \\ 0 & \text{annars}\end{cases}$$

"då och endast då"

dvs $\hat{g}_0(\omega) = \hat{f}(\omega)\,\chi_{[-\Omega,\Omega]}(\omega)$

Inversionsformeln ger $g_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{g}(\omega)e^{i\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\Omega}^{\Omega}\hat{f}(\omega)e^{i\omega t}\,d\omega$

Stürm-Liouville-problem (en viss typ av diff ekv)

$\underbrace{(rf')' + pf}_{L(f)} + \lambda\omega f = 0$ på $[a,b]$ (r, p & ω reguljära, kont fkner)

2 st (RV): $B_1(f) = B_2(f) = 0$

Hitta de λ för vilka $\exists$ icke-trivial lösn

3.5:4 $\quad f''+\lambda f=0$ på $[0,\ell]$

$f'(0)=0$

$f(\ell)=0$

I sådana här problem är alla egenvärden reella

fall $\lambda=0$: $f''=0 \Rightarrow f(x)=Ax+B$

$$\left.\begin{array}{l} f'(0)=0 \Rightarrow A=0 \\ f(\ell)=0 \Rightarrow B=0 \end{array}\right\} \Rightarrow f=0 \qquad \text{gav inget}$$

fall $\lambda>0$: $\lambda=\mu^2,\ \mu>0$:

$f(x)=A\cos\mu x+B\sin\mu x$

$f'(0)=0 \Leftrightarrow B=0$

$f(\ell)=0 \Leftrightarrow A\cos\mu\ell=0 \Rightarrow A=0$ eller $\cos\mu\ell=0$

$\mu\ell=(n-\frac{1}{2})\pi,$

$n=1,2,\ldots$

$\mu=\mu_n=(n-\frac{1}{2})\frac{\pi}{\ell}$

fall $\lambda<0$: $\lambda=-\nu^2,\ \nu>0$:

$f(x)=A\cosh\nu x+B\sinh\nu x$

$f'(0)=0 \Leftrightarrow B=0$

$f(\ell)=0 \Leftrightarrow A\underbrace{\cosh\nu\ell}_{\neq 0}=0 \Rightarrow A=0$ ger inget heller

Egenvärdena

$$\lambda_n=\left(n-\frac{1}{2}\right)^2\frac{\pi^2}{\ell}$$

Motsvarande egenfunktioner

$$\cos\mu_n x=\cos\left(n-\frac{1}{2}\right)\frac{\pi x}{\ell}$$

Normalisera

$$\left\|\cos\left(n-\frac{1}{2}\right)\frac{\pi x}{\ell}\right\|^2_{L^2(0,\ell)}=\int_0^\ell \cos^2\left(n-\frac{1}{2}\right)\frac{\pi x}{\ell}\,dx \underset{\uparrow}{=} \frac{\ell}{2}$$

ty

$$\int_a^b \left.\begin{array}{l}\sin^2 cx\\ \text{eller}\\ \cos^2 cx\end{array}\right\} dx = \frac{b-a}{2} \text{ ifall } \left.\begin{array}{l}\sin^2 x\\ \text{eller}\\ \cos^2 x\end{array}\right\} = 0 \text{ i ändpkterna}$$

$\Rightarrow$ Normerade egenfunktionerna

$$\sqrt{\frac{2}{\ell}}\cos\left(n-\frac{1}{2}\right)\frac{\pi x}{\ell}$$

7 $\begin{cases} f'' + \lambda f = 0 & \text{på } [0,1] \\ f(0) = 0 \\ f'(1) = -f(1) \end{cases}$

Alla egenvärden reella

fall $\lambda = 0: \Rightarrow f(x) = Ax + B$

$f(0) = 0 \Rightarrow B = 0$

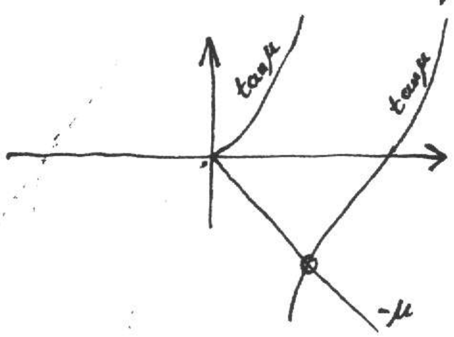
$f'(1) = -f(1) \Rightarrow A = -A \Rightarrow A = 0$ ger inget

fall $\lambda = \mu^2: f(x) = A\cos\mu x + B\sin\mu x$

$f(0) = 0 \Leftrightarrow A = 0$

$f'(1) = -f(1) \Leftrightarrow B\mu\cos\mu = -B\sin\mu$

$\Rightarrow B = 0$ eller $-\mu = \tan\mu$ (*)



Låt μ_n vara lösningarna till (*) $(\mu > 0)$

Egenvärden

$\lambda_n = \mu_n^2$, μ_n pos lösn till (*)

Egenfkner

$\sin\mu_n x$ $\qquad -\mu_n = \frac{\sin\mu_n}{\cos\mu_n}$

dubbla vinkeln för cos baklänges

Normera

$\|\sin\mu_n x\|^2 = \int_0^1 \sin^2\mu_n x\,dx = \int_0^1 \frac{1 - \cos 2\mu_n x}{2}\,dx =$

$= \left[\frac{x}{2} - \frac{\sin 2\mu_n x}{4\mu_n}\right]_0^1 = \frac{1}{2} - \frac{\sin 2\mu_n}{4\mu_n} = \frac{1}{2} - \frac{\cancel{2}\sin\mu_n\cos\mu_n}{\cancel{4}_2\,\mu_n} =$

$= \frac{1}{2} + \frac{1}{2}\cos^2\mu_n = \frac{1}{2}(1 + \cos^2\mu_n)$

$\Rightarrow$ Normerade egenfunktioner

$$\phi_n(x) = \sqrt{\frac{2}{1+\cos^2\mu_n}} \sin\mu_n x$$

fall $\lambda = -\nu^2$, $\nu > 0$

$f(x) = A\cosh\nu x + B\sinh\nu x$

$f(0) = 0 \Rightarrow A = 0$ $\quad$ >0 för $\nu>0$

$f'(1) = -f(1) \Rightarrow \tanh\nu = -\nu$ $\quad$ ger inget

Övning 6

3.5:10 $\quad (xf')' + \lambda \underbrace{x^{-1}}_{w(x)} f = 0$

$f(1) = f(b) = 0 \quad (b>1)$

Alla egenvärden λ är reella enligt teorin

$xf'' + f' + \lambda x^{-1} f = 0$

$x^2 f'' + x f' + \lambda f = 0$ (Eulerekvation) $\quad r(r-1) + r + \lambda = 0$

$r^2 + \lambda = 0$

lösn r_1, r_2

Allmän lösn $f(x) = Ax^{r_1} + Bx^{-r_2}$

om $r_1 = r_2$: $Ax^{r_1} + Bx^{r_1}\ln x$

fall $\lambda = 0$: $r_1 = r_2 = 0$

$f(x) = A + B\ln x$

$f(1) = f(b) = 0 \Rightarrow A = B = 0$, dvs $f \equiv 0$ $\quad$ ger inget

fall $\lambda = -\mu^2$, $(\mu > 0)$; $r = \pm\mu$

$f(x) = Ax^{\mu} + Bx^{-\mu}$

$$(RV): \begin{cases} A + B = 0 \\ Ab^{\mu} + Bb^{-\mu} = 0 \end{cases} \Leftrightarrow \begin{cases} B = -A \\ A(b^{\mu} - b^{-\mu}) = 0 \end{cases}$$

$A = 0$ eller $b^{\mu} - b^{-\mu} = 0$

$b^{2\mu} = 1$

$\mu = 0$ duger ej, ty $\mu > 0$

fall $\lambda = \nu^2$, $(\nu > 0)$; $r = \pm i\nu$

$f(x) = Ax^{i\nu} + Bx^{-i\nu} = Ae^{i\nu\ln x} + Be^{-i\nu\ln x}$

$$(RV): \begin{cases} f(1) = 0 \Leftrightarrow A + B = 0 \Leftrightarrow B = -A \Rightarrow f(x) = A(e^{i\nu\ln x} + e^{-i\nu\ln x}) = \\ = C\sin(\nu\ln x) \\ f(b) = 0 \Leftrightarrow C\sin(\nu\ln b) = 0 \end{cases}$$

$(C = 0)$ eller $\sin(\nu\ln b) = 0$

$\nu\ln b = n\pi$, $\quad n = 1, 2, 3, \ldots$

$$\nu = \nu_n = \frac{n\pi}{\ln b}$$

Egenvärdena: $\lambda_n = \left(\frac{n\pi}{\ln b}\right)^2$

Egenfunktioner: $\sin(\nu_n \ln x) = \sin \frac{n\pi \ln x}{\ln b}$

Egenfunktionerna är ortogonala m a p skalärprodukten

$$\langle f,g\rangle_w = \int_1^b f(x)\overline{g(x)}\,w(x)\,dx$$

Normera m a p denna skalärprodukt (dvs i $L^2_w(1,b)$)

$$\left\|\sin\frac{n\pi \ln x}{\ln b}\right\|_w^2 = \int_1^b \sin^2\left(\frac{n\pi \ln x}{\ln b}\right)\underbrace{\frac{1}{x}}_{w}\,dx = \left[\begin{matrix} y=\ln x \\ dy = \frac{dx}{x}\end{matrix}\right] = \int_0^{\ln b}\sin^2\left(\frac{n\pi y}{\ln b}\right)dy \overset{*}{=} \frac{\ln b}{2}$$

halva intervallängden

Normerade egenfunktionerna

$$\phi_n(x) = \sqrt{\frac{2}{\ln b}}\,\sin\left(\frac{n\pi \ln x}{\ln b}\right)$$

Utveckla $g(x)\equiv 1$ i dessa egenfunktioner

$\{\phi_n\}$ en ON-bas för $L^2_{1/x}(1,b)$

$\Rightarrow$ utvecklingen blir

$$g = \sum \langle g,\phi_n\rangle_{1/x}\,\phi_n$$

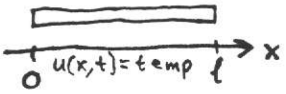
$$\langle g,\phi_n\rangle_{\frac{1}{x}} = \int_1^b 1\cdot\sqrt{\frac{2}{\ln b}}\,\sin\left(\frac{n\pi \ln x}{\ln b}\right)\frac{1}{x}\,dx = [y=\ln x] = \dots = \begin{cases} \frac{2\sqrt{2\ln b}}{n\pi}, & n \text{ udda} \\ 0, & n \text{ jämnt}\end{cases}$$

$$\therefore\; g(x) = \frac{4}{\pi}\sum_1^{\infty}\frac{1}{(2k-1)}\sin\frac{(2k-1)\pi \ln x}{\ln b}$$

