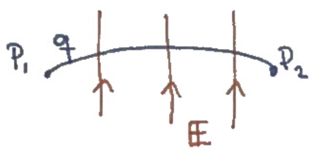
Storgruppsövning 1/11-13

Electric potential

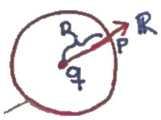
Identity: the curl of the gradient of any scalar field is zero.

$$\begin{cases} \nabla \times (\nabla V) = 0 \\ \nabla \times \mathbb{E} = 0 \end{cases} \Rightarrow \mathbb{E} = -\nabla \underbrace{V}_{\text{scalar electric potential}}$$



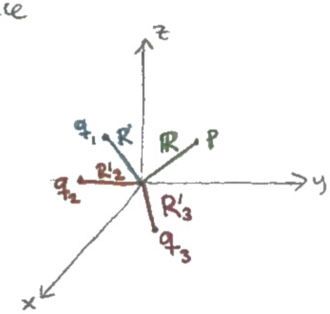
$$\frac{W}{q} = -\int_{P_1}^{P_2} \mathbb{E} \cdot d\mathbb{l} = \underbrace{V_2 - V_1}_{\text{potential difference between } P_1 \text{ and } P_2} \quad (V, J/C)$$

Potential at infinity is zero, $V_\infty = 0$.



equi-potential surface

$$V_R - \underbrace{V_\infty}_{=0} = V_R = -\int_\infty^R \underbrace{\mathbb{E}}_{\frac{q}{4\pi\varepsilon_0 R^2}} \cdot d\mathbb{l} = \frac{q}{4\pi\varepsilon_0 R}$$

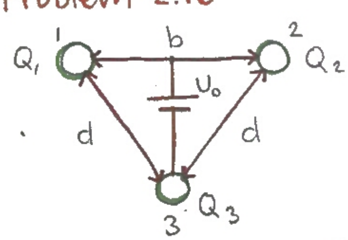


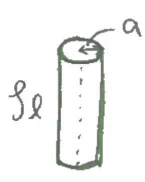
$$V_R = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{n} \frac{q_k}{|R - R'_k|}$$

$$V_R = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho}{R} dV', \quad \text{volume charge distribution}$$

$$V_R = \frac{1}{4\pi\varepsilon_0} \int_{S'} \frac{\rho_s}{R} dS' \quad \text{surface charge distribution}$$

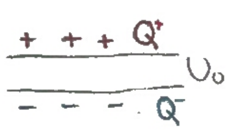
Problem 2.16



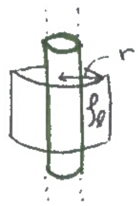


$$V_1 - V_3 = U_0$$

$$\begin{cases} Q_1 = Q_2 = Q \\ Q_3 = -2Q \end{cases}$$

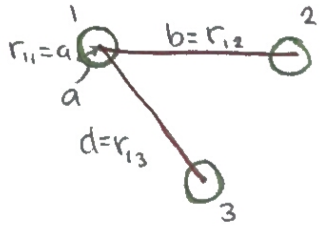


forts. →



Gauss law $\Longrightarrow$ $\mathbb{E} = \hat{a_r}\, E_r = \hat{a_r}\, \dfrac{\rho_\ell}{2\pi\varepsilon_0 r}$

$$V_r = -\int_\infty^r \mathbb{E}\cdot d\mathbb{l} = -\int_\infty^r \frac{\rho_\ell}{2\pi\varepsilon_0 r}\,dr = \frac{-\rho_\ell}{2\pi\varepsilon_0}\ln r$$

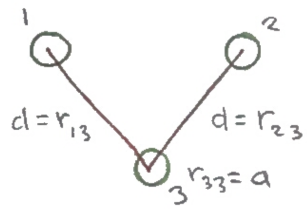


$$V_1 = V_{12} + V_{13} + V_{11} =$$

$$= \frac{-\rho_{\ell_2}}{2\pi\varepsilon_0}\ln r_{12} + \frac{-\rho_{\ell_3}}{2\pi\varepsilon_0}\ln r_{13} + \frac{-\rho_{\ell_1}}{2\pi\varepsilon_0}\ln r_{11}.$$

$$\Longrightarrow V_1 = \frac{-Q/\ell}{2\pi\varepsilon_0}\ln b + \frac{2Q/\ell}{2\pi\varepsilon_0}\ln d - \frac{Q/\ell}{2\pi\varepsilon_0}\ln a$$

$$V_1 = \frac{Q/\ell}{2\pi\varepsilon_0}\left[\ln d^2 - \ln(ab)\right] = \frac{Q/\ell}{2\pi\varepsilon_0}\ln\left(\frac{d^2}{ab}\right)$$



$$V_3 = V_{31} + V_{32} + V_{33} =$$

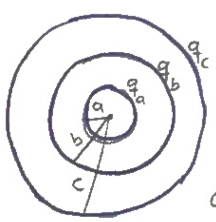
$$= -\frac{Q/\ell}{2\pi\varepsilon_0}\ln d - \frac{Q/\ell}{2\pi\varepsilon_0}\ln d + \frac{2Q/\ell}{2\pi\varepsilon_0}\ln a =$$

$$= \frac{Q/\ell}{2\pi\varepsilon_0}\left[\ln a^2 - \ln d^2\right] = \frac{Q/\ell}{2\pi\varepsilon_0}\ln\left(\frac{a^2}{d^2}\right)$$

$$V_1 - V_3 = U_0 \Longrightarrow \frac{Q/\ell}{2\pi\varepsilon_0}\left[\ln\left(\frac{d^2}{ab}\right) - \ln\left(\frac{a^2}{d^2}\right)\right] = \frac{Q/\ell}{2\pi\varepsilon_0}\ln\left(\frac{d^4}{a^3 b}\right) = U_0$$

$$\Longrightarrow Q = \frac{2\pi\varepsilon_0 V_0 \ell}{\ln\left(\frac{d^4}{a^3 b}\right)} \qquad \begin{cases} Q_1 = Q_2 = Q \\ Q_3 = -2Q \end{cases}$$

