Storgruppsövning 27/11-13

Source free wave equations

$$\begin{pmatrix} \nabla \times \mathbb{E} = -\mu \frac{\partial \mathbb{H}}{\partial t} & \nabla \cdot \mathbb{E} = 0 \\ \nabla \times \mathbb{H} = \varepsilon \frac{\partial \mathbb{E}}{\partial t} & \nabla \cdot \mathbb{H} = 0 \end{pmatrix} \text{Maxwell's equations} \Longrightarrow \begin{cases} \nabla^2 \mathbb{E} - \frac{1}{u^2} \frac{\partial^2 \mathbb{E}}{\partial t^2} = 0 \\ \nabla^2 \mathbb{H} - \frac{1}{u^2} \frac{\partial^2 \mathbb{H}}{\partial t^2} = 0 \end{cases} \text{homogenous wave equations}$$

u speed of wave

Phasors:

time harmonic $\mathbb{E}$-field $\mathbb{E}(x,y,z,t) = \text{Re}\{\overline{\mathbb{E}} e^{i\omega t}\}$ refers to cos

$\overline{\mathbb{E}}$ vector phasor depends on space coordinates

$$\begin{pmatrix} \nabla \times \overline{\mathbb{E}} = -i\omega\mu \overline{\mathbb{H}} & \nabla \cdot \overline{\mathbb{E}} = 0 \\ \nabla \times \overline{\mathbb{H}} = \underbrace{i\omega}_{\partial/\partial t} \varepsilon \overline{\mathbb{E}} & \nabla \cdot \overline{\mathbb{H}} = 0 \end{pmatrix}$$

$$\begin{Bmatrix} \nabla^2 \overline{\mathbb{E}} + k^2 \overline{\mathbb{E}} = 0 \\ \nabla^2 \overline{\mathbb{H}} + k^2 \overline{\mathbb{H}} = 0 \end{Bmatrix} \text{homogenous vector Helmholtz equations}$$

$$k = \omega\sqrt{\mu\varepsilon} = \omega/u$$

k wave number

12.3

an electromagnetic wave in vacuum, ω angular freq.

$\mathbb{E} = \hat{y} E_0 e^{-\alpha z} e^{-i\beta x} = \hat{y}\overline{E}_y$

$E_y(t) = \text{Re}\{\overline{E}_y e^{i\omega t}\} = E_0 e^{-\alpha z} \cos(\omega t - \beta x)$

a) Find $\mathbb{H}$ and $\mathbb{E}$ in real format

$$\overline{\mathbb{H}} = \frac{-1}{i\omega\mu_0} \nabla \times \overline{\mathbb{E}} = \frac{-1}{i\omega\mu_0} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & \overline{E}_y & 0 \end{vmatrix} = \frac{-1}{i\omega\mu_0} \left(-\hat{x} \frac{\partial \overline{E}_y}{\partial z} + \hat{z} \frac{\partial \overline{E}_y}{\partial x} \right)$$

$$\overline{\mathbb{H}} = \frac{-1}{i\omega\mu_0} \left(-\hat{x}(-\alpha E_0 e^{-\alpha z} e^{-i\beta x}) + \hat{z}(-i\beta E_0 e^{-\alpha z} e^{-i\beta x}) \right) =$$

$$= \frac{i\overline{E}_y}{\omega\mu_0} \left[\hat{x}\alpha - \hat{z} i\beta \right] \Longrightarrow H_x(t) = \text{Real}\{\overline{H}_x e^{i\omega t}\} = \frac{E_0 \alpha}{\omega\mu_0} e^{-\alpha z} \cos(\omega t - \beta x + \frac{\pi}{2})$$

$$H_z(t) = \text{Real}\{\overline{H}_z e^{i\omega t}\} = \frac{E_0 \beta}{\omega\mu_0} e^{-\alpha z} \cos(\omega t - \beta x)$$

b) What is the relation between α, β and ω to satisfy wave equation? In vacuum we have: Maxwell's eq.