$\left.\begin{matrix}\underline{3.5:11} \\ \underline{3.5:12}\end{matrix}\right\}$ se nästa övning

---

4.2:1a



isolerad stav

Värmeledningsekv: $u_t = k u_{xx}$

$x=0$: temp $u(0,t)=0$

$x=\ell$: isolerad $u_x(\ell,t)=0$

Begynnelsetemp: $u(x,0)=f(x)$

Variabelseparation Sök lösningar $X(x)T(t)$ som uppfyller de homogena randvillkoren $X(0)=X'(\ell)=0$

Insättning i ekv

$$X(x)T'(t) = K X''(x)T(t)$$

$$\frac{T'(t)}{K\,T(t)} = \frac{X''(x)}{X(x)} = \text{konst} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X'(\ell) = 0 \end{cases}$$ Ett S-L-problem, det blir oftast det

$$X_n(x) = \sin \underbrace{\left(n-\tfrac{1}{2}\right)\frac{\pi}{\ell}}_{\nu_n} x$$

$$\lambda_n = -\left(\left(n-\tfrac{1}{2}\right)\frac{\pi}{\ell}\right)^2 = -\nu_n^2$$

$$T'(t) + K\nu_n^2 T(t) = 0$$

$$T_n(t) = C e^{-K\nu_n^2 t}$$

$$X_n(x)\, T_n(t)$$

Ansätt en lösning

$$u(x,t) = \sum_1^\infty c_n X_n(x)\, T_n(t) = \sum_1^\infty c_n e^{-K\nu_n^2 t} \sin \nu_n x$$

Uppfyller (DE), uppfyller (RV)

Återstår (BV)

$$t = 0: \sum_1^\infty c_n \sin \nu_n x = f(x) \quad (*)$$

(VL) = Fourierserieutvecklingen av f

$$c_n = \underbrace{\frac{2}{\ell}}_{\text{normeringsfaktor}} \int_0^\ell f(x) \sin\left(n-\tfrac{1}{2}\right)\frac{\pi x}{\ell}\, dx$$

Alt: Multiplicera $(*)$ med $\sin \nu_m(x)$ och integrera över $[0,\ell]$

$$\sum_{n=1}^\infty c_n \int_0^\ell \sin \nu_n x \sin \nu_m x\, dx = \int_0^\ell f(x) \sin \nu_m x$$

$$c_m \frac{\ell}{2} \int_0^\ell f(x) \sin \nu_m x\, dx$$

Svar: $u(x,t) = \sum_1^\infty c_n e^{-K(n-\frac{1}{2})^2 t\pi^2/\ell^2} \sin\left(\left(n-\tfrac{1}{2}\right)\frac{\pi x}{\ell}\right)$

där $c_n = \frac{2}{\ell} \int_0^\ell f(x) \sin\left(n-\tfrac{1}{2}\right)\frac{\pi x}{\ell}\, dx$

b $u(x,t) = ?$ om $f(x) \equiv 50$

$$c_n = \frac{2}{\ell}\int_0^\ell 50 \sin\left(n-\tfrac{1}{2}\right)\frac{\pi x}{\ell}\, dx = \frac{100}{\pi(n-\frac{1}{2})}\left[\cos\left(n-\tfrac{1}{2}\right)\frac{\pi x}{\ell}\right]_0^\ell$$

$$c_n = \frac{100}{\pi(n-\frac{1}{2})} = \frac{200}{\pi(2n-1)}$$

2
$$\begin{cases} u_t = k u_{xx} \\ u(0,t) = C \\ u_x(\ell,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

Sätt $v(x,t) = u(x,t) - C$

v uppfyller då ekv:

$$\begin{cases} v_t = k v_{xx} \\ v(0,t) = 0 \\ v_x(\ell,t) = 0 \\ v(x,0) = f(x) - C \end{cases}$$

Samma ekv som i a): $u(x,t) = C + v(x,t) = C + \sum b_n e^{-k(n-\frac{1}{2})^2 \frac{\pi^2}{\ell^2} t} \sin(n-\frac{1}{2})\frac{\pi x}{\ell}$

där $b_n = \frac{2}{\ell}\int_0^\ell (f(x) - C) \sin(n-\frac{1}{2})\frac{\pi x}{\ell}\,dx = \underset{\substack{\uparrow \\ \text{från uppg 1}}}{C_n} - \frac{2C}{\ell}\int \sin(n-\frac{1}{2})\frac{\pi x}{\ell}\,dx =$

$= C_n - \frac{4C}{\pi(2n-1)}$

---

5 $\ell = \pi$
$$\begin{cases} u_t = k u_{xx} + e^{-2t}\sin x \\ u(0,t) = 0 \\ u(\pi,t) = 0 \\ u(x,0) = 0 \end{cases}$$

Betrakta motsvarande homogena problem:

$$\begin{cases} u_t = k u_{xx} \\ u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases}$$

Variabelseparation ger ekv

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(\pi) = 0 \end{cases}$$

Ansätt en lösning $u(x,t)$ utvecklad i motsvarande serie

$u(x,t) = \sum_1^\infty C_n(t)\sin nx$

Sätt in i ekvationen

$\sum_1^\infty C_n'(t)\sin nx = \sum_1^\infty -kn^2 C_n(t)\sin nx + e^{-2t}\sin x$

Identifiera koefficienter

$$C_n'(t) = -kn^2 C_n(t) + \begin{cases} e^{-2t}, & n=1 \\ 0, & \text{annars} \end{cases}$$

Begynnelsevillkor

$$(t=0) \quad \sum_1^\infty C_n(0) \sin nx = 0 \Leftrightarrow C_n(0) = 0$$

För $n \neq 1$ fås $C_n(t) = 0$

För $n=1$: $\begin{cases} C_1'(t) + kC_1(t) = e^{-2t} \\ C_1(0) = 0 \end{cases}$

Integrerande faktor $e^{kn^2 t}$, $n=1 \Rightarrow e^{kt}$

$$\frac{d}{dt}\left(e^{kt} C_1(t)\right) = e^{(k-2)t}$$

fall $k \neq 2$: $C_1(t) = e^{-kt} \int_0^t e^{(k-2)s} ds = e^{-kt} \dfrac{e^{(k-2)t} - 1}{k-2} = \dfrac{e^{-2t} - e^{-kt}}{k-2}$

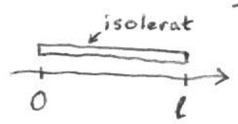
fall $k=2$: $C_1(t) = t e^{-2t}$

Svar: $u(x,t) = \dfrac{(e^{-2t} - e^{-kt}) \sin x}{k-2}$ om $k \neq 2$

$u(x,t) = t e^{-2t} \sin x$ om $k=2$

Övning 7

4.2:6.



$$\begin{cases} u_t = k u_{xx} + Re^{-ct} \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = 0 \end{cases}$$

Separation av motsvarande homogena ekvation i x-led ger egenfunktioner $\sin \frac{n\pi x}{l}$

Ansätt $u(x,t) = \sum_1^\infty A_n(t) \sin \frac{n\pi x}{l}$

Sätt in i ekv

$$\sum_1^\infty A_n'(t) \sin \frac{n\pi x}{l} = \sum_1^\infty -k \frac{n^2\pi^2}{l^2} A_n(t) \sin \frac{n\pi x}{l} + \sum_1^\infty \beta_n(t) \sin \frac{n\pi x}{l},$$

där $\beta_n = \dfrac{2}{l} \displaystyle\int_0^l R e^{-ct} \sin \frac{n\pi x}{l} dx = \ldots = \begin{cases} \dfrac{4Re^{-ct}}{n\pi} & \text{om } n \text{ udda} \\ 0 & \text{om } n \text{ jämnt} \end{cases} = \gamma_n e^{-ct}$

Identifiera koefficienter

$$A_n'(t) = -k \frac{n^2\pi^2}{l^2} A_n(t) + \gamma_n e^{-ct}$$

Begynnelsevillkor: $u(x,0) = \sum_{1}^{\infty} A_n(0) \sin \frac{n\pi x}{l} = 0 \Rightarrow A_n(0) = 0$

ekv typ $y' + ay = ke^{-ct} \Rightarrow$ löses med $\begin{cases} \text{homogen + partikulär lösning} \\ \text{eller integrerande faktor} \end{cases}$

Lösning av dessa ODE ger

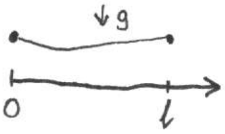
$$A_n(t) = \gamma_n \frac{e^{-ct} - e^{-kn^2\pi^2 t/l^2}}{\frac{kn^2\pi^2}{l^2} - C}$$

$$\therefore\ u(x,t) = \frac{4R}{\pi} \sum_{\substack{n=1 \\ n \text{ udda}}}^{\infty} \frac{e^{-ct} - e^{-kn^2\pi^2 t/l^2}}{n\left(\frac{kn^2\pi^2}{l^2} - C\right)} \sin \frac{n\pi x}{l} \quad \text{om } \frac{kn^2\pi}{l^2} \neq C \ \forall \text{ udda heltal}$$

För $y' - ay = Ae^{at}$ ansätter man kte^{at} som partikulärlösning $\Rightarrow$

$\Rightarrow$ om $\frac{kN^2\pi^2}{l^2} = C$ fås en term $\frac{te^{-kN^2\pi^2 t/l^2}}{N} \sin \frac{N\pi x}{l}$

