KOMPLEX MATEMATISK ANANALYS

Röv 2001

Sidor: 53

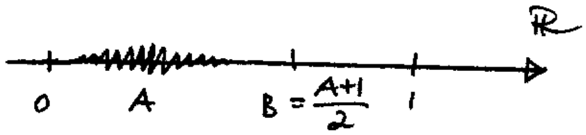
Pris: 25 kr

010910
Lv 2   sid 42
övn. 1   42.) $\sum_{0}^{\infty} c_n \quad (c_n \geq 0)$

$$\lim_{n \to \infty} c_n^{\frac{1}{n}} = A$$

Serien konvergerar om $0 \leq A \leq 1$

Bevis:



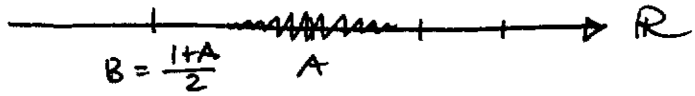
$\exists\, n_0$ så att $c_n^{\frac{1}{n}} < B, \quad n \geq n_0$

Får $c_n < B^n, \quad n \geq n_0$

$\sum_{0}^{\infty} B^n$ konv. om $0 < B < 1$

JK $\Rightarrow \sum_{0}^{\infty} c_n$ konv.

Serien divergerar om $A > 1$



$\exists\, n_0$ så att $c_n^{\frac{1}{n}} > B$ och $n \geq n_0$

Får $c_n > B^n, \quad n \geq n_0$

$B^n \to +\infty$ då $n \to +\infty$

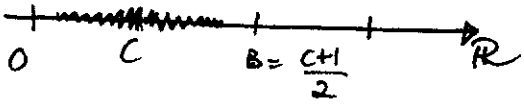
$c_n \to +\infty$ då $n \to +\infty$

$\sum_{0}^{\infty} c_n$ divergerar ty om serien konvergerar gäller att $c_n \to 0$ då $n \to +\infty$

sid 42

43) $\lim_{n\to+\infty} \frac{c_{n+1}}{c_n} = c \qquad (c_n > 0)$

serien konvergerar om $0 \le c \le 1$



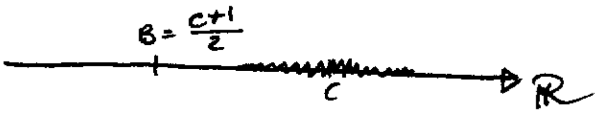
$\frac{c_{n+1}}{c_n} < B$ om $n \ge$ visst n_0

$\frac{c_{n+k}}{c_{n+k-1}} < B$ om $n \ge n_0$

$\frac{c_{n_0+k}}{c_{n_0}} < B^k \Rightarrow c_{n_0+k} < B^k c_{n_0}$

$\sum_0^\infty B^k$ konv.

JK $\Rightarrow \sum_{k=0}^{\infty} c_{n_0+k}$ konv, dvs $\sum_0^\infty c_k$ konv



$\exists n$ så att $\frac{c_{n+1}}{c_n} > B \qquad n \ge n_0$

$\frac{c_{n_0+k}}{c_{n_0}} > B^k$

$c_{n_0+k} > \underbrace{c_{n_0}}_{>0} B^k$

$B^k \to +\infty$ då $k \to +\infty$

$c_{n_0+k} \to +\infty$ då $k \to +\infty$

$\therefore \sum_0^\infty c_n$ divergerar

sid 54

16) $\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

$$\cosh z = \frac{1}{2}(e^{z} + e^{-z})$$

$$\sinh z = \frac{1}{2}(e^{z} - e^{-z})$$

$$\cos(x+iy) = \frac{1}{2}\left(e^{i(x+iy)} + e^{-i(x+iy)}\right) =$$

$$= \frac{1}{2}(e^{ix} \cdot e^{-y} + e^{-ix} \cdot e^{y}) =$$

$$= \frac{1}{2}\{e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i \sin x)\} =$$

$$= \cos x \{\tfrac{1}{2}(e^{-y} + e^{y})\} + \sin x \{\tfrac{i}{2}(e^{-y} - e^{y})\}$$

$$= \cos x \cosh y - i \sin x \sinh y$$

$$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

$$\sin(x+iy) = \frac{1}{2i}\left(e^{i(x+iy)} - e^{-i(x+iy)}\right) =$$

$$= \frac{1}{2i}\{e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i \sin x)\} =$$

$$= \frac{1}{2i}\{\cos x(e^{-y} - e^{y}) - i \sin x\{(e^{-y} + e^{y})\}\} =$$

$$= \sin x \{\tfrac{1}{2}(e^{-y} + e^{y})\} + \frac{1}{i}\cos\{\tfrac{1}{2}(e^{-y} - e^{y})\} =$$

$$= \sin x \cosh y - i \cos x \sinh y$$

sid 55

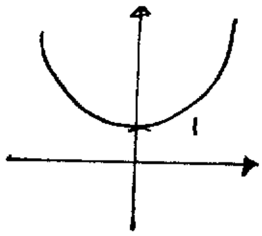
22) Visa att $\begin{cases} \sin(-z) = -\sin z \\ \cos(-z) = \cos z \end{cases}$ för alla z

$$\sin \xi = \frac{1}{2i}\left(e^{i\xi} - e^{-i\xi}\right)$$

$$\sin(-z) = \frac{1}{2i}\left(e^{-iz} - e^{iz}\right) = -\frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = -\sin z$$

$$\cos \xi = \frac{1}{2}\left(e^{i\xi} + e^{-i\xi}\right)$$

$$\cos(-z) = \frac{1}{2}\left(e^{-iz} + e^{iz}\right) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \cos z$$



$\cos iy = \frac{1}{2}(e^{-y} + e^{y})$

$\cos iy \geq 1$

$\square$

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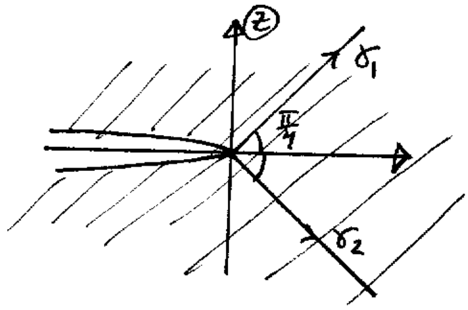
sid 55

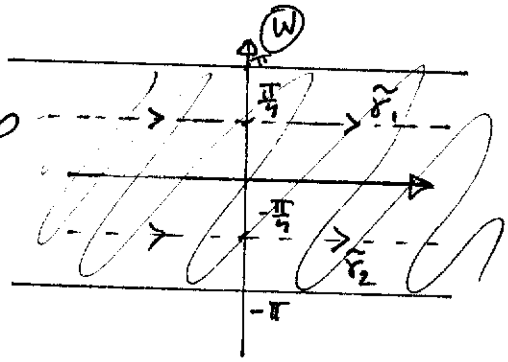
24) $G(z)$ gren av $\log z$

$G(z) = \ln z + i\theta = \ln r + i\theta \qquad r = |z|$

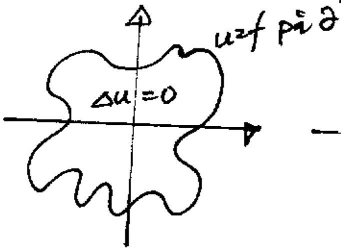
$z = |z| e^{i\theta}, \quad -\pi + n_0 2\pi < \theta < \pi + n_0 2\pi$

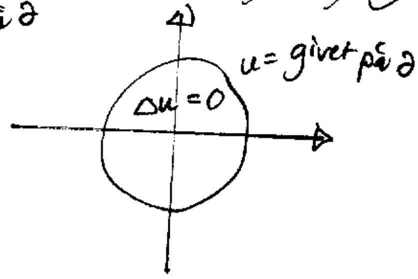


Tag först $n_0 = 0$



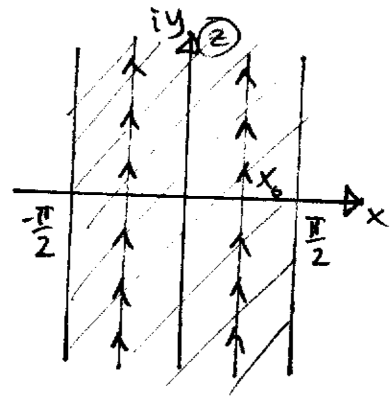
om $n_0 \neq 0$ parallellförflytta fig $n_0 2\pi$ enh. uppåt