forts. →

$$\nabla\times(\nabla\times\bar{E}) = -i\omega\mu_0(\nabla\times\bar{H}) = -i\omega\mu_0(i\omega\varepsilon_0\bar{E}) = \omega^2\underbrace{\mu_0\varepsilon_0}_{1/c^2}\bar{E}$$

$$\nabla(\underbrace{\nabla\cdot\bar{E}}_{=0}) - \nabla^2\bar{E} = \omega^2\mu_0\varepsilon_0\bar{E}$$

$$\Rightarrow \nabla^2\bar{E} + \frac{\omega^2}{c^2}\bar{E} = 0 \quad , \quad c = 1/\sqrt{\mu_0\varepsilon_0}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)E + \frac{\omega^2}{c^2}E = 0 \quad , \quad E = E_0 e^{-\alpha z} e^{-i\beta x}$$

$$\Longrightarrow (-i\beta)^2 + 0 + (-\alpha)^2 + \frac{\omega^2}{c^2} = 0 \Rightarrow \beta^2 - \alpha^2 = \frac{\omega^2}{c^2}$$

Another way to solve it:

$$E = \hat{y}E_0 e^{-\alpha z} e^{-i\beta x} = \hat{y}E_0 e^{-i\mathbb{k}\mathbb{R}}$$

$\mathbb{R} = \hat{x}x + \hat{y}y + \hat{z}z$ $\quad$ $\mathbb{R}$, radius vector from origin

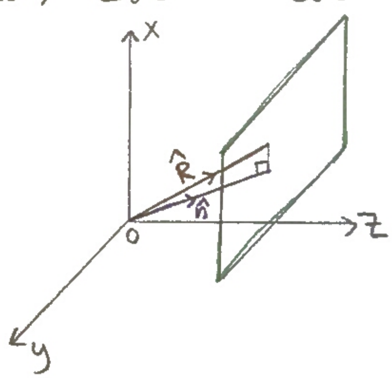
$$-i(\mathbb{k}\cdot\mathbb{R}) = -\alpha z - i\beta x$$

$$\Rightarrow \mathbb{k} = \hat{x}\beta - \hat{z}(i\alpha) = k_x\hat{x} + k_z\hat{z}$$

$$k^2 = k_x^2 + k_z^2 = \beta^2 - \alpha^2 = \frac{\omega^2}{c^2}$$

Uniform plane wave in lossless media

$$\bar{E}(\mathbb{R}) = \bar{E}_0 e^{-i\mathbb{k}\cdot\mathbb{R}} = E_0 e^{-ik\hat{n}\mathbb{R}}$$



$\mathbb{R}$: radius vector from origin

$\hat{n}$: direction of propagation

$k = \omega\sqrt{\mu\varepsilon}$, wave number

$\mathbb{k} = k\hat{n} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$

$|\mathbb{k}|^2 = k_x^2 + k_y^2 + k_z^2 = \omega^2\mu\varepsilon$

$$\bar{H}(R) = \frac{-1}{i\omega\mu}\nabla\times\bar{E}(R) = \frac{1}{\eta}\hat{n}\times\bar{E}(R) \quad , \quad \eta = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\varepsilon}}$$

wave impedance

$$\bar{E}(R) = \frac{1}{i\omega\varepsilon}\nabla\times\bar{H}(R) = -\eta\,\hat{n}\times\bar{H}(R) \quad , \quad \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \simeq 120\pi$$

11.8

A plane sinusoidal wave in vacuum.

$$\begin{cases} H_x = A\cos\omega\left(t - \frac{1}{c}(y\sin\alpha + z\cos\alpha)\right) = \mathrm{Re}\{\bar{H}e^{i\omega t}\} \\ H_y = H_z = 0 \end{cases}$$

Find $\mathbb{E}$!

$$\bar{\mathbb{H}} = \hat{x}Ae^{-i\frac{\omega}{c}(y\sin\alpha + z\cos\alpha)} = \hat{x}Ae^{-ik\hat{n}\mathbb{R}} = \hat{x}H_x$$

$$\mathbb{R} = \hat{x}x + \hat{y}y + \hat{z}z$$

$$\Rightarrow \begin{cases} k = \frac{\omega}{c} \\ \hat{n} = \hat{y}\sin\alpha + \hat{z}\cos\alpha \end{cases}$$ unit vector in direction of propagation

$$\bar{\mathbb{E}}(R) = -\eta\hat{n}\times\bar{\mathbb{H}}(R) = \left\{\eta = \frac{\omega\mu_0}{k} = \frac{k}{\omega\varepsilon_0}\right\} = \frac{-k}{\omega\varepsilon_0}(\hat{y}\sin\alpha + \hat{z}\cos\alpha)\times\hat{x}Ae^{-ik\hat{n}\cdot R} =$$