4.3:3



$u(x,t) =$ "utböjning" i pkten x vid tiden t

$u_{tt} = c^2 u_{xx} - g$

$u(0,t) = u(l,t) = 0$

a Sök en stationär lösning $u(x,t) = \phi(x)$

Ekv blir då:

$$\begin{cases} c^2\phi''(x) - g = 0 \\ \phi(0) = \phi(l) = 0 \end{cases}$$

$$\phi'' = \frac{g}{c^2} \Rightarrow \phi = \frac{gx^2}{2c^2} + Ax + B$$

RV ger $A, B \Rightarrow \phi(x) = \frac{g}{2c^2} x(x-l)$

b $u(x,0) = u_t(x,0) = 0$

Studera $v(x,t) = u(x,t) - \phi(x)$

Vi får då
$$\begin{cases} v_{tt} = c^2 v_{xx} \\ v(0,t) = v(l,t) = 0 \\ v(x,0) = -\phi(x) = \frac{g}{2c^2} x(l-x) \\ v_t(x,0) = 0 \end{cases}$$

Variabelseparation

Sök lösn $X(x) \cdot T(t)$

$X(x)T''(t) = c^2 X''(x) T(t)$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(\ell) = 0 \end{cases}$$

$$\left(\begin{array}{l} \lambda = 0 \text{ ger inget} \\ \lambda = -\mu^2 \text{ ger inget} \end{array}\right)$$

$\lambda = \nu^2$ ger $X(x) = A \sin \nu x + B \cos \nu x$, $X(0) = 0 \Rightarrow B = 0$

$X(\ell) = 0 \Rightarrow (A = 0)$ eller $\sin \nu \ell = 0 \Leftrightarrow \nu = \nu_n = \frac{n\pi}{\ell}$

Egenfunktioner $X_n(x) = \sin \frac{n\pi x}{\ell}$

Egenvärden $\lambda_n = \nu_n^2 = \frac{n^2\pi^2}{\ell^2}$

Motsvarande $T_n(t)$:

$$\begin{cases} T_n''(t) + \frac{n^2\pi^2c^2}{\ell^2} T_n(t) = 0 \\ T_n'(0) = 0 \end{cases}$$

$T_n(t) = A_n \cos \frac{n\pi c t}{\ell} + B_n \sin \frac{n\pi c t}{\ell}$

$T_n'(t) = 0 \Rightarrow B_n = 0$

$T_n(t) = A_n \cos \frac{n\pi c t}{\ell}$

Ansätt en lösning $v(x,t) = \sum_1^\infty X_n(x) T_n(t) = \sum_1^\infty A_n \sin \frac{n\pi x}{\ell} \cos \frac{n\pi c t}{\ell}$

Bestäm A_n så att $v(x,0) = \frac{g}{2c^2} x(\ell - x)$

Utveckla begynnelsevillkoret i Fouriersinusserie

$v(x,0) = -\phi(x) = \sum_0^\ell C_n \sin \frac{n\pi x}{\ell}$, $\quad C_n = \frac{2}{\ell}\int_0^\ell \frac{g}{2c^2} x(\ell - x) \sin \frac{n\pi x}{\ell}\, dx = [\text{P.I}] = \ldots$

$$\ldots = \begin{cases} \frac{4g\ell^2}{n^3\pi^3c^2}, & n \text{ udda} \\ 0, & n \text{ jämnt} \end{cases}$$

Begynnelsevillkor

$u(x,0) = \ldots = \sum A_n \sin \frac{n\pi x}{\ell} = \sum C_n \sin \frac{n\pi x}{\ell}$

$A_n = C_n$

$$v(x,t) = \frac{4g\ell^2}{\pi^3 c^2} \sum_{k=1}^\infty \frac{1}{(2k-1)^3} \sin \frac{(2k-1)\pi x}{\ell} \cos \frac{(2k-1)\pi c t}{\ell}$$

$$u(x,t) = \phi(x) + v(x,t) = \frac{g x(\ell - x)}{2c^2} + \ldots$$

Visa $u(x,t) = \phi(x) - \frac{1}{2}\left(\bar{\phi}(x+ct) + \bar{\phi}(x-ct)\right)$, där $\bar{\phi}$ är den udda 2ℓ-periodiska utvidgningen av $\phi(x)$, dvs $v(x,t) = -\frac{1}{2}\left(\bar{\phi}(x+ct) + \bar{\phi}(x-ct)\right)$

$$\sum_1^\infty C_n \sin\frac{n\pi x}{\ell} = \begin{cases} -\phi(x) \text{ på } [0,\ell] \\ \text{udda (ty sin udda)} \\ 2\ell\text{-periodisk, dvs} \\ = -\bar{\phi}(x) \end{cases}$$

$$v(x,t) = \sum_1^\infty C_n \sin\frac{n\pi x}{\ell}\cos\frac{n\pi c t}{\ell} = \left[\sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha+\beta) + \sin(\alpha-\beta)\right] =$$

$$= \frac{1}{2}\left(\sum_1^\infty C_n \sin\frac{n\pi(x+ct)}{\ell} + \sum_1^\infty C_n \sin\frac{n\pi(x-ct)}{\ell}\right) = \frac{1}{2}\left(-\bar{\phi}(x+ct) - \bar{\phi}(x-ct)\right)$$

VSV

4.3:7 

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = 0 \\ u(\pi,t) = \sin kt \\ u(x,0) = u_t(x,0) = 0 \end{cases}$$

Låt $u(x,t) = v(x,t) + \underbrace{\frac{x}{\pi}\sin kt}_{\text{uppfyller RV (i 0 och } \pi)}$

För v fås ekv

$$\begin{cases} v_{tt} = c^2 v_{xx} + \frac{k^2 x}{\pi}\sin kt \\ v(0,t) = v(\pi,t) = 0 \\ v(x,0) = 0 \\ v_t(x,0) = -\frac{kx}{\pi} \end{cases}$$

Motsvarande homogena problem ger egenfunktionerna $\sin\frac{n\pi x}{\ell} = \sin nx$ i x-led

Utveckla allt i dessa funktioner

$$v(x,t) = \sum_1^\infty A_n(t)\sin nx$$

$$x = \sum_1^\infty a_n \sin nx$$

$$a_n = \frac{2}{\pi}\int_0^\pi x\sin nx\,dx = \frac{2}{n}(-1)^{n+1}$$

Ekv ger $A_n''(t) = -c^2n^2A_n(t) + \frac{k^2}{\pi}\cdot\frac{2}{n}(-1)^{n+1}\sin kt$

och (BV) ger $\begin{cases} A_n(0) = 0 \\ A_n'(0) = -\frac{k}{\pi}\,\frac{2}{n}(-1)^{n+1} \end{cases}$

Allmän lösning $A_n(t) = D_n \sin nct + E_n\cos nct + \overbrace{\frac{2k^2(-1)^{n+1}}{\pi n}\,\frac{\sin kt}{n^2c^2-k^2}}^{\text{partikulärlösning}}$

$A_n(0)=0 \Rightarrow E_n=0$

$A_n'(0)=-\frac{k}{\pi}\,\frac{2}{n}(-1)^{n+1} \Rightarrow D_n=\frac{-2(-1)^{n+1}}{\pi(n^2c^2-k^2)}$

$A_n(t)=\frac{2(-1)^{n+1}k}{n\pi(n^2c^2-k^2)}(k\sin kt-nc\sin nct)$

$$u(x,t)=\frac{x}{\pi}\sin kt+v(x,t)=\frac{x}{\pi}\sin kt+\frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(k\sin kt-nc\sin nct)}{n(n^2c^2-k^2)}\sin nx$$

Dirichlets problem

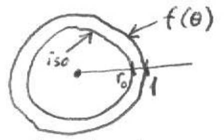
$\begin{cases}\Delta u=u_{xx}+u_{yy}=0\\ u(x)=f(x) \text{ på } \partial D \text{ (randen)}\end{cases}$

polära koordinater

$\Delta u=\frac{1}{r}(ru_r)_r+\frac{1}{r^2}u_{\theta\theta}=u_{rr}+r^{-1}u_r+r^{-2}u_{\theta\theta}$

4.4:5a

$\begin{cases}\Delta u=0\\ u_r(r_0,\theta)=0\\ u(1,\theta)=f(\theta)\end{cases}$ $\quad \frac{1}{r}(ru_r)_r+\frac{1}{r^2}u_{\theta\theta}=0$



$u(r,\theta)=$ temp t

Separera: Sök lösn $R(r)\Theta(\theta)$ till ekv och det homogena (RV)

$\frac{r^2R''+rR'}{R(r)}=-\frac{\Theta''(\theta)}{\Theta(\theta)}=\nu^2=\lambda$

θ-problemet: $\Theta''+\lambda\Theta=0$ $\quad \Theta$ 2π-periodisk $\Rightarrow$ $\Theta(2\pi)=\Theta(0)$, $\Theta'(2\pi)=\Theta'(0)$

Får trigonometriska fourierserier $\Theta_n(\theta)=e^{in\theta}$ $\quad \lambda_n=n^2$

motsvarande R_n: $\begin{cases}r^2R''+rR'-n^2R=0\\ R'(r_0)=0\end{cases}$

Nu talen som utlovades i förra övningen:

3.5:11: Bestäm egenvärden och normerade egenfunktioner för

$(x^2f')'+\lambda f=0$, $\quad f(1)=f(b)=0 \quad (b>1)$

Ekv kan skrivas

$x^2f''+2xf'+\lambda f=0$

Detta är en Euler-ekvation med lösn $C_1 x^{r_1} + C_2 x^{r_2}$, där r_1, r_2 löser

$$r(r-1)+2r+\lambda=0$$
$$r^2+r+\lambda=0$$
$$r=-1/2 \pm \sqrt{1/4-\lambda}$$

Dela upp i olika fall beroende på tecknet på $1/4-\lambda$

$1/4-\lambda=0$: $f(x)=C_1 x^{-1/2}+C_2 x^{-1/2}\ln x$

$$f(1)=0 \Rightarrow C_1=0$$
$$f(b)=0 \Rightarrow C_2=0$$

$1/4-\lambda>0$: $1/4-\lambda=\mu^2$, $\mu>0$

$$f(x)=C_1 x^{-1/2+\mu}+C_2 x^{-1/2-\mu}$$
$$f(1)=0 \Rightarrow C_1+C_2=0 \Rightarrow C_2=-C_1$$
$$f(b)=0 \Rightarrow C_1\left(b^{-1/2}-b^{-1/2}\ln b\right)=C_1 b^{-1/2}(1-\ln b)=0 \Rightarrow C_1=0$$