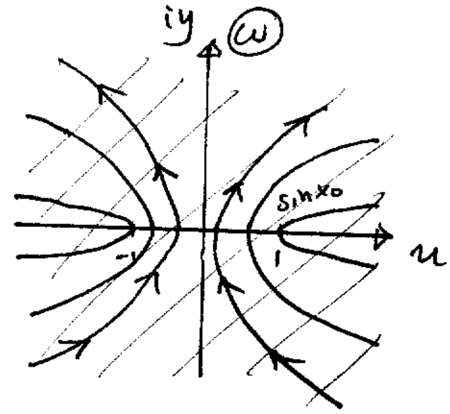




sid 55

25.) $w = \sin z$





$z = x + iy$

$\sin z = \sin x \cosh y + i \cos x \sinh y = w = u + iv$

$$\begin{cases} u = \sin x \cosh y \\ v = \cos x \sinh y \end{cases}$$

$$\frac{u}{\sin x_0} = \cosh y$$

$$\frac{v}{\cos x_0} = \sinh y$$

$$\cosh^2 y - \sinh^2 y = \frac{1}{4}\left\{(e^y + e^{-y})^2 - (e^y - e^{-y})^2\right\} = 1$$

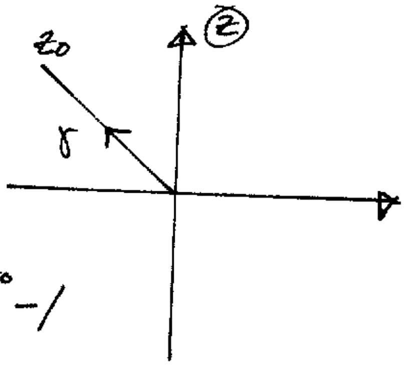
$$\frac{u^2}{\sin^2 x_0} - \frac{v^2}{\cos^2 x_0} = 1$$ hyperbel

$$u^2 = \sin^2 x_0 \left(1 + \frac{v^2}{\cos^2 x_0}\right)$$ antag $u > 0$, $\sin x_0 > 0$

$$\Rightarrow u = \sin x_0 \sqrt{1 + \frac{v^2}{\cos^2 x_0}}$$

sid 73

2) $\displaystyle\int_\gamma e^z\,dz$



$$\int_\gamma e^z\,dz = \left[e^z\right]_0^{z_0} = e^{z_0} - 1$$

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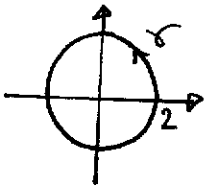
sid 73

6) $\displaystyle\int_\gamma (z^2+3z+4)\,dz$



$$\int_\gamma (z^2+3z+4)\,dz = \left[\frac{z^3}{3} + \frac{3z^2}{2} + 4z\right]_2^2 = 0$$

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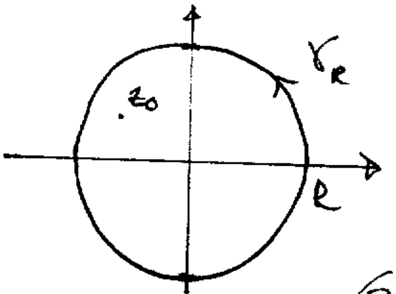
sid 74

15)



$$\int \frac{u(z)}{(z-z_0)^2}\,dz$$

Givet $|u(z)| \le C$

M.L $\displaystyle |I_R| \le \max_{\gamma_R} \left|\frac{u(z)}{(z-z_0)^2}\right| \cdot \ell(\gamma_R) \le$

$\displaystyle \le \frac{C}{(R-|z_0|)^2} \cdot 2\pi R \to 0$ då $R \to \infty$

Viktigt här: $|z+w| \ge \big||z|-|w|\big|$

010914 sid 84
lv2 1.e) $\frac{d}{dz}\tan z = \frac{1}{\cos^2 z}$
övn2

Bevis: $VL = \frac{d}{dz}\frac{\sin z}{\cos z} =$

$$= \frac{\cos z \frac{d}{dz}\sin z - \sin z \frac{d}{dz}\cos z}{\cos^2 z} = \frac{\cos^2 z + \sin^2 z}{\cos^2 z} =$$

$$= \frac{1}{\cos^2 z}$$

trigg-ettan: $\cos^2 z + \sin^2 z =$

$$= \left\{\frac{1}{2}(e^{iz}+e^{-iz})\right\}^2 + \left\{\frac{1}{2i}(e^{iz}-e^{-iz})\right\}^2 =$$

$$= \frac{1}{4}\left\{e^{iz}+e^{-iz} - (e^{iz}-e^{-iz})^2\right\} = 1$$

1.d) $\frac{d}{dz}\cosh z = \sinh z$

$$V.L = \frac{d}{dz}\frac{1}{2}(e^{z}+e^{-z}) = \frac{1}{2}(e^{z}-e^{-z}) = \sinh z$$

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sid 85 15) $f: D \to \mathbb{C}$ & $f'(z) = 0 \quad \forall z \in D$
analytisk holomorf

$P: f = \text{konst}$

$B: f(x+iy) = u(x,y) + iv(x,y)$

$$f'(x+iy) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) = 0$$

$$\therefore \frac{\partial u}{\partial x} = 0, \quad \frac{\partial v}{\partial x} = 0$$

$$0 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

CR!

$$0 = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

D sammanhängande

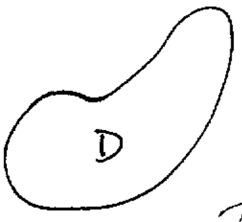
$\nabla u = 0 \Rightarrow u$ konst.

$\nabla v = 0 \Rightarrow v$ konst

$\therefore f$ konst

sid 85

17)



$f: D \to \mathbb{C}$

$f'(z) = \alpha f(z), \quad z \in D$

$f'(z) - \alpha f(z) = 0$

$IF = e^{-\alpha z}$

$\frac{d}{dz}\left\{ f(z)\, e^{-\alpha z} \right\} = 0$

I övn. 15 såg vi att: $f(z) \cdot e^{-\alpha z} = C$

$f(z) = C e^{\alpha z}$

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sid 85

20a) $f = u + iv$ analytisk

$u = x^2 - y^2$

$(x+iy)^2 = x^2 + 2ixy + (iy)^2 = x^2 - y^2 + 2ixy$

$V(x,y) = 2xy$ Finns fler v

$x^2 - y^2 - i2xy$ analytisk?

$u = x^2 - y^2$

$V = -2xy$

$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = -2x$ Nej

$\underset{\underset{2x}{\parallel}}{\frac{\partial u}{\partial x}} = \frac{\partial v}{\partial y} \Rightarrow V = 2xy + C(x);$

$\underset{\underset{-2y}{\parallel}}{\frac{\partial u}{\partial y}} = -\frac{\partial v}{\partial x} = -(2y + C'(x)) \Rightarrow C'(x) = 0 \Rightarrow C(x) = B$

Slutsats $V = 2xy + B$

sid 86

20 d)

$f = u + iv$ analytisk

$u = \cosh y \sin x$

Bestäm v

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow v = \sinh y \cos x + C(x)$$

$$\frac{\partial u}{\partial x} = \cosh y \cos x$$

$$\left.\begin{array}{l} \frac{\partial v}{\partial x} = -\sinh y \sin x + C'(x) \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \frac{\partial u}{\partial y} = -\sinh y \sin x \end{array}\right\} \Rightarrow C'(x) = 0 \Rightarrow C(x) = A$$

Svar: $v = \sinh y \cos x + A$

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sid 103

3) $\displaystyle\sum_{j=0}^{\infty} \frac{z^{3j}}{2^j}$ hitta konvergensradien

sätt $w = z^3$ Får: $\displaystyle\sum_{j=0}^{\infty} \frac{w^j}{2^j} = \sum_{j=0}^{\infty} \left(\frac{w}{2}\right)^j$

