Storgruppsövning 30/10-13

Electric field intensity and Gauss's law

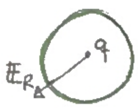
E: defined as the force per unit charge $\mathbb{E} = \lim_{q \to 0} \frac{F}{q} \left(\frac{V}{m}\right)$

Two fundamental postulates in electrostatic (free space):

$$\nabla \cdot \mathbb{E} = \frac{\rho}{\varepsilon_0} \Longrightarrow \oint_S \mathbb{E} \cdot d\mathbb{S} = \frac{Q}{\varepsilon_0} \quad \text{(Gauss's law)}$$

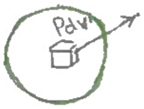
$$\nabla \times \mathbb{E} = 0 \Longrightarrow \oint_S \mathbb{E} \cdot d\ell = 0 \quad \text{(Kirchoff's voltage law)}$$

Coulomb's law:

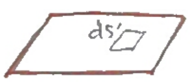


$$\mathbb{E} = a\hat{R} \frac{q}{4\pi\varepsilon_0 R^2}$$

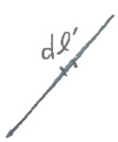
$$\mathbb{E} = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{N} \frac{q_k (\mathbb{R} - \mathbb{R}'_k)}{|\mathbb{R} - \mathbb{R}'_k|^3}$$ electric field due to a system of discrete charges.



$[\rho] = \left(\frac{c}{m^3}\right)$ $\quad \mathbb{E} = \frac{1}{4\pi\varepsilon_0} \int_{V'} \rho \frac{\mathbb{R}}{|R|^3} dV' = \frac{1}{4\pi\varepsilon_0} \int_{V'} \vec{aR} \frac{\rho}{|R|^2} dv'$



$[\rho_s] = \left(\frac{c}{m^2}\right)$ $\quad \mathbb{E} = \frac{1}{4\pi\varepsilon_0} \int_{S'} \vec{aR} \frac{\rho_s}{|R|^2} ds'$



$[\rho_\ell] = \left(\frac{c}{m}\right)$ $\quad \mathbb{E} = \frac{1}{4\pi\varepsilon_0} \int_{\ell'} \vec{aR} \frac{\rho_\ell}{|R|^2} d\ell'$

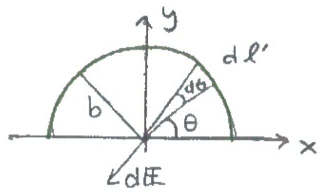
Gauss's law:

$\oint_S \mathbb{E} \cdot d\mathbb{S} = \frac{Q}{\varepsilon_0}$ outward flux of the electric field over any closed surface equals to the total charge in surface over ε_0.

It is useful to find $\mathbb{E}$ in symmetric problems.

Problem 3.8

line charge ρ_ℓ



$$\rho_\ell = dq/d\ell$$

$$dq = \rho_\ell \cdot d\ell' = \rho_\ell \cdot b \cdot d\theta$$

$$d\mathbf{E} = a\hat{R}\,\frac{dq}{4\pi\varepsilon_0 R^2} = -\hat{r}\,\frac{\rho_\ell \cdot b \cdot d\theta}{4\pi\varepsilon_0 b^2} \quad , \quad \hat{r} = \hat{x}\cos\theta + \hat{y}\sin\theta$$

$$d\mathbf{E} = (\hat{x}\cos\theta + \hat{y}\sin\theta)\,\frac{-\rho_\ell \cdot d\theta}{4\pi\varepsilon_0 b} = \hat{x}\,dE_x + \hat{y}\,dE_y$$

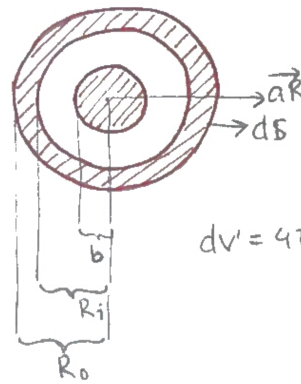
$$E_x = \int_{\theta=0}^{\pi} dE_x = 0$$

$$E_y = \int_{\theta=0}^{\pi} dE_y = \int_0^{\pi} \frac{-\rho_\ell}{4\pi\varepsilon_0 b}\sin\theta\,d\theta = \frac{-\rho_\ell}{2\pi\varepsilon_0 b}$$