$1/4-\lambda<0$: $1/4-\lambda=-\nu^2$, $\nu>0$

$$f(x)=C_1 x^{-1/2+i\nu}+C_2 x^{-1/2-i\nu}$$
$$f(1)=0 \Rightarrow C_1+C_2=0 \Rightarrow C_2=-C_1 \Rightarrow f(x)=C_1 x^{-1/2}\left(e^{i\nu\ln x}-e^{-i\nu\ln x}\right)=$$
$$=C x^{-1/2}\sin(\nu\ln x)=0$$

$f(b)=0 \Leftrightarrow (C=0)$ eller $\sin(\nu\ln b)=0$

$$\nu=\nu_n=\frac{n\pi}{\ln b}, \quad n=1,2,3,\ldots$$

Egenvärdena är alltså $\lambda_n=1/4+\nu_n^2=\frac{1}{4}+\left(\frac{n\pi}{\ln b}\right)^2$ och motsvarande egenfunktioner är $x^{-1/2}\sin\left(\frac{n\pi\ln x}{\ln b}\right)$

Återstår att normera:

$$\left\|x^{-1/2}\sin\left(\frac{n\pi\ln x}{\ln b}\right)\right\|^2=\int_1^b x^{-1}\sin^2\left(\frac{n\pi\ln x}{\ln b}\right)dx=[y=\ln x]=\int_0^{\ln b}\sin^2\frac{n\pi y}{\ln b}\,dy=\frac{\ln b}{2}$$

så de normerade egenfunktionerna blir $\phi_n(x)=\sqrt{\frac{2}{\ln b}}\,x^{-1/2}\sin\left(\frac{n\pi\ln x}{\ln b}\right)$

3.5:12 Betrakta Stürm-Liouvilleproblemet

$$(rf')'+pf+\lambda f=0, \quad f(a)=f(b)=0 \quad (*)$$

a Visa att om f uppfyller (*) så är

$$\lambda\int_a^b|f|^2dx=\int_a^b r|f'|^2dx-\int_a^b p|f|^2dx$$

Bevis: $\lambda\int_a^b|f|^2dx=\int_a^b\lambda f\cdot\bar f\,dx=[\lambda f=-(rf')'-pf \text{ enl}(*)]=$

$$=-\int_a^b(rf')'\bar f\,dx-\int_a^b pf\bar f\,dx=[\text{Part int}]=-\underbrace{\left[rf'\bar f\right]_a^b}_{=0 \text{ pga (RV)}}+\int_a^b rf'\bar f'\,dx-\int_a^b p|f|^2dx=$$

$$=\int_a^b r|f'|^2dx-\int_a^b p|f|^2dx$$

b Om $p(x) \leq C$ för alla x så uppfyller alla egenvärden till (*) $\lambda \geq -C$

Bevis: $\lambda \int_a^b |f|^2 dx = \underbrace{\int_a^b r|f'|^2 dx}_{\geq 0} - \int_a^b p|f|^2 dx \geq 0 - \int_a^b C|f|^2 dx = -C\int_a^b |f|^2 dx$

Division med $\int_a^b |f|^2 dx \,(>0)$ ger $\lambda \geq -C$

c. Slutsatsen i b/ är fortfarande riktig om randvillkoren $f(a)=f(b)=0$ ersätts med $f'(a) - \alpha f(a) = f'(b) - \beta f(b) = 0$ där $\alpha \geq 0$ och $\beta \geq 0$.

OBS! Fel i Folland. Det här är rätt.

Bevis: Den utintegrerade termen från a/ blir här

$$-\left[r f' \bar{f}\right]_a^b = -r(b)f'(b)\bar{f}(b) + r(a)f'(a)\bar{f}(a) = [\text{Randvillkor}] =$$

$$= -\beta r(b)|f(b)|^2 + \alpha r(a)|f(a)|^2$$

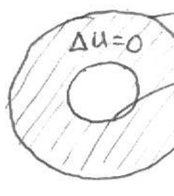
så vi får $\lambda \int_a^b |f|^2 dx = -\underbrace{\beta}_{\geq 0}\, \underbrace{r(b)}_{>0}\, \underbrace{|f(b)|^2}_{\geq 0} + \underbrace{\alpha}_{\geq 0}\, \underbrace{r(a)}_{>0}\, \underbrace{|f(a)|^2}_{\geq 0} + \underbrace{\int_a^b r|f'|^2 dx}_{\geq 0} -$

$- \int_a^b p|f|^2 dx \geq 0 - \int_a^b C|f|^2 dx = -C\int_a^b |f|^2 dx$ varav $\lambda \geq -C$ precis som i b/.

Övning 8

(forts)

4.4:5



a separation $u = R(r)\Theta(\theta)$

$$\Delta u = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = \frac{1}{r}(r u_r)_r + r^{-2}u_{\theta\theta}$$

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

Θ-problemet:

$$\begin{cases} \Theta'' + \lambda\Theta = 0 \\ \Theta \ \ 2\pi\text{-periodisk} \end{cases}$$

Man får trigonometriska fourierserier

egenfunktioner $\Theta_n = e^{in\theta}$, $n \in \mathbb{Z}$

egenvärden: $\lambda = n^2$, egen

R-problemet:

$r^2R''(r) + rR(r) - n^2R(r) = 0$ Eulerekv m lösn $Ar^{s_1} + Br^{s_2}$

där s_1 & s_2 lösn till $s(s-1) + s - n^2 = 0 \Leftrightarrow s^2 = n^2 \Leftrightarrow s = \pm n$

$R_n(r) = A_n r^n + B_n r^{-n}$

(RV): $u_r(r_0,\theta)=0$ ger $R'(r_0)=0$

$R_n'(r) = nA_n r^{n-1} - nB_n r^{-n-1}$ $\quad \underline{n\neq 0}$

$R_n'(r_0)=0 \Leftrightarrow B_n = A_n r_0^{2n}$

$R_n(r) = A_n(r^n + r_0^{2n} r^{-n})$

$\underline{n=0}$: $R_0(r) = A_0 + B_0 = konst$

$R_0'(r)=0$ OK Men formeln ovan ger också $R_0(r) = konst$ OK

Ansätt $u(r,\theta) = \sum_{-\infty}^{\infty} A_n(r^n + r_0^{2n} r^{-n}) e^{in\theta}$

(RV): $u(1,\theta) = f(\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta}$, där $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta$

ger $A_n(1+r_0^{2n}) = c_n \Leftrightarrow A_n = \frac{c_n}{1+r_0^{2n}}$

Alltså: $u(r,\theta) = \sum_{-\infty}^{\infty} c_n \frac{r^n + r_0^{2n} r^{-n}}{1+r_0^{2n}} e^{in\theta}$

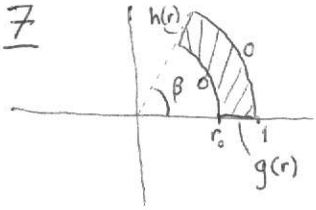
$\underline{b}$ $f(\theta) = 1 + 2\sin\theta = 1 + \frac{e^{i\theta} - e^{-i\theta}}{i}$

$u(r,\theta) = ?$

f givet som fourierserie

$u(r,\theta) = 1 + \frac{1}{i}\left(\frac{r + r_0^2 r^{-1}}{1+r_0^2} e^{i\theta} + \frac{r^{-1} + r_0^{-2} r}{1 + r_0^{-2}} e^{-i\theta}\right) = 1 + 2\,\frac{r^2 + r_0^2}{r(1+r_0^2)} \sin\theta$

$\underline{7}$



$$\begin{cases} \Delta u = 0 \\ u(r_0,\theta) = u(1,\theta) = 0 \\ u(r,0) = g(r) \\ u(r,\beta) = h(r) \end{cases}$$

Variabelseparation:

Sök lösningar $R(r)\Theta(\theta)$

$$\frac{-r^2 R''(r) + r R'(r)}{R(r)} = \frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda = konst$$

R-ekv: $\begin{cases} r^2 R''(r) + r R'(r) + \lambda R(r) = 0 \\ R(r_0) = R(1) = 0 \end{cases}$

$(rR')' + \lambda \underbrace{r^{-1}}_{=w} R(r) = 0$

övningstal 3.5:10 ger $\lambda_n = \left(\frac{n\pi}{\ln r_0}\right)^2$, $n = 1, 2, \ldots$

och egenfunktioner $\sqrt{\frac{2}{-\ln r_0}} \sin\left(\frac{n\pi \ln r}{\ln r_0}\right)$

Θ-ekv:

$$\Theta'' - \left(\frac{n\pi}{\ln r_0}\right)^2 \Theta = 0$$

$\Theta_n(\theta) = A_n e^{n\pi\theta/\ln r_0} + B_n e^{-n\pi\theta/\ln r_0}$ (se i ramen nedan)

Ansätt en lösning $u(r,\theta) = \sum_1^\infty \left(A_n e^{n\pi\theta/\ln r_0} + B_n e^{-n\pi\theta/\ln r_0}\right) \sin\left(\frac{n\pi \ln r}{\ln r_0}\right)$

Utveckla g och h i samma egenfunktionssystem

$$g(r) = \sum_1^\infty c_n \sin\left(\frac{n\pi \ln r}{\ln r_0}\right)$$

$$h(r) = \sum_1^\infty d_n \sin\left(\frac{n\pi \ln r}{\ln r_0}\right)$$

$$c_n = \underbrace{\frac{2}{-\ln r_0}}_{= \frac{1}{\|\sin(\text{--})\|_w^2}} \int_{r_0}^1 g(r) \sin\left(\frac{n\pi \ln r}{\ln r_0}\right) r^{-1}\, dr$$

$d_n = \text{----}$ (analogt)

Randvillkoren blir då

$$\begin{cases} A_n + B_n = c_n \\ A_n e^{n\pi\beta/\ln r_0} + B_n e^{-n\pi\beta/\ln r_0} = d_n \end{cases}$$

ur vilket A_n och B_n kan lösas ut

om lösningen till $\begin{cases} \Theta'' + \lambda\Theta = 0 \\ \Theta \ 2\pi\text{-periodisk} \end{cases}$

fall $\lambda = 0$: $\Theta'' = 0$, $\Theta = A\Theta + B$

2π-per $\Rightarrow A = 0$

$\Theta_0 = 1 = e^{i0\theta}$

fall $\lambda = -\mu^2$: $\Theta'' - \mu^2\Theta = 0$

$\Theta = Ae^{\mu\theta} + Be^{-\mu\theta}$ aldrig periodiskt

fall $\lambda = \nu^2$, $\nu > 0$:

