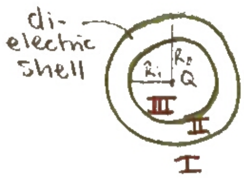
Storgruppsövning 5/11-13

example 3.12

a positive point charge, Q, is at the center of a spherical dielectric shell (inner radius R_i and outer radius R_o)



$E, V, D, P = ?$ as function of R.

$$\begin{cases} \text{Spherical symmetry} \Rightarrow \text{Gauss} \Rightarrow E \\ V = -\int E \cdot dl \ , \ D = \varepsilon E \ , \ P = D - \varepsilon_0 E \end{cases}$$

I) $R > R_o$, $E_1 = \frac{Q}{4\pi\varepsilon_0 R^2}$, $V_1 = \frac{Q}{4\pi\varepsilon_0 R}$, $D_1 = \varepsilon_0 E = \frac{Q}{4\pi R^2}$, $P_1 = 0$

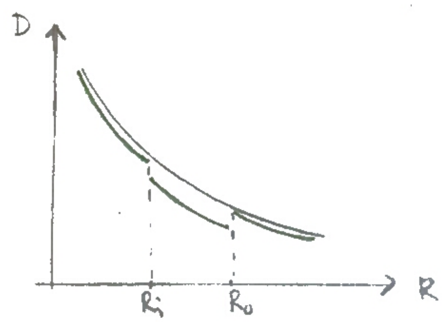
II) $R_i < R < R_o$, $E_{R2} = \frac{Q}{4\pi\varepsilon_0\varepsilon_r R^2}$, $D_2 = \varepsilon_0\varepsilon_r E = \frac{Q}{4\pi R^2}$, $P_2 = \frac{Q}{4\pi R^2}\left(1 - \frac{1}{\varepsilon_r}\right)$

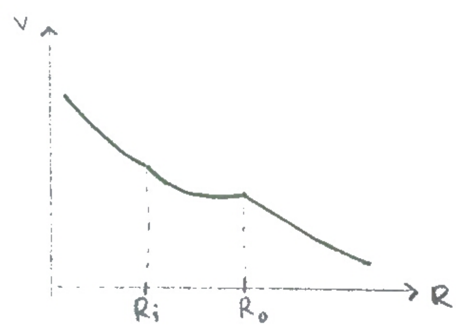
$$V_2 = -\int_{\infty}^{R} E \cdot dl = -\int_{\infty}^{R_o} E_1 \cdot dR - \int_{R_o}^{R} E_2 \, dR = V_1\Big|_{R=R_o} - \int_{R_o}^{R} \frac{Q}{4\pi\varepsilon R^2} \, dR =$$

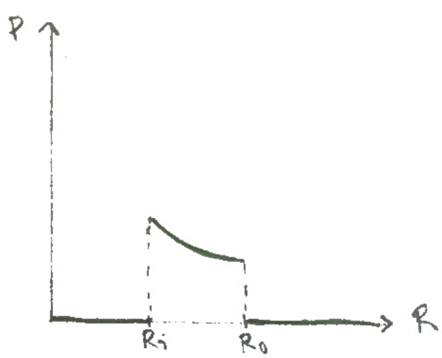
$$= \frac{Q}{4\pi\varepsilon_0}\left[\left(1 - \frac{1}{\varepsilon_r}\right)\frac{1}{R_o} + \frac{1}{\varepsilon_r R}\right]$$

III) $R < R_i$ $E_3 = \frac{Q}{4\pi\varepsilon_0 R^2}$, $D_3 = \varepsilon_0 E = \frac{Q}{4\pi R^2}$, $P_3 = 0$

$$V_3 = V_2\Big|_{R=R_i} - \int_{R_i}^{R} E_3 \, dR = \frac{Q}{4\pi\varepsilon_0}\left[\left(1 - \frac{1}{\varepsilon_r}\right)\frac{1}{R_o} - \left(1 - \frac{1}{\varepsilon_r}\right)\frac{1}{R_i} + \frac{1}{R}\right]$$