$$= \frac{-k}{\omega\varepsilon_0}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \sin\alpha & \cos\alpha \\ \bar{H}_x & 0 & 0 \end{vmatrix} = \frac{-k}{\omega\varepsilon_0}\left(\hat{y}\cos\alpha - \hat{z}\sin\alpha\right)\bar{H}_x =$$

$$= -\eta_0(\hat{y}\cos\alpha - \hat{z}\sin\alpha)Ae^{-i\frac{\omega}{c}(y\sin\alpha + z\cos\alpha)}$$

$$\begin{cases} E_y(t) = -\eta_0\cos\alpha A\cos\left(\omega t - \frac{\omega}{c}(y\sin\alpha + z\cos\alpha)\right) = \mathrm{Re}\{\bar{E}e^{i\omega t}\} \\ E_z(t) = \eta_0\sin\alpha A\cos\left(\omega t - \frac{\omega}{c}(y\sin\alpha + z\cos\alpha)\right) \end{cases}$$

Another way to solve the problem:

$$\mathbb{E} = \frac{1}{i\omega\varepsilon_0}\nabla\times\mathbb{H} = \frac{1}{i\omega\varepsilon_0}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \bar{H}_x & 0 & 0 \end{vmatrix} \quad \ldots$$

Plane waves in lossy media

In a source free: $\nabla^2\mathbb{E} + k_c^2 + \mathbb{E} = 0$

complex number

$k_c = \omega\sqrt{\mu\varepsilon_c}$ complex wave number

$$\gamma = ik_c = i\omega\sqrt{\mu\varepsilon_c} = \alpha + i\beta$$

propagation constant | attenuation constant | phase constant

$\nabla^2 \mathbb{E} - \gamma^2 \mathbb{E} = 0$ wave eq. in lossy media

Solution $\Longrightarrow$ $\bar{\mathbb{E}} = \hat{x}\bar{E}_x = \hat{x}E_0 e^{-\gamma z} = \hat{x}E_0 e^{-\alpha z} e^{-i\beta z}$

Good conductors: $\sigma/\omega\varepsilon \gg 1$, $\gamma = \alpha + i\beta = (1+i)\sqrt{\frac{\omega\mu\sigma}{2}}$

12.7

Calculate α, β, $Z^{=\eta}$ for a metal with permeability μ_r and conductivity σ, $\sigma \gg \omega\varepsilon_0\mu_r$

$$\nabla \times \bar{\mathbb{H}} = \bar{\mathbb{J}} + i\omega\bar{\mathbb{D}} = \sigma\bar{\mathbb{E}} + i\omega\varepsilon\bar{\mathbb{E}} = i\omega\underbrace{\left(\varepsilon + \frac{\sigma}{i\omega}\right)}_{\varepsilon_c = \varepsilon - i\frac{\sigma}{\omega}}\bar{\mathbb{E}} = i\omega\varepsilon_c\bar{\mathbb{E}}$$

$$\nabla^2\mathbb{E} - \gamma^2\mathbb{E} = 0 \quad , \gamma = ik_c = i\omega\sqrt{\mu\varepsilon_c} = i\omega\sqrt{\mu(\underbrace{\cancel{\varepsilon}}_{\text{neglect}} - i\frac{\sigma}{\omega})} = i\omega\sqrt{\mu(-i\frac{\sigma}{\omega})} =$$

$$= \sqrt{i\sigma\omega\mu} = \frac{1+i}{\sqrt{2}}\sqrt{\omega\mu\sigma} = \alpha + \beta i$$

$$\Longrightarrow \alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$$

$$Z_c = \eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\cancel{\varepsilon} - i\frac{\sigma}{\omega}}} = \sqrt{\frac{\mu}{-i\frac{\sigma}{\omega}}} = \sqrt{\frac{i\omega\mu}{\sigma}} \quad \left(\cdot\sqrt{\frac{\mu_0}{\varepsilon_0}}\cdot\sqrt{\frac{\varepsilon_0}{\mu_0}}\right)$$

$$\eta_c = \frac{1+i}{\sqrt{2}}\underbrace{\sqrt{\frac{\mu_0}{\varepsilon_0}}}_{\eta_0}\sqrt{\frac{\omega\mu\varepsilon_0}{\mu_0\sigma}} = (1+i)\eta_0\sqrt{\frac{\omega\mu_r\varepsilon_0}{2\sigma}} \quad \measuredangle Z_c = 45^\circ$$

– the magnetic field lags behind the E-field by 45°.

$$|\eta_c| = \frac{|\mathbb{E}|}{|\mathbb{H}|}$$