$\Theta = Ae^{i\nu\theta} + Be^{-i\nu\theta}$

$\Theta(\theta) = \Theta(\theta + 2\pi)$

$\Rightarrow e^{i\nu 2\pi} = 1 \Rightarrow \nu$ heltal, $\nu = n$ ger $e^{in\theta}$, $e^{-in\theta}$

Besselfunktioner lösn till Bessels ekv

$$x^2 f'' + x f' + (x^2 - \nu^2) f = 0$$

Lösn $J_\nu(x) = \sum_1^\infty \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}$

$$\Gamma(z) \overset{\text{def}}{=} \int_0^\infty t^{z-1} e^{-t}\, dt$$

$\Gamma(z+1) = z\Gamma(z)$ $\qquad$ $(\ast)$: $J_1(s) = \frac{1}{\Gamma(2)} \frac{s}{2} + \dots$

$\Gamma(n+1) = n!$

5.1:1 f_1 och f_2 lösn till Bessels ekv av ordning ν

$$W = \text{Wronskianen} = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = f_1 f_2' - f_1' f_2$$

a/ Visa $W'(x) = -\frac{W(x)}{x}$

$$W' = (f_1 f_2' - f_1' f_2)' = f_1 f_2'' + \cancel{f_1' f_2'} - \cancel{f_1' f_2'} - f_1'' f_2 = [\text{Bessels ekv}] =$$

$$= f_1\left(-\frac{(x f_2' + (x^2-\nu^2) f_2)}{x^2}\right) - f_2\left(-\frac{(x f_1' + (x^2-\nu^2) f_1)}{x^2}\right) = -\frac{(f_1 f_2' - f_1' f_2)}{x} = -\frac{W(x)}{x}$$

$W'(x) + \frac{W(x)}{x} = 0$ diff ekv löses tex mha int faktor $e^{\int 1/x\, dx} = x$

$$\frac{d}{dx}(x W(x)) = 0 \Leftrightarrow W(x) = \frac{c}{x}$$

b/ $f_1 = J_\nu$, $f_2 = J_{-\nu} \Rightarrow W(x) = -\frac{2 \sin \nu\pi}{\pi x}$

$J_\nu(x) \sim \frac{x^\nu}{2^\nu \Gamma(\nu+1)}$ då $x \to 0$

betyder "kommer att bete sig som"

$J_\nu'(x) = \sum_0^\infty \frac{(-1)^k (2k+\nu)}{k!\,\Gamma(k+\nu+1)} \frac{x^{2k+\nu-1}}{2^{2k+\nu}} \sim \frac{\nu x^{\nu-1}}{2^\nu \Gamma(\nu+1)}$ då $x \to 0$

$$W(x) = f_1 f_2' - f_1' f_2 = J_\nu J_{-\nu}' - J_\nu' J_{-\nu} \sim \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \frac{(-\nu) x^{-\nu-1}}{2^{-\nu}\Gamma(-\nu+1)} -$$

$$- \frac{\nu x^{\nu-1}}{2^\nu \Gamma(\nu+1)} \frac{x^{-\nu}}{2^{-\nu}\Gamma(-\nu+1)} = -\frac{2\nu x^{-1}}{\Gamma(1+\nu)\Gamma(1-\nu)} = -\frac{\cancel{\nu}\, 2 x^{-1}}{\cancel{\nu}\,\Gamma(\nu)\Gamma(1-\nu)} = -\frac{2x^{-1}}{\left(\frac{\pi}{\sin \nu\pi}\right)} =$$

$$= \frac{-2 \sin \nu\pi}{\pi x}$$

ty $\Gamma(z+1) = z\Gamma(z)$

c/ $f_1 = J_\nu$, $f_2 = Y_\nu$

$Y_\nu(x) = \frac{\cos(\nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi} \Rightarrow W(x) = \frac{2}{\pi x}$ Lösn: Räkna på och använd b/

Avsnitt 5.2 Rekurrensformler s 133 i Folland

5.2:6 Visa $\int_0^x J_3(s)\,ds = 1 - J_2(x) - 2x^{-1}J_1(x)$

formel 5.13: $\frac{d}{dx}\left(x^{-\nu}J_\nu(x)\right) = -x^{-\nu}J_{\nu+1}(x)$

$$\int_0^x J_3(s)\,ds = \int_0^x s^2 s^{-2} J_3(s)\,ds = [PI] = \left[\cancel{s^2}\left(-\cancel{s^{-2}}J_2(s)\right)\right]_0^x + 2\int_0^x s^{-1}J_2(s)\,ds =$$

$$= -J_2(x) + \underbrace{J_2(0)}_{=0} + 2\left[-s^{-1}J_1(s)\right]_0^x = -J_2(x) - 2x^{-1}J_1(x) + 2\underbrace{\lim_{s\to 0}\frac{J_1(s)}{s}}_{} =$$

se (✱) förra sidan $\Rightarrow \frac{1}{\Gamma(2)\cdot 2} = \frac{1}{2}$

$$= 1 - J_2(x) - 2x^{-1}J_1(x)$$

5.2:8 $\int_0^x s^n J_0(s)\,ds = x^nJ_1(x) + (n-1)x^{n-1}J_0(x) - (n-1)^2\int_0^x s^{n-2}J_0(s)\,ds$

$$\int_0^x s^n J_0(s)\,ds = \left[\begin{array}{c}\text{använd formlerna}\\ (xJ_1(x))' = xJ_0(x)\\ J_0'(x) = -J_1(x)\end{array}\right] = \int_0^x s^{n-1}\, sJ_0(s)\,ds = [PI] = \left[s^{n-1}sJ_1(s)\right]_0^x -$$

$$-(n-1)\int_0^x s^{n-2}\underbrace{sJ_1(s)}_{=-J_0'}\,ds = x^nJ_1(x) - 0 - (n-1)\left(\left[-s^{n-1}J_0(s)\right]_0^x + \right.$$

$$\left. +(n-1)\int_0^x s^{n-1}J_0(s)\,ds\right) \quad \text{vsv}$$

$$\int_0^x s^3 J_0(s)\,ds = x^3J_1(x) + 2x^2J_0(x) - 4\underbrace{\int_0^x s\,J_0(s)\,ds}_{\left[sJ_1(s)\right]_0^x = xJ_1(x)} =$$

$$= (x^3 - 4x)J_1(x) + 2x^2J_0(x)$$

$$\int_0^x s^5 J_0(s)\,ds = x^5J_1(x) + 4x^4J_0(x) - 16\underbrace{\int_0^x s^3J_0(s)\,ds}_{} = \dots$$

Övning 9

$$\left.\begin{array}{l} x^2 f'' + x f' + (\underbrace{\mu^2}_{\lambda = \text{egenvärde}} x^2 - \nu^2) f = 0 \\ f(x) = g(\mu x) \end{array}\right\} \ldots \Rightarrow f(x) = a J_\nu(\mu x) + b Y_\nu(\mu x)$$

Vanliga RV: $f(0^+)$ ändligt $\Rightarrow$ strykt Y_ν, ty $\to \infty$ i 0

$\beta f(b) + \beta' f'(b) = 0$

<u>Sats 5.3:</u> $\beta' = 0$ Man får $\left\{ J_\nu\left(\lambda_k \frac{x}{b}\right) \right\}_{k=1}^{\infty}$ där λ_k är pos nollst t J_ν

Dessa egenfkner är ortogonala, bas för $L^2_w(0,b)$ där $w(x) = x$

<u>5.4:2</u> utveckla $f(x) = b^2 - x^2$, $0 \le x \le b$ i en fourierserie

$f(x) = \sum C_k \underbrace{J_0\left(\lambda_k \frac{x}{b}\right)}_{\phi_k}$ där $\{\lambda_k\}$ är positiva nollställen till J_0

$C_k = \frac{1}{\|\phi_k\|^2_w} \langle f, \phi_k \rangle_w$, $\quad \langle \phi_k, \phi_k \rangle_w = [\text{Th } 5.3] = \frac{b^2}{2}\left(J_1(\lambda_k)\right)^2$

$$\langle f, \phi_k \rangle_w = \int_0^b f(x) J_0\left(\lambda_k \frac{x}{b}\right) x \, dx = \int_0^b x(b^2 - x^2) J_0\left(\lambda_k \frac{x}{b}\right) dx = \left[\begin{array}{l} y = \frac{\lambda_k x}{b} \\ x = \frac{y b}{\lambda_k} \end{array}\right] =$$

$$= \frac{b^4}{\lambda_k^4} \int_0^{\lambda_k} \underbrace{y(\lambda_k^2 - y^2) J_0(y)\, dy}_{\underbrace{\lambda_k^2 y J_0(y)}_{(y J_1(y))'} - \underbrace{y^3 J_0(y) dy}_{\text{se övning 8}}} = \frac{b^4}{\lambda_k^4} \left(\lambda_k^2 \left[y J_1(y) \right]_0^{\lambda_k} - \left((\lambda_k^3 - 4\lambda_k) J_1(\lambda_k) + \right.\right.$$

$$\left.\left. + 2\lambda_k^2 \underbrace{J_0(\lambda_k)}_{=0} \right)\right) = \frac{4b^4}{\lambda_k^3} J_1(\lambda_k), \quad C_k = \frac{\frac{4b^4}{\lambda_k^3} J_1(\lambda_k)}{\frac{b^2}{2}\left(J_1(\lambda_k)\right)^2} = \frac{8b^2}{\lambda_k^3 J_1(\lambda_k)}$$

$$f(x) = 8b^2 \sum_{k=1}^{\infty} \frac{J_0\left(\lambda_k \frac{x}{b}\right)}{\lambda_k^3 J_1(\lambda_k)}$$

<u>5.4:5</u> Utveckla $f(x) = 1$ på $[0, b]$

$f(x) = \sum C_k J_0\, \lambda_k\left(\frac{x}{b}\right) = \sum C_k \Psi_k$ där $\{\lambda_k\}$ pos nollst t $c J_0(x) + x J_0'(x)$, $c > 0$