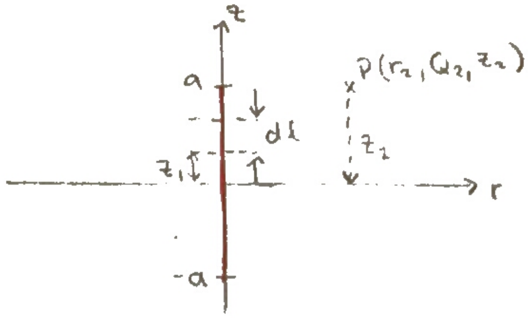




example 2.14

a homogenous line charge, ρ_ℓ, is located on z-axis, between $z=-a$ and $z=a$.

Find $E_r(r,z)$ and $E_z(r,z)$ of the electric field in point $P(r_2, \varphi_2, z_2)$



- not Gauss law!
- $dq = \rho_\ell \, d\ell = \rho_\ell \, dz_1$
- charge at $(\mathbb{R}_1 = z_1 \hat{z})$ gives rise E-field at field point $\mathbb{R}_2$.

$$\mathbb{R}_2 = r_2 \hat{r} + z_2 \hat{z} \qquad \mathbb{R}_{12} = \mathbb{R}_1 - \mathbb{R}_2 = r_2 \hat{r} + \hat{z}(z_2 - z_1)$$

$$dE = \frac{dq}{4\pi\varepsilon_0 R_{12}^2} \hat{R}_{12} \qquad R_{12} = \sqrt{r_2^2 + (z_2 - z_1)^2}$$

Summing the contribution of dE from all dq:

$$\mathbb{E}(\mathbb{R}_2) = \frac{1}{4\pi\varepsilon_0} \int_{z=-a}^{a} \frac{\hat{r} r_2 + \hat{z}(z_2 - z_1)}{[r_2^2 + (z_2 - z_1)^2]^{3/2}} \underbrace{\rho_\ell \, dz_1}_{dq}$$

First we find the $\hat{r}$ component:

$$\hat{r} \cdot \mathbb{E}(\mathbb{R}_2) = \frac{\rho_\ell r_2}{4\pi\varepsilon_0} \int_{-a}^{a} \frac{1}{[r_2^2 + (z_2 - z_1)^2]^{3/2}} dz_1 = \left\{ \begin{array}{l} \text{subs. } \xi = z_2 - z_1 \\ z_1 = -a \Rightarrow \xi = z_2 + a \\ z_1 = a \Rightarrow \xi = z_2 - a \\ d\xi = -dz_1 \end{array} \right\} =$$

$$= -\frac{\rho_\ell r_2}{4\pi\varepsilon_0} \int_{z_2+a}^{z_2-a} \frac{1}{[r^2 + \xi^2]^{3/2}} d\xi = \left\{ \begin{array}{l} \int \frac{dx}{u^{3/2}} = \frac{x}{b\sqrt{u}} , \; u = ax^2 + b \\ a = 1 , \; b = r_2^2 , \; x = \xi \end{array} \right\} =$$

$$= -\frac{\rho_\ell r_2}{4\pi\varepsilon_0} \left[\frac{\xi}{r_2^2 \sqrt{r_2^2 + \xi^2}} \right]_{z_2+a}^{z_2-a} = -\frac{\rho_\ell r_2}{4\pi\varepsilon_0 r_2^2} \left[\frac{z_2 - a}{\sqrt{r_2^2 + (z_2 - a)^2}} - \frac{z_2 + a}{\sqrt{r_2^2 + (z_2 + a)^2}} \right] =$$

$$= \frac{\rho_\ell}{4\pi\varepsilon_0 r_2} \left[\frac{z_2 + a}{\sqrt{r_2^2 + (z_2 + a)^2}} - \frac{z_2 - a}{\sqrt{r_2^2 + (z_2 - a)^2}} \right] \Rightarrow E_r \text{ component of P.}$$