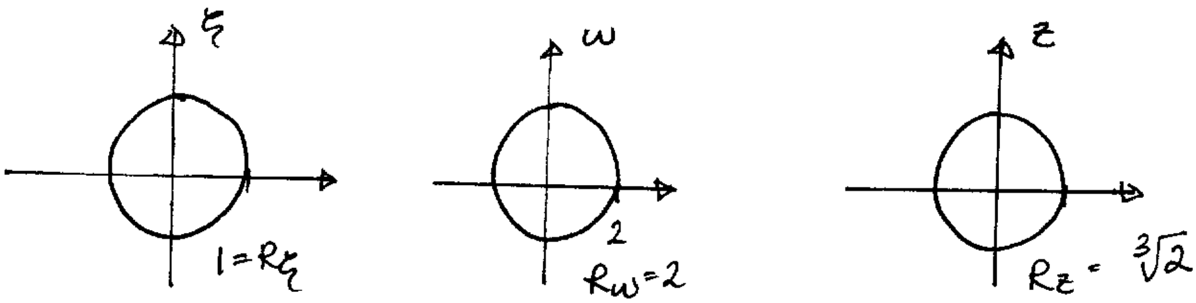
sätt $\xi = \frac{w}{2}$ Får: $\displaystyle\sum_{j=0}^{\infty} \xi^j$ geometrisk serie med kvoten ξ

$$\left[\begin{array}{l} 1 + \ldots + \xi^n = S_n \\ \xi + \ldots + \xi^{n+1} = \xi S_n \end{array}\right]$$

$1 - \xi^{n+1} = (1-\xi) S_n \qquad S_n = \dfrac{1-\xi^{n+1}}{1-\xi} \qquad \xi \neq 1$

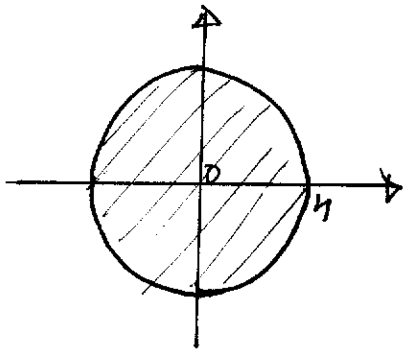
$|\xi| < 1 \Rightarrow \lim\limits_{n \to \infty} S_n = \dfrac{1}{1+\xi}$

$|\xi| > 1 \Rightarrow |S_n| \to \infty$ då $n \to +\infty$



sid 103

11) $f(z) = \dfrac{z^2}{(4-z)^2} \quad |z| < 4$



$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

$$\frac{1}{4-z} = \frac{1}{4}\left(\frac{1}{1-\frac{z}{4}}\right) = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n = \sum_{n=0}^{\infty} \frac{z^n}{4^{n+1}}$$

<u>derivera:</u>

$$\frac{1}{(4-z)^2} = \sum_{n=1}^{\infty} \frac{n z^{n-1}}{4^{n+1}} \quad , \quad |z| < 4$$

$$f(z) = \frac{z^2}{(4-z)^2} = \sum_{n=1}^{\infty} \frac{n z^{n+1}}{4^{n+1}} \underset{n+1=k}{=} \sum_{k=2}^{\infty} \frac{(k-1) z^k}{4^k} \quad |z| < 4$$

Alt: $z^2 (4-z)^{-2}$ kör Liebnitz

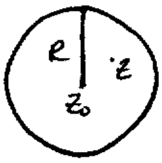
$$\left(\frac{d}{dz}\right)^n (fg) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

010917
sid 103
ör 3
övn 3

19a) $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ konv. $|z-z_0|<R$
div. $|z-z_0|>R$

Visa att serien $\sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^n$ konvergerar för $|z-z_0|<R$

Bevis:



$$\sum_0^{\infty} |a_n(z-z_0)^n| < +\infty \xleftarrow{\text{mult. m. } |z-z_0|}$$

$$\sum_0^{\infty} |a_n(z-z_0)^{n+1}| < +\infty$$

$$\sum_0^{\infty} \frac{|a_n(z-z_0)^{n+1}|}{n+1} < +\infty \quad \text{(Jämförelsekriteriet)}$$

$$\sum_0^{\infty} \frac{a_n(z-z_0)^{n+1}}{n+1} \text{ konv.}$$

Klart!

19.b) $F(z) = \sum_0^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1} \quad |z-z_0|<R$

$$F'(z) = \sum_0^{\infty} \frac{d}{dz}\frac{a_n}{n+1}(z-z_0)^{n+1} = \sum_0^{\infty} a_n(z-z_0)^n = f(z)$$

19c) Sätt $h(z) = \mathrm{Log}(1-z) + z + \frac{z^2}{2} + \frac{z^3}{3} + \dots, \ |z|<1$

Påst.: $h=0$

Bevis: $h(0) = 0+0 = 0$

$$h'(z) = -\frac{1}{(1-z)} + \underbrace{1+z+z^2+\dots}_{\frac{1}{1-z}} = 0$$

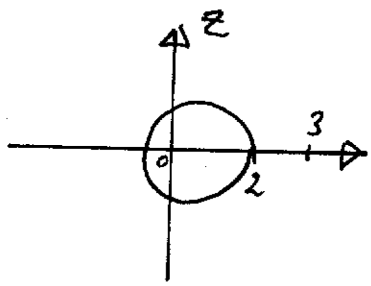
$\therefore h = 0$

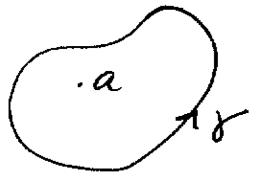
$$\mathrm{Log}(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}, \quad |z|<1$$

sid 116

2.3.2

$$\int_{|z|=2} \frac{e^z}{z(z-3)}\,dz$$



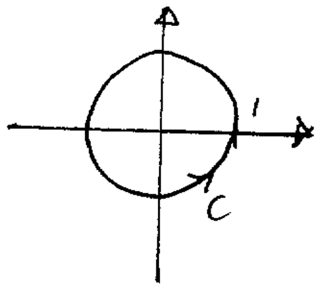


$$\int_{\gamma} \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$\int_{|z|=2} \frac{e^z}{z(z-3)}\,dz = \int_{C} \frac{\frac{e^z}{z-3}}{z-0}\,dz = 2\pi i \frac{e^0}{0-3} =$$

$$= -\frac{2\pi i}{3}$$

2.3.7



$z = e^{i\theta} \qquad 0 \le \theta \le 2\pi$

$dz = i e^{i\theta}\, d\theta \qquad \left(\frac{dz}{iz} = d\theta\right)$

$$\int_{C} g(z)\,\frac{dz}{iz} = \int_{0}^{2\pi} g(e^{i\theta})\,d\theta$$

$$\int_{0}^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_{0}^{2\pi} \frac{d\theta}{a+\frac{b}{2}\left(e^{i\theta}+\frac{1}{e^{i\theta}}\right)} =$$

$(a>b>0)$

$$= \int_C \frac{dz}{iz\left(a+\frac{b}{2}\left(z+\frac{1}{z}\right)\right)} = \frac{1}{i}\int \frac{dz}{az+\frac{b}{2}(z^2+1)} =$$

$$= \frac{2}{bi}\int_C \frac{dz}{z^2+\frac{2a}{b}z+1}$$

$$z^2+\frac{2a}{b}z+1=0$$

$$z = -\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2}-1} = z_{\pm}$$

$$z_+ = -\frac{a}{b}+\sqrt{\frac{a^2}{b^2}-1} < 0$$

$$z_+ > -1 \iff -\frac{a}{b}+\sqrt{\frac{a^2}{b^2}-1} > -1$$

$$\iff \sqrt{\frac{a^2}{b^2}-1} > \frac{a}{b}-1 \iff \sqrt{\frac{a}{b}-1}\,\overbrace{\sqrt{\frac{a}{b}+1}}^{>1} > \frac{a}{b}-1$$