$$\mathbf{E} = \hat{y}E_y = -\hat{y}\,\frac{\rho_\ell}{2\pi\varepsilon_0 b}$$

Problem 3.11

$$\rho = \rho_0\left[1 - \frac{R^2}{b^2}\right] \quad , \quad 0 \le R < b$$



$dV' = 4\pi R'^2 dR'$

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{Q_{in}}{\varepsilon_0} \Rightarrow \oint E_R(R)\cdot dS = \frac{Q_{in}}{\varepsilon_0}$$

$$E_R(R)\oint_S dS = \frac{Q_{in}}{\varepsilon_0}$$

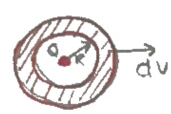
$$E_R(R)\,4\pi R^2 = \frac{Q_{in}}{\varepsilon_0}$$

$$E_R(R) = \frac{Q_{in}}{4\pi\varepsilon_0 R^2}$$

1) $0 \le R \le b$

$$Q_{in} = \int_0^R \rho\,dV' = \int_0^R \rho_0\left[1 - \frac{R'^2}{b^2}\right]4\pi R'^2\,dR' = 4\pi\rho_0\int_0^R\left[R'^2 - \frac{R'^4}{b^2}\right]dR' =$$

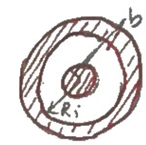
$$= 4\pi\rho_0\left[\frac{R^3}{3} - \frac{R^5}{5b^2}\right]$$



$$E_R(R) = \frac{Q_{in}}{4\pi\varepsilon_0 R^2} = \frac{4\pi\rho_0}{4\pi\varepsilon_0 R^2}\left[\frac{R^3}{3} - \frac{R^5}{5b^2}\right] = \frac{\rho_0}{\varepsilon_0}\left[\frac{R}{3} - \frac{R^3}{5b^2}\right]$$

2) $b \le R \le R_i$

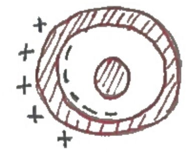
$$Q_{in} = Q_{in-1}\Big|_{R=b} = 4\pi\rho_0\left[\frac{b^3}{3} - \frac{b^5}{5b^2}\right] = 4\pi\rho_0 b^3\,\frac{2}{15}$$



$$E_{R2}(R) = \frac{4\pi\rho_0 b^3}{4\pi\varepsilon_0 R^2}\cdot\frac{2}{15} = \frac{\rho_0 b^3}{\varepsilon_0 R^2}\cdot\frac{2}{15}$$

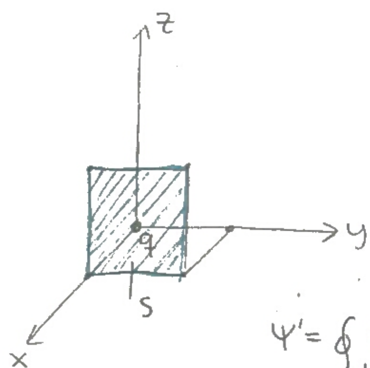
3) $R_i \le R \le R_o$

$$Q_{in} = 0 \Longrightarrow E_{R3}(R) = 0$$



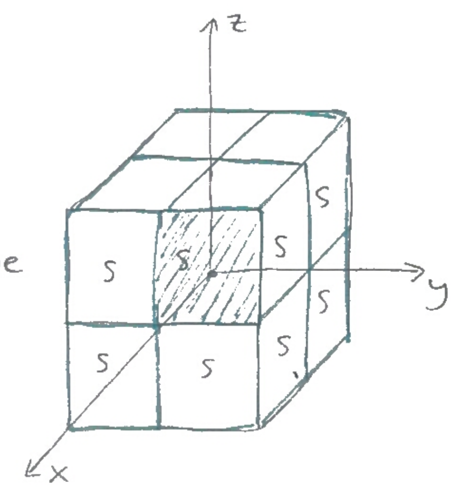
4) $R \ge R_o \Longrightarrow Q_{in} = Q_{in2} \Longrightarrow E_{R4}(R) = E_{R2}(R) = \frac{2\rho_0 b^3}{\varepsilon_0 R^2 15}$

Problem 2.6



$$\Psi = \int_S \varepsilon_0 E \cdot dS$$

Make the problem symmetric.
Build a closed surface.



$$\Psi' = \oint_{S'_t} \varepsilon_0 E \cdot dS = q \quad (*)$$

$S_t = 6 \cdot 4S = 24S$, total area of cube

$$\Psi' = \oint_{S_t} \varepsilon_0 E\, dS = 24\int_S \varepsilon_0 E\, dS = 24\Psi$$

$$(*) \Rightarrow 24\Psi = q \Longrightarrow \Psi = q/24$$