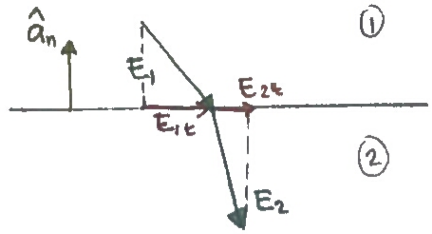
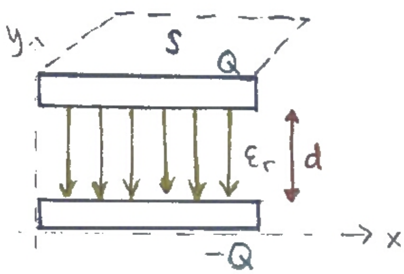
Storgruppsövning 8/11-13

- Boundary conditions for electrostatic field
- $E_{1t} = E_{2t}$
- $D_{1n} - D_{2n} = \rho_s$

  $\hat{a}_{n2} \cdot (D_1 - D_2) = \rho_s$

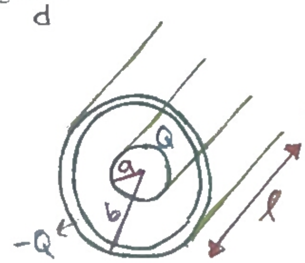


Capacitance:



$C = \dfrac{Q}{V_{12}}$ — charge on each conductor / — voltage difference between conductors.

$C = \varepsilon \dfrac{S}{d}$

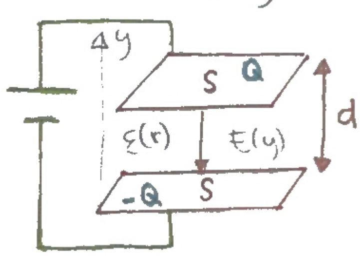


$C = \dfrac{Q}{V_{12}} = \dfrac{2\pi\varepsilon \ell}{\ln\left(\frac{b}{a}\right)}$

1. Choose a right coordinate system
2. Assume Q, -Q on conductors
3. Find E-field
4. $V_{12} = -\int E \, dl$
5. $C = \dfrac{Q}{V_{12}}$

P 3.30

The space between a parallel-plate capacitor of area S is filled with $\varepsilon_r(y)$.



$$\begin{cases} \varepsilon(y) = \varepsilon_1 + \dfrac{\varepsilon_2 - \varepsilon_1}{d} y \\ c = ? \end{cases}$$

- Cartesian coordinate

$$\nabla \cdot \underset{\varepsilon E}{D} = \underset{\text{zero}}{\rho} \Rightarrow \nabla(\varepsilon(y) E(y)) = \rho = 0$$

$$\varepsilon(y) E(y) = C_1 = \text{constant} \Rightarrow E(y) = \frac{C_1}{\varepsilon(y)}$$

$$V_0 = -\int_{y=0}^{y=d} E_y(y)\,dy = -\int_{y=0}^{y=d} \frac{C_1}{\varepsilon_1 + \frac{\varepsilon_2 - \varepsilon_1}{d} y}\,dy =$$

$$= -C_1 \left[\frac{d}{\varepsilon_2 - \varepsilon_1} \ln\left(\frac{\varepsilon_2 - \varepsilon_1}{d} y + \varepsilon_1 \right) \right]_{y=0}^{y=d} = -C_1 \frac{d}{\varepsilon_2 - \varepsilon_1} \ln \frac{\varepsilon_2}{\varepsilon_1}$$

$$\Longrightarrow C_1 = -\frac{V_0}{d} \frac{(\varepsilon_2 - \varepsilon_1)}{\ln\left(\frac{\varepsilon_2}{\varepsilon_1}\right)}$$

$$E(y) = -\frac{V_0}{d} \frac{\varepsilon_2 - \varepsilon_1}{\ln\left(\frac{\varepsilon_2}{\varepsilon_1}\right)} \cdot \frac{1}{\varepsilon(y)}$$

$$D_y = \varepsilon E_y(y) = -\frac{V_0}{d} \frac{\varepsilon_2 - \varepsilon_1}{\ln\left(\frac{\varepsilon_2}{\varepsilon}\right)}$$

normal boundary condition: $D_{1n} - \cancel{D_{2n}} = \rho_s \Rightarrow D_y = \rho_s$

$$\Longrightarrow Q = \rho_s \cdot S = D_y \cdot S$$

$$C = \frac{Q}{V_0} = \frac{D_y \cdot S}{V_0} = \frac{1}{d} \frac{\varepsilon_2 - \varepsilon_1}{\ln\left(\frac{\varepsilon_2}{\varepsilon_1}\right)} \cdot S$$

$$C = \frac{Q}{V_0} \Rightarrow \rho_s \Rightarrow D_y = D_{12} \Rightarrow E_y \Rightarrow V_0 = -\int E \cdot dl$$
$$\Leftarrow \quad \Leftarrow \quad \Leftarrow \quad \Leftarrow$$