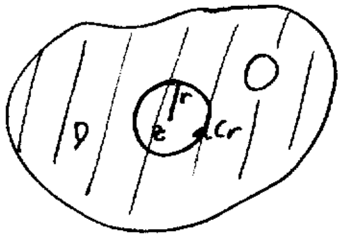
$$I = \frac{2}{bi}\int_C \frac{dz}{(z-z_+)(z-z_-)}$$

$$\frac{2}{bi}\int_C \frac{\frac{1}{(z-z_-)}}{(z-z_+)}\,dz = \frac{2}{bi}\cdot 2\pi i\,\frac{1}{\sqrt{z_+-z_1}} =$$

$$= \frac{4\pi}{b}\cdot\frac{1}{2\sqrt{\frac{a^2}{b^2}-1}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

sid 117

2.3.14



$f: D \to \mathbb{C}$ analytisk

CIF: $f(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\xi)}{\xi - z} d\xi$

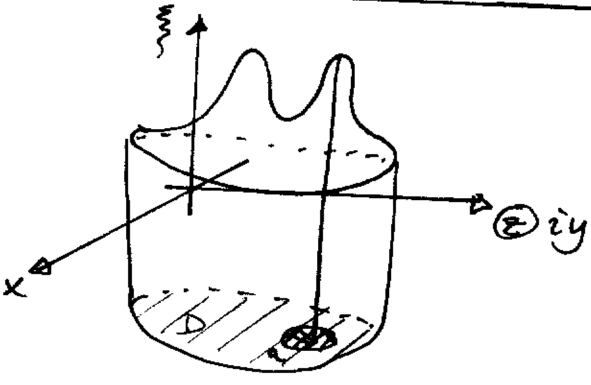
$C_r: \xi = z + re^{it}$

$0 \le t \le 2\pi$

$d\xi = rie^{it} dt$

$\Rightarrow f(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z+re^{it})}{z+re^{it}-z} rie^{it} dt = \frac{1}{2\pi} \int_0^{2\pi} f(z+re^{it}) dt$

2.3.15



$\xi = |f(z)|$

Antag $\exists \delta > 0 \ni |f(z)| < |f(a)|$, om $0 < |z-a| \le \delta$

Har $f(a) = \frac{1}{2\pi i} \int_0^{2\pi} f(a+\delta e^{it}) dt$

Då $|f(a)| = \frac{1}{2\pi} \left| \int_0^{2\pi} f(a+\delta e^{it}) dt \right| \overset{ML}{\le} \frac{1}{2\pi} \cdot 2\pi \max_{0 \le t \le 2\pi} |f(a+\delta e^{it})| =$

$= \max_{0 \le t \le 2\pi} |f(a+\delta e^{it})| < |f(a)|$ (enl. ovan) motsägelse

slutsats: strikt lokala max för $|f|$ $\exists$ ej

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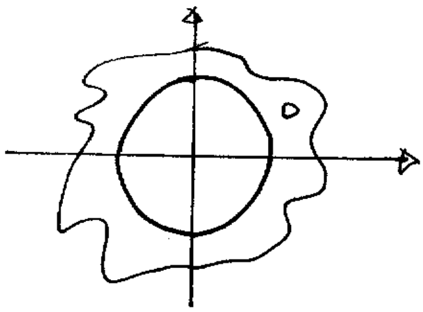
21.e) Poisson: $f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it}) \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\,dt$



$z = re^{i\theta} \quad 0<r<1$

$f: D \to \mathbb{C}$ analytisk

a) $\mathrm{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right) = \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}$

fall $t=0$:

$$\mathrm{Re}\left(\frac{1+z}{1-z}\right) = \frac{1}{2}\left(\frac{1+z}{1-z} + \frac{1+\bar{z}}{1-\bar{z}}\right) =$$

$$= \frac{1}{2}\,\frac{1-\bar{z}+z-|z|^2+1+\bar{z}-z-|z|^2}{1-\bar{z}-z+|z|^2} = \frac{1-|\bar{z}|^2}{1-2\mathrm{Re}\,z+|z|^2} =$$

$$= \frac{1-r^2}{1-2r\cos\theta+r^2}$$

godtyckligt t:

$$VL: \mathrm{Re}\left(\frac{1+ze^{-t}}{1-ze^{-it}}\right) = \mathrm{Re}\,\frac{1+re^{i(\theta-t)}}{1-re^{i(\theta-t)}} =$$

$$= \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}$$

21.b) $\dfrac{e^{it}}{e^{it}-z} + \dfrac{\bar{z}e^{it}}{1-e^{it}\bar{z}} = Re\left(\dfrac{e^{it}+z}{e^{it}-z}\right)$

Lösn: $V.L = \dfrac{e^{it}}{e^{it}-z} + \dfrac{\bar{z}}{e^{-it}-\bar{z}} =$

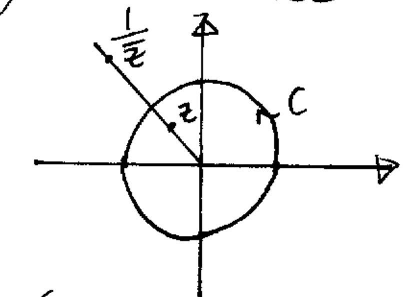
$$= \frac{1-\bar{z}e^{it} + \bar{z}e^{it} - |z|^2}{(e^{it}-z)\overline{(e^{it}-z)}} = \frac{1-r^2}{|e^{it}-z|^2} =$$

$$= \frac{1-r^2}{|1-re^{i(\theta-t)}|^2} = \frac{1-r^2}{|1-r\cos(\theta+t) + ir\sin(\theta-t)|^2} =$$

$$= \frac{1-r^2}{(1-r\cos(\theta-t))^2 + r^2\sin^2(\theta-t)} = \frac{1-r^2}{1-2r\cos(\theta-t)+r^2} =$$

$$= Re\left(\frac{e^{it}+z}{e^{it}-z}\right)$$

21.c) $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(e^{it})\,\frac{e^{it}\bar{z}}{1-\bar{z}e^{it}}\,dt = 0$



$\xi = e^{it}$

$d\xi = ie^{it}dt$

$V.L: \dfrac{1}{2\pi i}\displaystyle\int_C f(\xi)\,\frac{\bar{z}}{1-\bar{z}\xi}\,d\xi = \frac{1}{2\pi\bar{z}i}\int \underbrace{\frac{f(\xi)}{\frac{1}{\bar{z}}-\xi}}_{\text{analytisk i en omgivning av } |\xi|\leq 1} d\xi \overset{CIS}{=} 0$

21.d) $\frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\frac{e^{it}}{e^{it}-z}\,dt = f(z)$

$$V.L = \frac{1}{2\pi i}\int_C f(\xi)\,\frac{1}{\xi - z}\,d\xi \overset{\text{CIF}}{=} f(z)$$

21.e) $f(e^{it})\left(\frac{e^{it}}{e^{it}-z} + \frac{\bar{z}e^{it}}{1-e^{it}\bar{z}}\right) =$

$$= f(e^{it})\,\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(z) + 0 = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\,\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\,dt$$

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lv 3

övn. 4

2.) • $f(z) = (z-z_0)^m g(z)$, där $g(z)$ analytisk och $g(z) \neq 0$ så är z_0 nollställe av mult. m till $f(z)$

• $f(z) = \overset{\neq 0}{a_m}(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \ldots$
z_0 nollställe av mult. m

• Om $f(z_0) = f'(z_0) = \ldots = f^{(m-1)}(z_0) = 0$, men $f^{(m)}(z_0) \neq 0$ så är z_0 nollställen av mult. m

Bestäm multipliciteten av varje nollställe till $f(z) = (e^z - 1)^2$

$1 = e^z = e^x \cdot e^{iy}$, $1 = e^x$ (längden) ger $x = 0$

Nollställen $\underset{z_0}{z = in2\pi}$ n heltal $e^{z_0} = 1$ $y = 2n\pi$, n heltal

$f''(z) = 2e^z$

$f'(z_0) = 2(e^{z_0} - 1) = 0$

$f''(z_0) = 2e^{z_0} = 2$

Alltså: $z_0 = i2n\pi$ är nollställena av multiplicitet 2.