Find the $\hat{z}$ component:

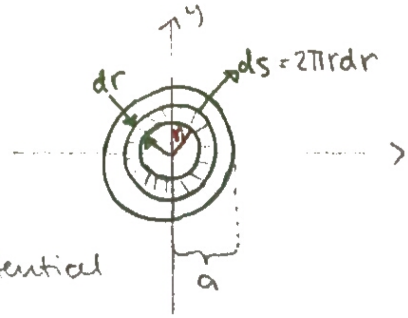
$$\hat{z} \cdot \mathbb{E}(\mathbb{R}_2) = \frac{\rho_\ell}{4\pi\varepsilon_0} \int_{z_1=-a}^{a} \frac{z_2 - z_1}{[r_2^2 + (z_2 - z_1)^2]^{3/2}} dz_1 = \{ \text{subs } \xi = z_2 - z_1 \} =$$

$$= \frac{\rho_\ell}{4\pi\varepsilon_0} \int_{z_2+a}^{z_2-a} \frac{\xi}{[r_2^2 + \xi^2]^{3/2}} d\xi , \text{ p.s.s} \Rightarrow E_z = \frac{\rho_\ell}{4\pi\varepsilon_0} \left[\frac{1}{\sqrt{r_2^2 + (z_2 - a)^2}} - \frac{1}{\sqrt{r_2^2 + (z_2 + a)^2}} \right]$$

example 2.11

a thin circular metal disk of radius a, is located at very large distance from other bodies. A charge Q is distributed as a surface charge density on each side of the disk.

$$\rho_s(r) = \frac{Q}{4\pi a\sqrt{a^2-r^2}}$$



Find the potential of metal disk if $V(\infty) = 0$.

- Potential is the same all over the disk.
- we compute the potential at center (V_0).

$$V = \int_S \frac{dq}{4\pi\varepsilon_0 R_{12}}$$

$$\begin{cases} \mathbf{R}_1 = \hat{r}\, r_1 \\ \mathbf{R}_2 = 0 \end{cases} \Rightarrow \mathbf{R}_{12} = -\hat{r}\, r \ , \ R_{12} = r$$

$$dq = (\rho_s\, dS)\cdot 2 = \left(\frac{Q}{4\pi a\sqrt{a^2-r^2}}\cdot 2\pi r\, dr\right)\cdot 2$$, is charge on two sides.

$$V(\mathbf{R}_2) = \int_S \frac{1}{4\pi\varepsilon_0 R_{12}}\, dq = \int_{r_1=0}^{a} \frac{1}{4\pi\varepsilon_0 \cancel{r_1}}\cdot\frac{Q\,\cancel{r_1}}{a\sqrt{a^2-r_1^2}}\, dr_1 = \frac{Q}{4\pi\varepsilon_0 a}\int_0^a \frac{1}{\sqrt{a^2-r_1^2}}\, dr_1 =$$

$$= \left\{\begin{array}{l} \displaystyle\int \frac{dx}{\sqrt{\tilde{b}-\tilde{a}x^2}} = \frac{1}{\sqrt{\tilde{a}}}\arcsin\left(x\sqrt{\frac{\tilde{a}}{\tilde{b}}}\right) \\ \tilde{b} = a^2,\ \tilde{a} = 1,\ x = r_1 \end{array}\right\} = \frac{Q}{4\pi\varepsilon_0 a}\left[\arcsin\left(\frac{r_1}{a}\right)\right]_0^a =$$

$$= \frac{Q}{4\pi\varepsilon_0 a}\left[\underbrace{\arcsin(1)}_{=\pi/2} - \underbrace{\arcsin(0)}_{=0}\right] = \frac{Q}{4\cancel{\pi}\varepsilon_0 a}\cdot\frac{\cancel{\pi}}{2} = \frac{Q}{8\varepsilon_0 a}$$