Electrostatic energy and forces

$W_e = \frac{1}{2} \sum_{k=1}^{n} Q_k V_k \longrightarrow$ Potential energy of a group of N discrete charge

$W_e = \frac{1}{2} \int_{V'} \rho V \, dV' \longrightarrow$ continuous charge dist.

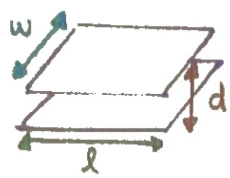
$W_e = \frac{1}{2} \int D \cdot E \, dV$ (in terms of E-field)

$W_e = \frac{1}{2} C V^2 = \frac{Q^2}{2C} = \frac{1}{2} Q V$ (J)

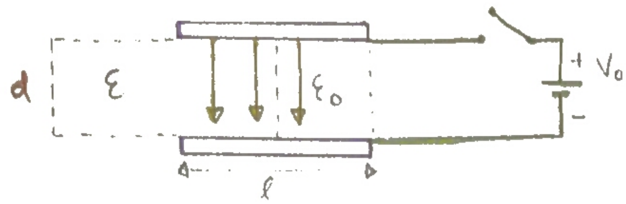
Electrostatic forces:

1. fixed charge system $F_Q = -\nabla W_e$
   - isolated
2. fixed potential system $F_V = +\nabla W_e$
   - connected to external sources

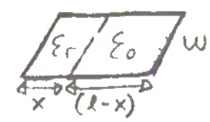
P 3.48



A parallell-plate capacitor, has a dielectric slab (ε) in the space between the plates. The dielectric slab is moved to a new position



a) when the switch closed.



$W_e = \frac{1}{2} Q_{in} V_0$, $\oint D \cdot dS = Q_{in}$, $D \rightarrow E$ *

$E_y = -\frac{V_0}{d} \Rightarrow D_y = \begin{cases} \text{air} & \varepsilon_0 E_y = (-\varepsilon_0 V_0)/d \\ \text{dielectric} & \varepsilon E_y = (-\varepsilon V_0)/d \end{cases}$
εE_y

$\oint D \cdot ds = Q_{in} \Rightarrow Q_{in} = \frac{\varepsilon_0 \varepsilon_r V_0}{d} x w + \frac{\varepsilon_0 V_0}{d} (l - x) w =$

$= \frac{w \varepsilon_0 V_0}{d} \left(\varepsilon_r x + (l - x) \right)$ total charge

$W_e = \frac{1}{2} Q_{in} V_0 = \frac{1}{2} \frac{\varepsilon_0 V_0^2 w}{d} \left(l + (\varepsilon_r - 1) x \right)$

$F_V = + \nabla W_e = \frac{\partial W_e}{\partial x} = \frac{\varepsilon_0 V_0^2 w}{2d} (\varepsilon_r - 1)$, $C = \frac{Q_{in}}{V_0} = \frac{\varepsilon_0 w}{d} (\varepsilon_r x + (l - x))$

b) switch is open. (fixed charge system)

$W_e = \frac{Q^2}{2C} = \frac{Q^2}{2} \frac{d}{w} \frac{1}{\varepsilon x + \varepsilon_0 (l - x)}$

$F_Q = -\nabla W_e = \frac{Q^2}{2} \frac{d}{w} \frac{\varepsilon - \varepsilon_0}{[\varepsilon x + \varepsilon_0 (l - x)]^2}$

$Q = CV \rightarrow F_Q = F_V$

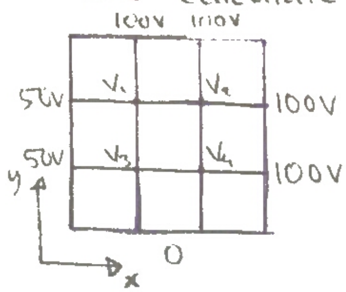
Poisson's and Laplace equation

$\nabla \cdot D = \rho$, $D = \varepsilon E$
$\nabla \times E = 0 \rightarrow E = -\nabla V$
$\Big\}$ $\nabla^2 V = -\frac{\rho}{\varepsilon}$ Poissons eq.