7. $f(z) = e^{2z} - 3e^{z} - 4$

Nollställen: $e^{z} = w$

$$0 = w^2 - 3w - 4 = (w+1)(w-4)$$

i) $e^{z} = -1$ $\quad -1 = e^{x} \cdot e^{iy}$ längden ger $x=0$,

$-1 = e^{iy}$, $y = (2n+1)\pi$

$z = z_0 = i(2n+1)\pi \qquad e^{z_0} = -1$

$f'(z) = 2e^{2z} - 3e^{z}$, $f(z_0) = 5$

z_0 nollställe av mult. 1

ii) $4 = e^{z} = e^{x} e^{iy}$, längden ger $e^{x} = 4$, $x = \ln 4$,

$e^{iy} = 1$ ger $y = 2n\pi$

$z = z_1 = \ln 4 + i2n\pi$, n heltal $\quad e^{z} \leq 4$

$f'(z_1) = 2 \cdot 4^2 - 3 \cdot 4 = 20 \neq 0$

z nollst. med mult. 1

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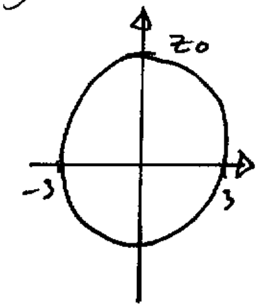
13. $f(z) = \dfrac{z+z}{z+3}$ utveckla i potensserie kring $z_0 = -1$



$f(z) = \sum a_n (z - z_0)^n$

konvergensradie 2

$n!\, a_n = f^{(n)}(z_0)$

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi$$

$$\frac{1}{1-z} = 1+z+z^2+\dots = \sum_{n=0}^{\infty} z^n$$

$$e^z = 1+\frac{z}{1!}+\frac{z^2}{2!}+\dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\frac{(z+2)}{z+3} = \frac{z+2}{2+(z+1)} = \frac{z+2}{2}\cdot\frac{1}{1-\left(\frac{z+1}{-2}\right)}$$

$$\frac{1}{1-\left(\frac{z+1}{-2}\right)} = \frac{1}{1-w} = \sum w^n = \sum \frac{(-1)^n}{2^n}(z+1)^n$$

går tillbaka lite $\rightarrow$ $\frac{z+2}{z+3} = 1-\frac{1}{z+3} =$

$$= 1-\frac{1}{2}\cdot\frac{1}{1-\left(\frac{z+1}{-2}\right)} = 1-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}(z+1)^n =$$

$$= \frac{1}{2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{n+1}}(z+1)^n$$

---

17. f är analytisk i D har nollställen av ordning m i z_0

a) visa att $f'(z)$ har nollställen av ordningen $m-1$ i z_0

(*) $f(z)=(z-z_0)^m g(z)$, $g(z)$ analytisk, $g(z_0)\neq 0$

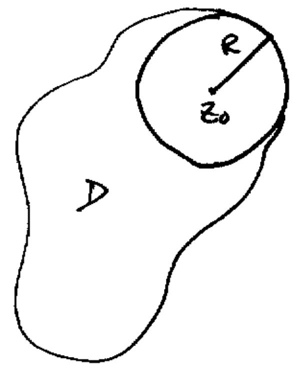
$$f'(z) = m(z-z_0)^{m-1}g(z)+(z-z_0)^m g'(z)$$
$$= (z-z_0)^{m-1}\underbrace{\left[m g(z)+(z-z_0)g'(z)\right]}_{\text{analytiskt}}$$

$f^2(z)$ har z_0 som nollställe av mult. $2m$

(*) ger $f^2(z) = (z-z_0)^{2m} g^2(z)$ $\leftarrow$ analytisk, $\neq 0$ i $z=z_0$

18. $f(z) = \sum a_n (z-z_0)^n$

$f^{(n)}(z_0) = n! a_n$

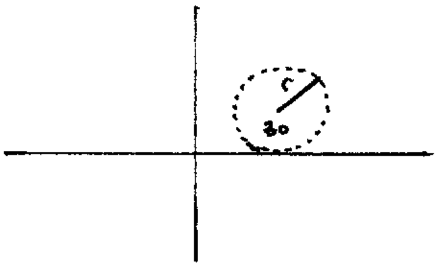


f analytisk i D

$$\frac{f(z)}{(z-z_0)^{n+1}} = a_0 + a_1 (z-z_0)^1 + \dots$$
$$\dots + a_{n-1}(z-z_0)^{n-1} +$$
$$+ a_n (z-z_0)^n + a_{n+1}(z-z_0)^{n+1} + \dots$$

$$\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}} dz = 2\pi i a_n, \quad a_n = \frac{1}{2\pi i}\int \frac{f(z)}{(z-z_0)^{n+1}} dz$$

Visa att $|f^{(n)}(z_0)| \le \frac{n!}{r} \max_{|z-z_0|=r} |f(z)|$



$$|a_n| = \frac{1}{2\pi}\left|\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz\right| \le$$

$$\le \frac{1}{2\pi} \cdot 2\pi r \cdot \max_{|z-z_0|=r} \frac{|f(z)|}{|z-z_0|^{n+1}} = \frac{1}{r^n} \max_{|z-z_0|=r} |f(z)|$$

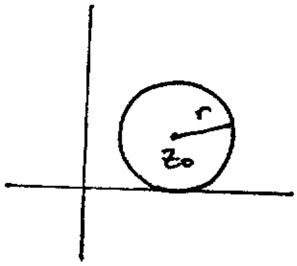
$$|f^{(n)}(z_0)| = n! |a_n| = \frac{n!}{r} \max_{|z-z_0|=r} |f(z)|$$

---

19. Visa Liouville's sats: Om $f(z)$ är hel och $|f(z)| \le M$ för något konstant M så är $f(z)$ konstant.

$|f^{(n)}(z_0)| \le \frac{n!}{r} M \to 0$ när $r \to \infty$

(utnyttja att $f(z)$ är <u>hel</u>)



Alltså: $f^{(n)}(z_0) = 0$ för alla n.

f i potensserie runt z_0: $n!\,a_n = f^{(n)}(z_0)$

$$f(z) = \underbrace{f(z_0)}_{=a_0} + 0\cdot z + 0\cdot z^2 + \dots = f(z_0)$$

Ger $f(z)$ konstant

---

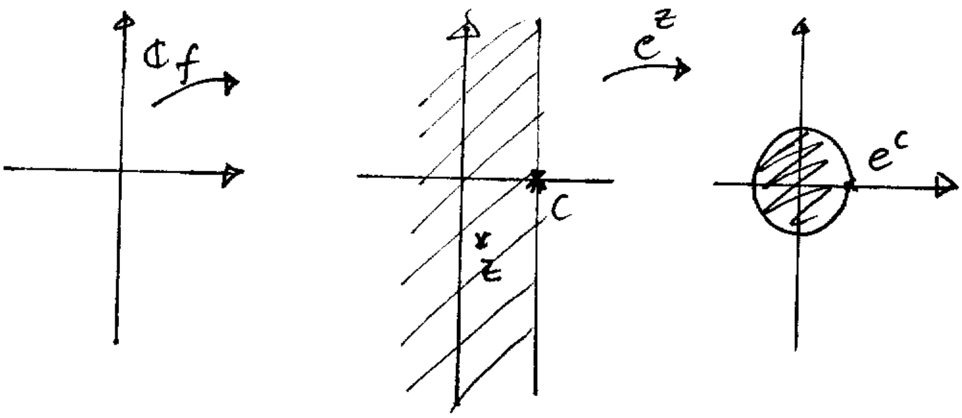
20. f är hel, $\operatorname{Re} f(z) \le c$

Visa $f(z)$ konstant

$g(z) = e^{f(z)}$ har $|g(z)| \le e^c$

Alltså $g(z)$ konstant,

ger $f(z)$ konstant.



$z = x + iy$

$x \le c$

$e^z = e^x e^{iy}$

$|e^z| = e^x \le e^c$

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21. från uppgift 18:

Om f är analytisk i D som innehåller $|z-z_0| \leq r$ så gäller:

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n} \max_{|z-z_0|=r} |f(z)|$$

Vet: $|f(z)| \leq A|z|^m$ när $|z| \geq R_0$
där A, m, R_0 är konstanter. f är hel.
Visa att $f(z)$ är ett polynom av grad högst m.