<u>Th 5.3(b)</u>: Om $c > -\nu$ så är $\{\Psi_k\}_1^\infty$ en ortogonalbas för $L^2_w(0,b)$, $w(x) = x$

och $\|\Psi_k\|^2_w = \frac{b^2 \lambda_k^2 - \nu_k^2 + c^2}{2\lambda_k^2} \left(J_0(\lambda_k)\right)^2$

$$c_k = \frac{1}{\|\Psi_k\|^2} \int_0^b \underbrace{f(x)\,\Psi_k(x)\,x}_{(*)}\,dx$$

$$\int (*)\,dx = \int_0^b 1\, J_0(\lambda_k \tfrac{x}{b})\,x\,dx = \frac{b^2}{\lambda_k^2} \int_0^{\lambda_k} \underbrace{y\,J_0(y)}_{(y J_1(y))'}\,dy = \frac{b^2}{\lambda_k^2}\Big[y\,J_1(y)\Big]_0^{\lambda_k} = \frac{b^2}{\lambda_k} J_1(\lambda_k)$$

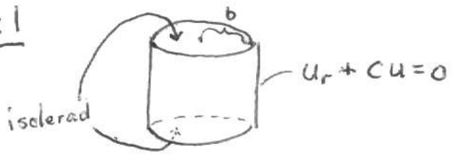
$$c_k = \frac{2\lambda_k^2}{b^2(\lambda_k^2 + c^2)(J_0(\lambda_k))^2} \cdot \frac{b^2}{\lambda_k^2} J_1(\lambda_k) = \frac{2\lambda_k J_1(\lambda_k)}{(\lambda_k^2 + c^2)\,J_0(\lambda_k)^2}$$

$$f(x) = 2\sum_{k=1}^{\infty} \frac{\lambda_k J_1(\lambda_k)}{(\lambda_k^2 + c^2)} \, \frac{J_0(\lambda_k \frac{x}{b})}{(J_0(\lambda_k))^2}$$

för $c = 0$? $\therefore J_1(\lambda_k) = -J_0'(\lambda_k) = 0 \quad k = 1, 2, 3, \dots$

Sats 5.3(b): Om $c = -\nu$ måste man lägga till x^ν för att få en ortogonalbas:
Här lägger man till $\Psi_0(x) \equiv 1$, och Fourier-Bessel utv av $f(x) \equiv 1$ blir $f(x) = 1$

5.5:1



$u_t = \Delta u = u_{rr} + r^{-1}u_r + r^{-2}\cancel{u_{\theta\theta}} + \cancel{u_{zz}}$
$u = A$ vid $t = 0$

Lösningen kommer av symmetriskäl att vara θ-oberoende
Vidare är lösningen z-oberoende

Variabelsep: $u = R(r)T(t)$

$$RT' = R''T + \frac{1}{r}R'T$$

$$\frac{T'}{T} = \frac{R'' + \frac{1}{r}R'}{R} = -\mu^2$$

R-ekv $\quad r^2R''(r) + rR'(r) + \mu^2 r^2 R(r) = 0$

Besselekv med $\nu = 0$ $\therefore R(r) = a\,J_0(\mu r) + B\,Y_0(\mu r)$

$R(0^+)$ ändlig $\Rightarrow B = 0$

$R'(b) + c\,R(b) = 0$

$\mu J_0'(\mu b) + c\,J_0(\mu b) = 0$

Sätt $\lambda = \mu b \Rightarrow \lambda J_0'(\lambda) + bc\,J_0(\lambda) = 0 \qquad (*)$

Låt $\{\lambda_k\}_1^\infty$ vara de positiva lösningarna till $(*)$

Vi får då egenvärden $\mu^2 = \left(\frac{\lambda_k}{b}\right)^2$ och egenfkner $\left\{J_0\left(\lambda_k \frac{r}{b}\right)\right\}_1^\infty \quad (\circledast)$

Th 5.3 $\Rightarrow (\circledast)$ ortogonalbas

T-led: $T'(t) + (\lambda_k/b)^2 T(t) = 0$

$$T(t) = P e^{-(\lambda_k/b)^2 t}$$

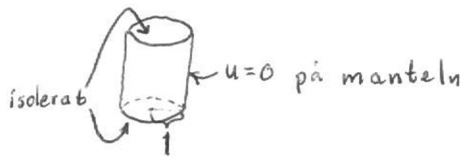
Allmän lösning: $u(r,t) = \sum_1^\infty c_k J_0(\lambda_k r/b)\, e^{-(\lambda_k/b)^2 t}$

(B V) $\sum_1^\infty c_k J_0(\lambda_k \frac{r}{b}) = A$

Förra uppg: $c_k = A \dfrac{2\lambda_k J_1(\lambda_k)}{(\lambda_k^2 + (bc)^2)(J_0(\lambda_k))^2}$

$$u(r,t) = 2A \sum_1^\infty \frac{\lambda_k J_1(\lambda_k)}{(\lambda_k^2 + (bc)^2)(J_0(\lambda_k))^2} \cdot J_0\left(\frac{\lambda_k r}{b}\right) e^{-\lambda_k^2 t/b^2}$$

5.5:4



$u_t = \Delta u + a$

$u(1,t) = 0$

Lösningen oberoende av θ & z $\quad u = r(r,t)$

a) Finn stationär lösning av temperaturfördelningen

$\Delta v + a = 0 \Leftrightarrow \frac{1}{r}(r v_r)_r + a = 0$

$$r v_r = -\frac{ar^2}{2} + A$$

$$v = -\frac{ar^2}{4} + A \log r + B$$

$v(0^+)$ ändlig $\Rightarrow A = 0$

$v(1) = 0 \Rightarrow B = a/4$

$\therefore\ v(r) = \frac{a}{4}(1 - r^2)$

b) Sätt $u = v + w$, $w = u - v$ (B V): $u(r,0) = 0$

w uppfyller: $w_t = \Delta w$

$w(1,t) = -v(1) = 0$

$w(r,0) = -v(r) = \frac{a}{4}(r^2 - 1)$

Separera $\quad W = RT \Rightarrow r^2 R'' + rR' + \mu^2 r^2 R = 0$

$R(1) = 0$, $R(0^+)$ ändlig

$T'(t) + \mu^2 T(t) = 0$

R-led: Egenfkner $\{J_0(\lambda_k r)\}_1^\infty$ där $\{\lambda_k\}_1^\infty$ pos nollst till J_0

Egenvärden $\mu^2 = \lambda_k^2$

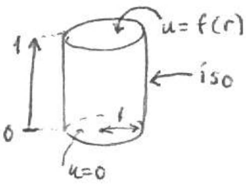
$w(r,t) = \sum c_k J_0(\lambda_k r) e^{-\lambda_k^2 t}$

(BV): $\sum c_k J_0(\lambda_k r) = \frac{a}{4}(r^2 - 1) = -\frac{a}{4}(1 - r^2)$

$$w(r,t) = -2a \sum \frac{J_0(\lambda_k r)}{\lambda_k J_1(\lambda_k)} e^{-\lambda_k^2 t}$$

$u(r,t) = \frac{a}{4}(1 - r^2) + $ — " —

5.5:6



inget φ-beroende

$$\begin{cases} \Delta u = 0 \\ u_r(1,z)=0 \\ u(r,0)=0 \\ u(r,1)=f(r) \end{cases} \qquad sep \Rightarrow u = RZ$$

$$(R''+r^{-1}R')Z + RZ'' = 0$$

$$-\frac{R''+r^{-1}R'}{R} = \frac{Z''}{Z} = \mu^2$$

R-led: $r^2R''(r) + rR'(r) + \mu^2 r^2 R(r) = 0$

$$\left.\begin{array}{l} R(0^+) \text{ ändligt} \\ R'(1)=0 \end{array}\right\} \text{ har egenfkner } \{J_0(\lambda_k r)\}_1^\infty$$

där $\{\lambda_k\}$ är pos lösn till $J_0'(\lambda_k)=0$

Sats 5.3 $\Rightarrow$ ortogonalbas om man lägger till $\Psi_0(x) \equiv 1 = J_0(0\cdot r)$

$J_0'(0)=0$ Denna kommer med om vi tar alla lösn till $J_0'(\lambda_k)=0$ istället ovan

Övning 10

5.5:7
$$\begin{cases} u_{tt} = c^2\Delta u = c^2(u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz}) \\ u(r,\theta,0,t) = u(r,\theta,1,t) = 0 \\ u_r(1,\theta,z,t) = 0 \\ u_t(r,\theta,z,0) = 0 \quad \text{(dvs startar i vila)} \end{cases}$$

Variabelseparation: $u = f(r,\theta,z)T(t)$

$$\frac{T''(t)}{c^2T(t)} = \frac{\Delta f}{f} = -\alpha$$

$\Delta f + \alpha f = 0$ (Helmholz ekv)

separera vidare: $f = R(r)\Theta(\theta)Z(z)$

$$\frac{R''(r)+r^{-1}R'(r)}{R(r)} + \frac{r^{-2}\Theta''(\theta)}{\Theta(\theta)} + \underbrace{\frac{Z''(z)}{Z(z)}}_{=-m^2\pi^2} = -\alpha$$

$$-\frac{Z''(z)}{Z(z)} = \text{resten} = \beta$$

$$\begin{cases} Z''(z) + \beta Z(z) = 0 \\ Z(0) = Z(1) = 0 \end{cases}$$

Lösningarna blir $\{\sin m\pi z\}_{m=1}^{\infty}$ egenvärden $\beta = m^2\pi^2$

$$\frac{r^2R''(r) + rR'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} + (\alpha - m^2\pi^2)r^2 = 0$$

$$-\frac{\Theta''(\theta)}{\Theta(\theta)} = \text{resten} = \gamma$$

$$\begin{cases} \Theta''(\theta) + \gamma\Theta(\theta) = 0 \\ \Theta \ 2\pi\text{-periodisk} \end{cases}$$

Lösn: $\{\sin n\theta\}_1^{\infty} \cup \{\cos n\theta\}_0^{\infty}$ Egenvärden $\gamma = n^2$

$$\begin{cases} r^2R''(r) + rR'(r) + \Big(\underbrace{(\alpha - m^2\pi^2)}_{\mu^2}r^2 - n^2\Big)R(r) = 0 \quad \text{Bessels ekv} \\ R'(1) = 0 \\ R(0^+) \text{ ändlig} \end{cases}$$

$J_n(\mu r)$

egenvärden $\mu = \lambda_{n,k}$

Lösningar: $J_n(\lambda_{n,k} r)$, där $\lambda_{n,k}$ är de pos nollst till J_n'

Sats 5.3 $\Rightarrow$ I fallet $n=0$ måste man dessutom lägga till fknen 1

Men $1 = J_0(0) = J_0(0r)$ så om vi plockar med $\lambda_{n,0} = 0$ får vi med det ovan

$\alpha = \lambda_{n,k}^2 + m^2\pi^2 = \mu_{m,n,k}^2$

Återstår T: $\dfrac{T''}{c^2T} = -\alpha = -\mu_{m,n,k}^2$

$$T'' + (c\mu_{m,n,k})^2 T = 0$$

$$T'(0) = 0$$

$$T(t) = \cos(\mu_{m,n,k}ct)$$

Slutligen blir alltså lösn: $u(r,\theta,z,t) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(a_{mnk}\cos n\theta + b_{mnk}\sin n\theta)\cdot$
$\cdot J_n(\lambda_{n,k} r)\sin m\pi z\cos(\mu_{m,n,k}ct)$ där $\mu_{m,n,k}^2 = m^2\pi^2 + \lambda_{n,k}^2$ och $J_n'(\lambda_{n,k}) = 0$

5.5:8 $\begin{cases} (xf'(x))' - \nu^2x^{-1}f(x) + \lambda^2xf(x) = 0 \quad \text{Bessels ekv} \\ f(a) = f(b) = 0, \quad 0 < a < b \end{cases}$

Visa att egenvärdena är talen λ^2 så att $J_\nu(\lambda a)Y_\nu(\lambda b) = J_\nu(\lambda b)Y_\nu(\lambda a)$

Vilka är egenfknerna?