Cartesian coord. system: $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

when there is no free charge $\Rightarrow \nabla^2 V = 0$ Laplace eq.

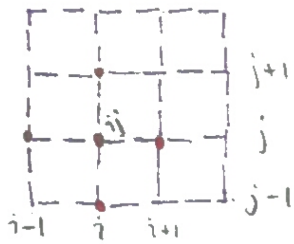
5.2 calculate the potential distribution numerically



$$\nabla^2 V = 0 \Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \cancel{\frac{\partial^2 V}{\partial z^2}} = 0$$

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \cancel{\frac{d^2 V}{dz^2}} = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{dV}{dx}\right) + \frac{d}{dy}\left(\frac{dV}{dy}\right) = 0$$



derivative can be approximated by differences between neighbouring points on a grid.

$$\frac{d}{dx} V = \lim_{h \to 0} \frac{(V_{i+1,j} - V_{i,j})}{h} \Rightarrow \frac{d}{dx}\left(\frac{dV}{dx}\right) = \lim_{h \to 0}\left(\frac{(V_{i+1,j} - V_{i,j}) - (V_{i,j} - V_{i-1,j})}{h^2}\right)$$

$$= \lim_{h \to 0}\left(\frac{V_{i+1,j} - 2_{i,j} + V_{i-1,j}}{h^2}\right)$$

$$\nabla^2 V = \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{h^2} + \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{h^2} = \frac{V_{i,j+1} + V_{i,j-1} + V_{i+1,j} + V_{i-1,j} - 4V_{i,j}}{h^2}$$

a) $V_{i,j+1} + V_{j,j-1} + V_{i+1,j} + V_{i-1,j} - 4V_{i,j} = 0$

$P_1: 100 + V_3 + V_2 + 50 - 4V_1 = 0$

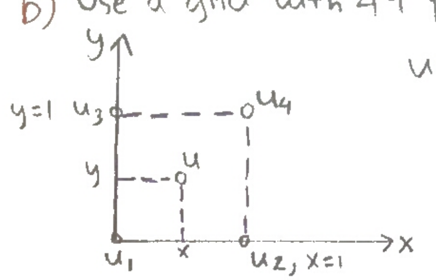
$P_2: 100 + V_4 + 100 + V_1 - 4V_2 = 0$

$P_3: V_1 + 0 + V_4 + 50 - 4V_3 = 0$

$P_4 = V_2 + 0 + 100 + V_3 - 4V_4 = 0$

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} -150 \\ -200 \\ -50 \\ -100 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 68{,}75 \\ 81{,}25 \\ 43{,}75 \\ 56{,}25 \end{bmatrix}$$

b) Use a grid with 49 points:

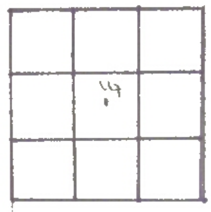


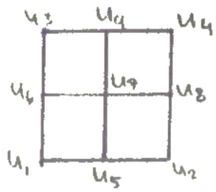
$u(x,y) = u_1(1-x)(1-y) + u_2(1-y)x + u_3(1-x)y + u_4 xy$

$0 < x < 1$, $0 < y < 1$

$x = 0{,}5$, $y = 0{,}5$ in the center

$$\Rightarrow u(0.5, 0.5) = \frac{u_1 + u_2 + u_3 + u_4}{4}$$





$$u_7 = \frac{u_1 + u_2 + u_3 + u_4}{4} \qquad u_6 = \frac{u_3 + u_1}{2} \qquad u_8 = \frac{u_4 + u_2}{2}$$

$(x=0, y=0.5)$ $\quad$ $(x=1, y=0.5)$

c) When we iterate more to find the exact potential
$\Rightarrow$ we use Gauss-Seidel method.

d) Analytic solution $V_0 = V_7 = 50 + \frac{200}{\pi} \sum_{n=1,3,\ldots}^{\infty} \frac{\sinh(n\pi/2)}{\sinh(n\pi)} \sin\left(\frac{n\pi}{2}\right) = u_7$

≈ 62