Skall välja $z_0 = 0$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ (för alla z)
Skall visa att $0 = f^{(n)}(0) = n! a_n$ när $n > m$

$$|f^{(n)}(0)| \leq \frac{n!}{r^n} \max_{|z|=r} |f(z)| \leq \{\text{väljer } r > R_0\}$$

$$\leq \frac{n!}{r^n} \max_{|z|=r} A|z|^m = \frac{n!}{r^n} A \cdot r^m \overset{n>m}{=} \frac{n! A}{r^{n-m}} \to 0 \text{ då } r \to \infty$$

Alltså $(f^{(n)}(z)) = 0$ så $f^{(n)}(z) = 0$ när $n > m$. Därför är $f(z)$ ett polynom av grad högst m.

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25) Lös $f''(z)+\beta^2 f(z)=0$ f är hel (analytisk i hela planet)

$$(*)\quad f(z)=\sum_{n=0}^{\infty} a_n z^n$$

$$f'(z)=\sum_{n=1}^{\infty} n a_n z^{n-1}$$

$$(**)\quad f''(z)=\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}=\sum_{n=0}^{\infty}(n+2)(n-1)a_{n+2}z^n$$

$f''(z)=-\beta^2 f$ jmfr $(*)$ med $(**)$

$$-\beta^2 f(z)=\Sigma(-\beta^2 a_n)z^n$$

Ser: $-\beta^2 a_n=(n+2)(n+1)a_{n+2}$

$$a_{n+2}=\frac{-\beta^2 a_n}{(n+2)(n+1)}$$

$$a_2=-\frac{\beta^2 a_0}{2} \qquad a_4=\frac{\beta^4 a_0}{4!}$$

Induktivt: $a_{2n}=(-1)^n\dfrac{\beta^{2n}}{(2n)!}a_0$

$$a_3=\frac{-\beta^2 a_1}{3\cdot 2}\,,\quad a_5=\frac{\beta^4 a_1}{5\cdot 4\cdot 3\cdot 2}$$

$$a_{2n+1}=(-1)^n\frac{\beta^{2n}a_1}{(2n+1)!} \qquad a_0=f(0)$$

$$a_1=f'(0)$$

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2.) $\dfrac{z^2}{\sin z}$

Om $f(z)$ är analytisk i $0 < |z-z_0| < r$

i) $|f(z)|$ begränsad när $z \to z_0$

$f(z)$ kan då definieras i z_0 så att $f(z)$ är analytisk även i z_0.

ii) $|f(z)| \to \infty$ när $z \to z_0$

$$f(z) = \frac{H(z)}{(z-z_0)^m} \quad \text{där } H(z_0) \neq 0$$

$H(z)$ analytisk i $|z-z_0| < r$, f har pol av ordning m i z_0

iii) Annars (väsentlig singularitet)

$$f(z) = \frac{z^2}{\sin z}$$

Analytisk utom i z_0 där $\sin(z_0) = 0$

$$0 = \frac{1}{2i}\left(e^{iz_0} - e^{-iz_0}\right), \quad e^{iz_0} = e^{-iz_0},$$

$$1 = e^{2iz_0} \text{ ger } e^{iz_0} = 1 \text{ eller } e^{iz_0} = -1$$

$$z_0 = n2\pi, \quad z_0 = (\pi + n2\pi) \quad z_0 = n\pi$$

Ordning av nollstället z_0.

$\sin z = \cos z$

$\cos z_0 \neq 0$

z_0 nollställe av ordningen 1 till $\sin(z)$

$\sin(z) = (z - z_0)\, g(z)$, $g(z)$ analytisk

$g(z_0) \neq 0$

$$\frac{z^2}{\sin z} = \frac{z^2}{(z-z_0)(g(z))} = \left\{ H(z) = \frac{z^2}{g(z)} \right\} \frac{H(z)}{z-z_0}$$

pol av ordning 1 om $z_0 \neq 0$

$$\frac{z^2}{\sin(z)} \Rightarrow \{ z_0 = 0 \} \quad \frac{z^2}{z\, g(z)} = \frac{z}{g(z)}$$

analytisk i $z_0 = 0$ för $g(0) \neq 0$

8.) $f(z) = \dfrac{z^2}{z^2-1}$ $\quad z_0 = 1$ $\quad$ ingen hävbar singularitet.

$f(z) = \dfrac{z^2}{(z-1)(z+1)}$ pol av ordn. 1 i $z_0 = 1$

Utveckla i Laurent-serie

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-1)^n$$

$$f(z) = 1 - \frac{1}{z^2-1} = 1 + \frac{1}{z-1} \cdot \frac{1}{z+2} = \{ w = z-1 \} =$$

$$= \frac{1}{2+w} = \frac{1}{2} \cdot \frac{1}{1-\left(\frac{-w}{2}\right)} = \left\{ \frac{1}{1-\xi} = \sum \xi^n \right\} =$$

$$= \frac{1}{2} \sum \frac{(-1)^n}{2^n} w^n = \sum \frac{(-1)^n}{2^{n+1}} (z-1)^n$$

$$f(z) = 1 + \frac{1}{z-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-1)^n =$$

$$= 1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(z-1)^{n-1} = 1 + \sum_{n=-1}^{\infty} \frac{(-1)^{n+1}}{2^{n+2}}(z-1)^n$$

$$a_0 = 1 - \frac{1}{4} = \frac{3}{4}$$

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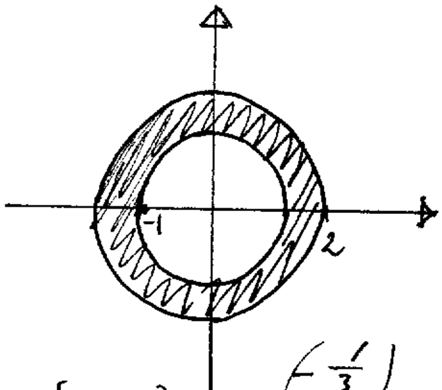
23.a) Bestäm Laurent-serie för

$$f(z) = \frac{z+2}{z^2-z-2} \qquad \begin{array}{l} 1)\ 1<|z|<2 \\ 2)\ |z|>2 \end{array}$$

$$f(z) = \frac{z+2}{(z+1)(z-2)}$$ analytisk överallt utom i $z=2$ och $z=-1$



1) $$\frac{z+2}{(z+1)(z-2)} = \{PBU\} = \frac{\left(-\frac{1}{3}\right)}{z+1} + \frac{\frac{4}{3}}{z-2}$$

$$\frac{1}{z-2} = \frac{1}{-2} \cdot \frac{1}{1-\frac{z}{2}} \overset{g}{=} -\frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} \qquad \left|\frac{z}{2}\right|<1$$

$$\frac{1}{z+1} = \frac{1}{z} \cdot \frac{1}{1-\left(\frac{-1}{z}\right)} \overset{g}{=} \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^n} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{n+1}} \qquad \left|\frac{1}{z}\right|<1$$

när $|z|>1$

Svar: $$f(z) = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{3}\cdot\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{-4}{3\cdot 2^{n+1}}z^n$$

2) $$\frac{1}{z-2} = \frac{1}{z}\cdot\frac{1}{1-\frac{2}{z}} \overset{g}{=} \frac{1}{z}\sum_{n=0}^{\infty}2^n\frac{1}{z^n} = \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}} \qquad |z|>2$$

$$f(z) = \sum_{n=0}^{\infty}\left(\frac{(-1)^{n+1}}{3} + \frac{2^{n+2}}{3}\right)\frac{1}{z^{n+1}}$$

22a) Bestäm Laurent-serien till

$e^{\frac{1}{z}}$ i $0<|z|$

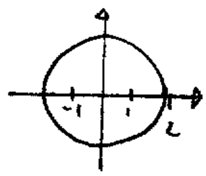
$$e^{w} = \sum_{n=0}^{\infty} \frac{w^n}{n!} \quad \text{för alla } w$$

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{z^n} \quad \text{för alla } z \neq 0$$