Problem 3.1



a) V_0 when $V_\infty = 0$?

b) b and c are connected by conductor, V_0?

$$\text{a)}\quad V_R - \underbrace{V_\infty}_{=0} = -\int_\infty^R \mathbb{E}\cdot d\mathbb{l} = V_0 = -\int_\infty^0 \mathbb{E}\cdot d\mathbb{l}$$

Gauss's law: $\mathbb{E}_R(R) = \dfrac{Q_{in}}{4\pi\varepsilon_0 R^2}$

forts. $\longrightarrow$

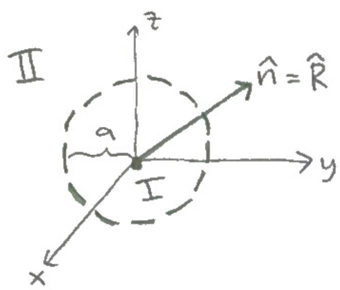
Electric flux density $\mathbb{D}$

$\nabla \cdot \mathbb{E} = \dfrac{\rho}{\varepsilon_0}$ free space

$\nabla \cdot \mathbb{E} = \dfrac{\rho + \rho_p}{\varepsilon_0}$ Polarized dielectric $\xRightarrow{\rho_p = -\nabla \cdot \mathbb{P}}$ $\nabla \cdot \underbrace{(\varepsilon_0 E + \mathbb{P})}_{\mathbb{D}} = \rho$ free charge

Problem 3.2

A dielectric sphere of radius a, has a constant polarization $\mathbb{P} = P\hat{R}$, V_0 ?



$\left(\begin{array}{l} 1.\ V \Longrightarrow \mathbb{E} \overset{\text{Gauss}}{\Longrightarrow} Q_{in} \begin{cases} \rho_p \\ \rho_{ps} \end{cases} \\ 2.\ V \Longrightarrow \begin{cases} \rho_p \\ \rho_{ps} \end{cases} \end{array}\right)$ methods...

$$\rho_p = -\nabla \cdot \mathbb{P} = -\frac{1}{R^2}\frac{\partial}{\partial R}(R^2 P) = -\frac{2P}{R} \quad (R<a)$$ volume charge density

$$\rho_{ps} = \mathbb{P} \cdot \hat{n} = (P\hat{R}) \cdot \hat{R} = P$$ surface charge density

1a) $\oint_s E \cdot ds = \dfrac{Q_{in}}{\varepsilon_0} \Longrightarrow E_R(R) = \dfrac{Q_{in}}{4\pi\varepsilon_0 R^2}$

$$\underset{0<R<a}{Q_{in}} = \int_{R'=0}^{R} \rho_p(R')\,dV' = \int_{R'=0}^{R} -\frac{2P}{R} 4\pi R'^2\, dR' = -8P\pi \int_{R'=0}^{R} R'\,dR' =$$

$$= -4\pi P\left[R'^2\right]_0^R = -4\pi P R^2$$

$R>a$ in II : $Q_{in}^{I}(R=a) + \oint_s \rho_{ps}\, ds' = -4\pi P a^2 + P \overbrace{\oint_s ds'}^{4\pi a^2} = 0$

$$\Longrightarrow \begin{cases} E_{RI}(R) = \dfrac{-4\pi P R^2}{4\pi\varepsilon_0 R^2} = \dfrac{-P}{\varepsilon_0} & (R<a) \\[2ex] E_{RII}(R) = \dfrac{0}{4\pi\varepsilon_0 R^2} = 0 & (R>a) \end{cases}$$

1b) Use the Gauss's law

$$\oint_s \mathbb{D} \cdot d\mathbb{S} = Q \Longrightarrow D_R \oint_s ds = Q = 0 \Longrightarrow D_R = 0$$

$$0 = D = \varepsilon_0 E + P \Longrightarrow E = -\frac{P}{\varepsilon_0}\ , \quad E_R^{I}(R) = \frac{-P}{\varepsilon_0}\ , \quad E_R^{II}(R) = 0$$

$$V_0 = V(R=0) - \underbrace{V(R=\infty)}_{=0} = -\int_\infty^0 E_R(R)\,dR =$$

$$= -\underbrace{\int_\infty^a E_R^{II}(R)\,dR}_{=0} - \int_a^0 E_R^{I}(R)\,dR = \left[\frac{PR}{\varepsilon_0}\right]_{R=a}^{0} = \frac{-Pa}{\varepsilon_0}$$

2. $$V = \frac{1}{4\pi\varepsilon_0}\oint_s \frac{\rho_{ps}}{R}\,ds' + \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_p}{R}\,dv' =$$

$$= \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \frac{P}{4\pi\varepsilon_0 a}\, a^2 \sin\theta\, d\theta\, d\varphi + \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\int_{R=0}^{a} \frac{-2P}{4\pi\varepsilon_0 R^2}\, R^2 \sin\theta\, dR\, d\theta\, d\varphi =$$

$$= \frac{Pa}{4\pi\varepsilon_0}\, 2\pi \int_0^\pi \sin\theta\, d\theta - \frac{Pa}{2\pi\varepsilon_0}\cdot 2\pi \int_0^\pi \sin\theta\, d\theta =$$

$$= \frac{Pa}{2\varepsilon_0}\cdot 2 - \frac{Pa}{\varepsilon_0}\cdot 2 = \frac{-Pa}{\varepsilon_0}$$