$x^2 f'' + x f' + (\lambda^2 x^2 - \nu^2) f = 0$ (2:a ordn)

Allmän lösn: $f(x) = A J_\nu(\lambda x) + B Y_\nu(\lambda x)$

Randvillkor: $A J_\nu(\lambda a) + B Y_\nu(\lambda a) = 0$

$A J_\nu(\lambda b) = B Y_\nu(\lambda b) = 0$

Homogent linjärt ekv syst i A, B

Icke-triviál lösn finns omm $\begin{vmatrix} J_\nu(\lambda a) & Y_\nu(\lambda a) \\ J_\nu(\lambda b) & Y_\nu(\lambda b) \end{vmatrix} = 0 \Leftrightarrow J_\nu(\lambda a) Y_\nu(\lambda b) - J_\nu(\lambda b) Y_\nu(\lambda a) =$

$= 0$ VSV

Egenfknerna?

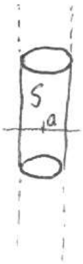
$A J_\nu(\lambda a) + B Y_\nu(\lambda a) = 0$

$A = Y_\nu(\lambda a) \cdot k$

$B = -J_\nu(\lambda a) \cdot k$

dvs $f(x) = Y_\nu(\lambda a) J_\nu(\lambda x) - J_\nu(\lambda a) Y_\nu(\lambda x)$

7.3:5



$\Delta u = 0$ i S

$u(r, \varphi, z) = \begin{cases} 1, & z < \ell \\ 0 & \text{annars} \end{cases} = \chi_{[-\ell, \ell]}(z)$

Lösningen φ-oberoende, $u = u(r, z)$ $\begin{cases} u_{rr} + r^{-1} u_r + u_{zz} = 0 \\ u(a, z) = \chi_{[-\ell,\ell]}(z) \end{cases}$

Fouriertransformera i z-led

$\begin{cases} \hat{u}_{rr} + r^{-1} \hat{u}_r + (i\xi)^2 \hat{u} = 0 \\ \hat{u}(a, \xi) = \hat{\chi}_{[-\ell,\ell]} = 2\xi^{-1} \sin \ell\xi \end{cases}$

$r^2 \hat{u}_{rr} + r \hat{u}_r - \xi^2 r^2 \hat{u} = 0$, $\hat{u}(0^+)$ ändlig

Bessels ekv: $r^2 f_{rr} + r f_r + (\mu^2 r^2 - \nu^2) f = 0$

Modifierade Bessels ekv (fås genom att byta x mot ix):

$r^2 f_{rr} + r f_r - (\mu^2 r^2 + \nu^2) f = 0$

Lösningarna till denna kallas modifierade Besselfkner $I_\nu(x) = i^{-\nu} J_\nu(ix)$

I vårt fall fås $\hat{u} = c_\xi I_0(\xi r)$ $\left(= c_\xi J_0(i\xi r) \right)$

(RV): $c_\xi I_0(\xi a) = 2\xi^{-1} \sin \ell\xi \Rightarrow \hat{u} = 2\xi^{-1} \sin \ell\xi \dfrac{I_0(\xi r)}{I_0(\xi a)}$

Inverstransformera (i ξ-led)

$u = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(r, \xi) e^{i\xi z} d\xi = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \ell\xi}{\xi} \frac{I_0(r\xi)}{I_0(a\xi)} e^{i\xi z} d\xi$

Legendrepolynom P_n

$\{P_n(x)\}_{n=0}^{\infty}$ ortogonalbas för $L^2(-1,1)$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$$

$$\|P_n\|^2_{L^2(-1,1)} = \frac{2}{2n+1}$$

Sats 6.5: $\sum_0^{\infty} P_n(x) z^n = (1-2xz+z^2)^{-1/2} = F(x,z)$

$-1 \leq x \leq 1$, $|z|<1$ ← Genererande fkn

6.2:5 Visa: $(1-2xz+z^2)\frac{\partial F}{\partial z} = (x-z)F$

Härled $(n+1)P_{n+1}(x) - (2n+1)\,x\,P_n(x) + n\,P_{n-1}(x) = 0$ $(*)$

$$\frac{\partial F}{\partial z} = -\frac{1}{2}(1-2xz+z^2)^{-3/2}\cdot(-2x+2z) = \frac{x-z}{(1-2xz+z^2)}F$$

$$F(x,z) = \sum_0^{\infty} P_n(x) z^n$$

$$\frac{\partial F}{\partial z}(x,z) = \sum_0^{\infty} n P_n(x) z^{n-1}$$

Sätt in detta i formeln ovan $\Rightarrow (1-2xz+z^2) = \sum_0^{\infty} n P_n(x) z^{n-1} =$

$= (x-z)\sum_0^{\infty} P_n(x) z^n$

$$\sum_{-1}^{\infty}(n+1)P_{n+1}(x)z^n - 2x\sum_0^{\infty} n P_n(x) z^n + \sum_1^{\infty}(n-1)P_{n-1}(x)z^n =$$

$$= \sum_0^{\infty} x P_n(x) z^n - \sum_1^{\infty} P_{n-1}(x) z^n$$

Identifiera koeff: $(n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x) =$

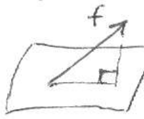
$= xP_n(x) - P_{n-1}(x) \Leftrightarrow (*)$

EÖ 12 Minimera $\int_0^1 (P(x))^2\,dx$ då $P(x) = x^3 + ax^2 + bx + c = x^3 - Q(x)$

där $Q(x)$ andragradspolynom

Min $\|x^3 - Q(x)\|_{L^2(0,1)}$ då $Q(x) \in P_2 =$ [andragradspolynom]

Bästa Q är ortogonala projektionen av x^3 på P_2



Om P_0, P_1, P_2 är ortogonalbas blir då $Q(x) = \sum_{k=0}^{2} \frac{\langle x^3, P_k(x)\rangle}{\|P_k\|^2} P_k(x)$

$\{P_n\}$ ortogonala på $(-1,1) \Rightarrow \{\tilde{P}_n(x)\} = \{P_n(2x-1)\}$ ortogonala på $L^2(0,1)$

$$\int_0^1 \tilde{P}_m(x)\tilde{P}_n(x)\,dx = \int_0^1 P_n(2x-1)\,P_n(2x-1)\,dx = \frac{1}{2}\int_{-1}^{1} P_m(y)P_n(y)\,dy$$

$$\|\tilde{P}_n\|^2_{L^2(0,1)} = \frac{1}{2}\|P_n\|_{L^2(-1,1)} = \frac{1}{2n+1}$$

$$\left.\begin{array}{l} P_0(x) = 1 \\ P_1(x) = x \\ P_2(x) = \frac{1}{2}(3x^2-1) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \tilde{P}_0(x) = 1 \\ \tilde{P}_1(x) = 2x-1 \\ \tilde{P}_2(x) = \frac{1}{2}\left(3(2x-1)^2-1\right) = \dots = 6x^2-6x+1 \end{array}\right.$$

$$\langle x^3, \tilde{P}_0 \rangle = \int_0^1 x^3 \cdot 1 \, dx = 1/4$$

$$\langle x^3, \tilde{P}_1 \rangle = \int_0^1 x^3 (2x-1) dx = 2/5 - 1/4 = 3/20$$

$$\langle x^3, \tilde{P}_2 \rangle = \dots = 1/20$$

$$Q(x) = \frac{1/4}{1} \cdot 1 + \frac{3/20}{1/3}(2x-1) + \frac{1/2}{1/5}(6x^2-6x+1) = \dots = \frac{3}{2}x^2 - \frac{3}{5}x + \frac{1}{20}$$

$$P(x) = x^3 - Q(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{-5}x - \frac{1}{20}$$

Sista övningen

Hermitepolynom

$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$

$\{H_n\}_0^\infty$ ortogonalbas för $L^2_w(\mathbb{R})$, $w(x) = e^{-x^2}$

$\left(\langle f, g\rangle_w = \int_{-\infty}^{\infty} f(x)\, g(x)\, e^{-x^2} dx\right)$

$(*)$: $e^{-x^2} H_n(x) = -\frac{d}{dx}\left(e^{-x^2} H_{n-1}(x)\right)$

6.4:6 $f(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$ utveckla f i serie av Hermitepolynom

$f = \sum_0^\infty \frac{\langle f, H_n\rangle_w H_n}{\|H_n\|_w^2}$ $\qquad \|H_n\|_w^2 = 2^n n! \sqrt{\pi}$

$\langle f, H_n\rangle_w = \int_{-\infty}^{\infty} f(x) H_n(x) w(x)\, dx = \int_0^\infty H_n(x) e^{-x^2} dx \overset{\text{enl}(*)}{=} \left[-H_{n-1}(x) e^{-x^2}\right]_0^\infty = H_{n-1}(0)$

för $n \geq 1$, ty H_n; $n<0$ inte definierat

$\langle f, H_0\rangle_w = \int_0^\infty 1\, e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$