22.c) Bestäm Laurentserien till

$$f(z) = \frac{1}{z-1} - \frac{1}{z+1} \quad \text{i } 2<|z|$$

analytisk överallt, utom i 1 och -1



$$\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$

när $\left|\frac{1}{z}\right| < 1$ och $|z| > 1$

$$\frac{1}{z+1} = \frac{1}{z}\left(\frac{1}{1-\left(\frac{1}{z}\right)}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}}$$

$$f(z) = \sum_{n=0}^{\infty}\left(\left(1-(-1)^n\right)\right)\frac{1}{z^{n+1}} = \sum_{n=1}^{\infty}\left(\left(1+(-1)^n\right)\frac{1}{z^n}\right)$$

17.) f har väsentlig singularitet i z_0 ($|f(z)|$ ej begränsad nära z_0 och går heller inte mot ∞ när $z = z_0$)

Visa: $g(z) = \frac{A}{f(z)-w}$ inte är begränsad i någon enda punkterad skiva

$$0<|z-z_0|<\varepsilon$$

Antag $g(z)$ begränsad i $0 < |z-z_0| < \varepsilon$
$|g(z)|$ är begränsad i närheten av z_0
$g(z)$ har en hävbar singularitet i z_0
$g(z)$ kan förutsättas analytisk i z_0.
$g(z_0) = (z-z_0)^m h(z)$ där $h(z)$ analytisk
och $h(z_0) \neq 0$

1.*) Ger $f(z) - w = \dfrac{1}{(z-z_0)^m h(z)}$
där $\frac{1}{h(z)}$ analytisk i z_0!
Ger $f(z)$ saknar väsentlig singularitet
i z_0

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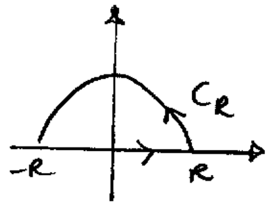
Lv 5   2.6.) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2+6x+10}\,dx = \operatorname{Im}\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+6x+10}\,dx$

övn. 6

$f(z) = \dfrac{e^{iz}}{z^2+6z+10}$ integrera runt konturen



$f(z)$ har poler då

$z^2+6z+10=0$

$z = -3 \pm \sqrt{9-10} = -3 \pm i$

I övre halvplanet ligger enkelpolen $z=-3+i$

$$\operatorname*{Res}_{z=-3+i} f(z) = \left.\frac{e^{iz}}{2z+6}\right|_{z=-3+i} = \frac{e^{i(-3+i)}}{2i}$$

Residuesatsen:

$$\oint_C = 2\pi i \sum_{z_i} \operatorname{Res} \Rightarrow \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz =$$

$$= 2\pi i \cdot \frac{e^{i(-3+i)}}{2i} = \pi e^{-1-3i}$$ för alla R stora nog ($\sqrt{10}$)

$$\left|\int_{C_R} f(z)\,dz\right| \underset{M.L}{\leq} \int_{C_R} \frac{\overbrace{|e^{iz}| \leq e^{-y} \leq 1}}{\underbrace{|z^2+6z+10|}_{\geq R^2-6R-10}} |dz| \leq \frac{1}{R^2-6R-10}\cdot \pi R$$

då $R \to \infty$ så får

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{e} e^{-3i}$$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+6x+10}\,dx = \operatorname{Im}\left(\frac{\pi}{e} e^{-3i}\right) = -\frac{\pi}{e}\sin 3$$

27) $C(z) = \pi\cot\pi z = \dfrac{\pi\cos\pi z}{\sin\pi z}$

har poler då $\sin\pi z = 0$, $\pi z = n\pi$, n heltal, $\underline{\underline{z=n}}$

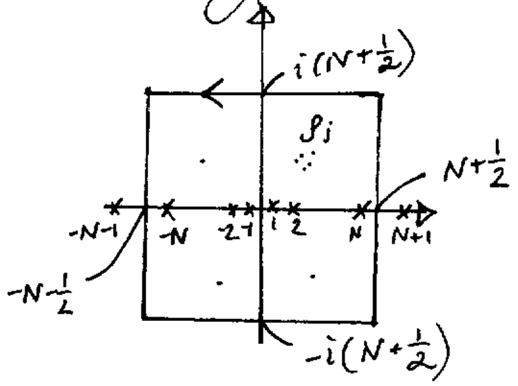
Enkelpoler, ty $\dfrac{d}{dz}\sin\pi z\Big|_{z=n} = \pi\cos n\pi = \pi(-1)^n \neq 0$

Låt $f(z)$ vara analytisk i $z=n$

$$\operatorname{Res}\left(f(z)\cdot C(z); n\right) = \frac{\pi f(z)\cos\pi z}{\underbrace{\pi\cos\pi z\big|_{z=n}}_{\uparrow \text{ OBS}}} = f(n)$$

---

28) Visa att $C(z) = \pi\cot\pi z$ begr. i kvadrat γ_N med vertikal i $\pm(N+\frac{1}{2}) \pm i(N+\frac{1}{2})$ med gränsvärden oberoende av N



titta på $|\cot\pi z| = \left|\dfrac{\cos\pi z}{\sin\pi z}\right| = \dfrac{|e^{i\pi z}+e^{-i\pi z}|}{|e^{i\pi z}-e^{-i\pi z}|}$

$\underline{x = \pm(N+\frac{1}{2}):}$

$$|\cot\pi z| \leq \frac{|e^{i\pi z}|+|e^{-i\pi z}|}{\left||e^{i\pi z}|-\underbrace{|e^{-i\pi z}|}_{\text{störst}}\right|} =$$

$$= \frac{e^{-(N+\frac{1}{2})\pi}+e^{(N+\frac{1}{2})\pi}}{e^{(N+\frac{1}{2})\pi}-e^{-(N+\frac{1}{2})\pi}} = \coth\left(\left(N+\tfrac{1}{2}\right)\pi\right)$$

$\leq \coth\frac{3\pi}{2} < 2 \quad \therefore\ |C(z)| \leq 2\pi$ på hela γ_N

29) Antag $f(z)$ analytisk överallt utom i ett ändligt antal isolerade singulära punkter.

$$\xi_j \, , \, j = 1, \ldots, m$$

och antag $|z f(z)| \to 0$ då $z \to \infty$

Integrera $f(z)\,c(z)$ runt γ_N där N är så stort att alla ξ_j ligger innanför γ_N

<u>Residusatsen</u>

$$\frac{1}{2\pi i}\int_{\gamma_N} f(z)c(z)\,dz = \sum_{\substack{n=-N \\ n \neq \xi_j}}^{N} \underbrace{\mathrm{Res}\,(f(z)c(z), n)}_{f(n) \text{ enl. uppg. 27}} +$$

$$+ \sum_{j=1}^{m} \mathrm{Res}\,(f(z)c(z), \xi_j)$$

$M_N : \max_{z \in \gamma_N} |f(z)|$ enl. förut. är $(N+\frac{1}{2})M_N \to 0, \; N \to \infty$

$$\left| \int_{\gamma_N} f(z)c(z)\,dz \right| \leq M_N \cdot \underset{\text{uppg. 28}}{2\pi \cdot 8(N+\tfrac{1}{2})} = 16\pi(N+\tfrac{1}{2})M_N \underset{N\to\infty}{\to} 0$$

Låt $N \to \infty$ så fås

$$\sum_{\substack{n=-\infty \\ n \notin \{\xi_1, \ldots, \xi_m\}}}^{\infty} f(n) = -\sum_{j=1}^{m} \mathrm{Res}\,(f(z)c(z); \xi_j)$$

Ex $f(z) = \dfrac{1}{z^{2p}}$; $p \geq 1$ heltal

$z f(z) \to 0$, $|z| \to \infty$

$$\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n^{2p}} = -\operatorname{Res}\left(\frac{\pi \cot \pi z}{z^{2p}} ; 0\right)$$

$$\sum_{1}^{\infty} \frac{1}{n^{2p}} = -\frac{1}{2} \operatorname{Res}\left(\frac{\pi z \cot \pi z}{z^{2p+1}} ; 0\right)$$