$H_n(0) = (-1)^n \cdot 1 \cdot \frac{d^n}{dx^n} e^{-x^2}\Big|_{x=0} = \begin{cases} 0 & \text{om } n \text{ udda} \\ H_{2m}(0) = (-1)^{2m} (-1)^m \frac{(2m)!}{m!} = (-1)^m \frac{(2m)!}{m!} & \text{om } n \text{ jämnt} \end{cases}$

Detta ger utvecklingen:

$f = \frac{\frac{1}{2}\sqrt{\pi}}{\sqrt{\pi}} H_0 + \sum_{m=0}^\infty \frac{(-1)^m \frac{2m!}{m!} H_{2m+1}}{2^{2m+1}(2m+1)\sqrt{\pi}} = \frac{1}{2} H_0 + \frac{1}{2\sqrt{\pi}} \sum_{m=0}^\infty \frac{(-1)^m}{2^{2m}(2m+1)\, m!} H_{2m+1}$

Laguerrepolynom $L_n^\alpha(x) = \frac{x^{-\alpha} e^x}{n!} \left(\frac{d}{dx}\right)^n \left(x^{\alpha+n} e^{-x}\right)$ $\quad$ ($\alpha > -1$ parameter)

$\{L_n^\alpha\}_{n=0}^\infty$ ortogonalbas för $L^2_w(0,\infty)$, $w(x) = x^\alpha e^{-x}$

$\|L_n^\alpha\|_w^2 = \frac{\Gamma(n+\alpha+1)}{n!}$ $\quad$ där $\begin{cases} \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, & \operatorname{Re} z > 0 \\ \Gamma(n+1) = n! \quad \Gamma(z+1) = z\Gamma(z) \end{cases}$ (se Beta)

6.3:5 Utveckla $f(x) = x^\nu$ $(\nu \geq 0)$ i serie av Laguerrepolynom

$f = \sum_0^\infty \frac{\langle f, L_n^\alpha\rangle_w L_n^\alpha}{\|L_n^\alpha\|_w^2}$

$\langle f, L_n^\alpha\rangle = \int_0^\infty x^\nu \frac{x^{-\alpha} e^x}{n!} \left(\frac{d}{dx}\right)^n \left(x^{\alpha+n} e^{-x}\right) x^\alpha e^{-x} dx = \frac{1}{n!} \int_0^\infty x^\nu \left(\frac{d}{dx}\right)^n \left(x^{\alpha+n} e^{-x}\right) dx = [\text{P.I.}] =$

$= \frac{1}{n!}\left(\left[\underbrace{x^\nu \left(\frac{d}{dx}\right)^{n-1} \left(x^{\alpha+n} e^{-x}\right)}_{x^{\alpha+1}(n-1:\text{egradspolynom})e^{-x}}\right]_0^\infty - \int_0^\infty \nu x^{\nu-1} \left(\frac{d}{dx}\right)^{n-1} \left(x^{\alpha+n} e^{-x}\right) dx \right) =$

$$= -\frac{1}{n!}\int_0^\infty \nu x^{\nu-1}\left(\frac{d}{dx}\right)^{n-1}\left(x^{\alpha+n}e^{-x}\right)dx = \left[\text{P.I. ytterligare } n-1 \text{ ggr}\right] = \frac{(-1)^n}{n!}\nu(\nu-1)\ldots(\nu-n+1)\cdot$$

$$\int_0^\infty \underbrace{x^{\nu-n}x^{\alpha+n}}_{x^{\nu+\alpha}}e^{-x}\,dx = \frac{(-1)^n}{n!}\nu(\nu-1)\ldots(\nu-n+1)\Gamma(\nu+\alpha+1)$$

$$f = \sum_{n=0}^\infty \frac{(-1)^n\nu(\nu-1)\ldots(\nu-n+1)\Gamma(\nu+\alpha+1)}{\Gamma(\alpha+n+1)}L_n^\alpha = \Gamma(\nu+1)\Gamma(\nu+\alpha+1)\sum_{n=0}^\infty \frac{(-1)^n L_n^\alpha}{\Gamma(\alpha+n+1)\Gamma(\nu-n+1)}\frac{\Gamma(\nu+1)}{\Gamma(\nu-n+1)}$$

$\maltese$

Eö10 Bestäm det polynom av grad ≤ 2 som minimerar $\int_0^\infty \left(\sqrt{x} - P(x)\right)^2 e^{-x}\,dx =$

$$= \|\sqrt{x} - P(x)\|^2_{L^2_w}, \quad w(x) = e^{-x}$$

$\min \|\sqrt{x} - P(x)\|^2_{L^2_w}$ $\quad P \in P_2$ fås då $P(x)$ är ortogonala projektionen av $\sqrt{x}$ på P_2

L_0^0, L_1^0, L_2^0 (eller L_1, L_2, L_3) är en ortogonalbas för $P_2 \Rightarrow P(x) = \sum_0^2 \frac{\langle x^{1/2}, L_n\rangle_w}{\|L_n\|_w^2}L_n$

Första termerna i utvecklingen i förra uppgiften med $\nu = \frac{1}{2}$, $\alpha = 0$, se $\maltese$

$$P(x) = \underbrace{\Gamma(3/2)}_{=1/2\,\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}}\sum_{n=0}^2 \frac{(-1)^n\nu(\nu-1)\ldots(\nu-n+1)}{\Gamma(n+1) = n!}L_n = \frac{\sqrt{\pi}}{2}\sum_{n=0}^2 \frac{(-1)^n\,1/2(1/2-1)\ldots(1/2-n+1)}{n!}L_n =$$

$$= \frac{\sqrt{\pi}}{2}\left(L_0 - \frac{1/2}{1}L_1 + \frac{1/2(-1/2)}{2}L_2\right) = \frac{\sqrt{\pi}}{2}\left(L_0 - \frac{1}{2}L_1 - \frac{1}{8}L_2\right)$$

$$L_n = \frac{e^x}{n!}\left(\frac{d}{dx}\right)^n\left(x^n e^{-x}\right) \implies \begin{aligned} L_0 &= 1 \\ L_1 &= 1 - x \\ L_2 &= 1 - 2x + x^2/2 \end{aligned}$$

Sätt in och förenkla

$$P(x) = \ldots = \frac{\sqrt{\pi}}{32}\left(6 + 2x - x^2\right)$$

Eö11 Minimera $\int_{-\infty}^\infty \left(x^4 - P(x)\right)^2 e^{-x^2/2}\,dx$ då P polynom av grad ≤ 2

Hermitepolynom ortogonalbas för $L^2_w(\mathbb{R})$, $w(x) = e^{-x^2}$

$$\int_{-\infty}^\infty \ldots = \left[x = \sqrt{2}\,y\right] = \sqrt{2}\int_{-\infty}^\infty \left(4y^4 - \underbrace{P(\sqrt{2}y)}_{Q(y)}\right)^2 e^{-y^2}dy = \sqrt{2}\,\|4y^4 - Q(y)\|^2_{L^2_w} \qquad Q \in P_2$$

minimeras av ortogonala projektionen av $4y^4$ på P_2 $\qquad H_0, H_1, H_2$ ortogonalbas för $P_2 \Rightarrow$

$$\Rightarrow Q = \sum_{n=0}^2 \frac{\langle 4y^4, H_n\rangle_w}{\|H_n\|^2_{L^2_w}}H_n$$

Metod I: Räkna ut skalärprodukterna

Metod II: (Bättre) uttryck $4y^4$ i Hermitepolynom

$$\begin{cases} H_0 = 1 \\ H_1 = 2x \\ H_2 = 4x^2 - 2 \\ H_3 = 8x^3 - 12x \\ H_4 = 16x^4 - 48x^2 + 12 = 4(4x^4 + 12x^2 + 3) \end{cases}$$

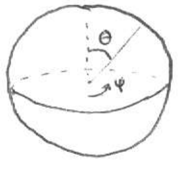
(se tabeller)

$4y^4 = \frac{1}{4} H_4(y) - 12y^2 - 3 = \frac{1}{4} H_4(y) + 3H_2(y) + 3H_0(y)$

$Q(y) = 3H_2(y) + 3H_0(y) = 12y^2 - 3$

$P(x) = Q\left(\frac{x}{\sqrt{2}}\right) = 6x^2 - 3$

<u>6.3:1</u> (Dirichlets problem i en sfär)



$\Delta u = 0, \quad r < 1$

$u(1, \theta, \varphi) = \begin{cases} \cos\theta, & 0 \le \theta \le \pi/2 \\ 0, & \pi/2 \le \theta \le \pi \end{cases}$

(θ och φ tvärtom i Folland)

Sats 6.8: lösningen ges av $u(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} c_{mn} r^n \underbrace{e^{im\varphi} P_n^{|m|}(\cos\theta)}_{\text{klotytfkner } Y_{nm}(\theta, \varphi)}$

P_n^m associerade Legendrefkner

$P_n^m(s) = (1-s^2)^{m/2} \left(\frac{d}{ds}\right)^m P_n(s)$

Här är lösn φ-oberoende

Så $u(r, \theta) = \sum_{n=0}^{\infty} C_n r^n P_n(\cos\theta)$

(RV): $\sum_0^{\infty} C_n P_n(\cos\theta) = \begin{cases} \cos\theta, & 0 \le \theta \le \pi/2 \\ 0, & \pi/2 \le \theta \le \pi \end{cases}$

Sätt $s = \cos\theta$

$\sum_0^{\infty} C_n P_n(s) = \begin{cases} s, & s \ge 0 \\ 0, & s \le 0 \end{cases} = g(s)$

Utveckla $g(s)$ i Legendrepolynom

$$C_n = \frac{\langle g, P_n \rangle}{\|P_n\|^2} = \frac{2n+1}{2} \int_{-1}^{1} g(s) P_n(s)\, ds = \frac{2n+1}{2} \int_0^1 s\, P_n(s)\, ds =$$

$$= \left[P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)\right] \left[P.\ I\right] = \frac{1}{2}\left[P'_{n+1}(x) - P'_{n-1}(x)\right]_0^1 -$$

$$- \frac{1}{2} \int_0^1 \left(P_{n+1}(x) - P_{n-1}(x)\right) dx = \ldots \text{ etc}$$

Behandla fallen $n=0$ & $n=1$ separat