Laurentserieutveckla $\{\omega = \pi z\}$ $\omega \cot \omega =$

$$= \frac{\cos \omega}{\frac{\sin \omega}{\omega}} = \frac{1 - \frac{1}{2}\omega^2 + \frac{\omega^4}{24} - \dots}{1 - \frac{1}{6}\omega^2 + \frac{1}{20}\omega^4 - \dots}$$

$$\frac{1}{1-\omega_0} = 1 + \omega_0 + \omega_0^2 + \omega_0^3 + \dots \qquad |\omega_0| < 1$$

$$\omega_0 = \frac{1}{6}\omega^2 - \frac{1}{120}\omega^4 + \dots$$

$$\omega \cot \omega = \left(1 - \frac{1}{2}\omega^2 + \frac{1}{24}\omega^4 - \dots\right)\left(1 + \frac{1}{6}\omega^2 - \frac{1}{120}\omega^4 + \dots\right) =$$

$$= 1 + \frac{1}{6}\omega^2 - \frac{1}{120}\omega^4 - \frac{1}{2}\omega^2 - \frac{1}{12}\omega^4 + \frac{1}{24}\omega^4 + O(\omega^5) =$$

$$= 1 - \frac{1}{3}\omega^2 - \frac{1}{45}\omega^4 - \dots$$

$$\omega = \pi z \Rightarrow \pi z \cot \pi z = 1 - \frac{\pi^2 z^2}{3} - \frac{\pi^4 z^4}{45}$$

$p = 1$ $$\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{2} \cdot \underbrace{\frac{\pi^2}{3}}_{\text{(Residum delar med } z^{2p+1} = z^3)} = -\frac{\pi^2}{6}$$

31.d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = -\frac{1}{2}\left(-\frac{\pi^4}{45}\right) = \frac{\pi^4}{90}$

31.a) $f(z) = \dfrac{1}{z^2+a^2}$ $\quad$ ($a \neq in$, n heltal

$z \cdot f(z) \to 0$ då $|z| \to \infty$)

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = -\left[\operatorname{Res}\left(\frac{\pi \cot \pi z}{z^2+a^2};\, ia\right) + \operatorname{Res}\left(\frac{\pi \cot \pi z}{z^2+a^2};\, -ia\right)\right]$$

$$\boxed{\sum_{\substack{n=-\infty \\ n \notin \{\xi_1,\dots,\xi_m\}}}^{\infty} f(n) = -\sum_{j=1}^{m} \operatorname{Res}\left(f(z)\,c(z);\, \xi_j\right)}$$

$$\operatorname{Res}\left(\frac{\pi \cot \pi z}{z^2+a^2};\, ia\right) = \frac{\pi \cot(\pi i a)}{2ia} =$$

jämn

$$= -\frac{\pi}{2a} \coth(\pi a) = \operatorname{Res}\left(\frac{\pi \cot \pi z}{z^2+a^2};\, -ia\right)$$

$$\sum_{-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a} \coth(\pi a)$$

jämn $\Rightarrow$ $\displaystyle\sum_{1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{2a} \coth(\pi a) - \frac{1}{2a^2}$

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sid 169

30) $\sum_{-\infty}^{\infty} f(n) = -\sum_{j=1}^{K} \mathrm{Res}(fC; \xi_j)$

- f analytisk men poler i $\xi_1, \ldots, \xi_K$ varav inget är heltal.
- $|z f(z)| \to 0$ när $|z| \to \infty$

$C(z) = \pi \cot(\pi z)$ analytisk men med enkelpoler i n, n heltal.

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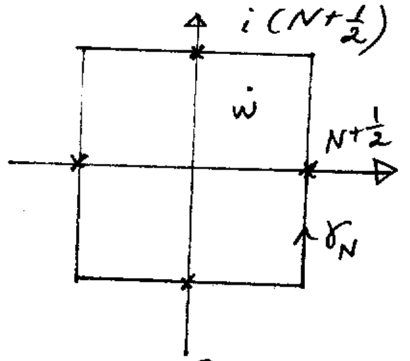
32)



$w \neq 0, \pm 1, \pm 2, \ldots, \pm N$
(w fix)

Visa: $\frac{1}{2\pi i} \int_{\gamma_N} \frac{C(z)}{z(z-w)}\, dz = \frac{\pi \cot(\pi w)}{w} + \sum_{\substack{k=-N \\ k \neq 0}}^{N} \frac{1}{k(k-w)} - \frac{1}{w^2}$

Residue-satsen

$$\frac{1}{2\pi i} \int_{\gamma_N} \frac{C(z)}{z(z-w)}\, dz = \sum_{\substack{z_k \text{ innanför } \gamma \\ \text{pol till fkt}}} \mathrm{Res}\left(\frac{C(z)}{z(z-w)}; z_k\right)$$

Poler till $\frac{C(z)}{z(z-w)}$: $\underset{\text{ordn. 2}}{0}, \underbrace{\pm 1, \pm 2, \ldots, \pm N}_{\text{ordn. 1}}$ (från $C(z)$)

w pol av ordning 1

Residue i $z=0$: (ordning 2)

$$C(z) \overset{\text{laurent serie}}{\overset{\downarrow}{=}} a_{-1} z^{-1} + a_0 + a_1 z + \dots = z^{-1} + 0 - \frac{\pi^2}{3} z + \dots$$

$$\frac{1}{(z-w)} = -\frac{1}{1-\frac{z}{w}} \cdot \frac{1}{w} = -\frac{1}{w}\left(1 + \frac{z}{w} + \frac{z^2}{w^2} + \dots\right)$$

$$\frac{C(z)}{(z-w)} = -\frac{1}{w} z^{-1} + \left(\frac{1}{-w^2}\right) + \dots$$

$$\frac{1}{z} \frac{C(z)}{z-w} = -\frac{1}{w} z^{-2} + \left(\frac{-1}{w^2}\right) z^{-1} + \dots$$

$$\operatorname{Res}\left(\frac{C(z)}{z(z-w)}; 0\right) = -\frac{1}{w^2}$$

2) Residue i $z=n$ $n = \pm 1, \dots, \pm N$

$$\frac{C(z)}{z(z-w)} = \frac{\pi \cos(\pi z)}{z(z-w)\sin(\pi z)}$$

$$\operatorname{Res}\left(\frac{C(z)}{z(z-w)}; n\right) = \frac{\pi \cos(\pi n)}{(2n-w)\underbrace{\cancel{\sin \pi n}}_{=0} + n(n-w)\pi \cos(\pi t))}$$

$$= \frac{1}{n(n-w)}$$

Residue i $z=w$

$$\operatorname{Res}\left(\frac{C(z)}{z(z-w)}; w\right) = \frac{\pi \cos(\pi w)}{(2w-w)\sin(\pi w)} = \frac{\pi}{w}\cot(\pi w)$$

Ger $\frac{1}{2\pi i}\int \frac{C(z)}{z(z-w)} dz = \frac{\pi}{w}\cot(\pi w) +$

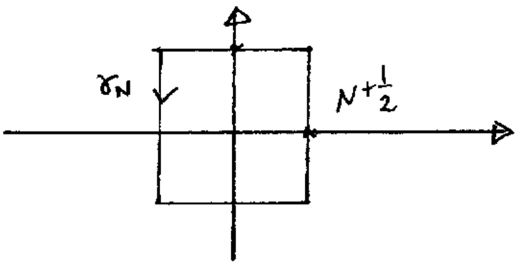
$$+ \sum_{\substack{k=-N \\ k \neq 0}}^{N} \frac{1}{k(k-w)} - \frac{1}{w^2}$$

33) Låt $N \to \infty$ och visa

$$C(\omega) = \pi\cot(\pi\omega) = \frac{1}{\omega} + \sum_{k=1}^{\infty} \frac{2\omega}{\omega^2 - k^2}$$

Har sen förra uppgiften:

$$\frac{1}{2\pi i}\int_{\gamma_N} \frac{C(z)}{z(z-\omega)}\,dz = \frac{\pi}{\omega}\cot(\pi\omega) + \sum_{\substack{k=-N \\ k=0}}^{N} \frac{1}{k(k-\omega)} = \frac{1}{\omega^2}$$



$$\left|\int_{\gamma_N} \frac{C(z)}{z(z-\omega)}\,dz\right| \le \underbrace{\text{längden av } (\gamma_N)}_{4(2N+1)} \cdot \max_{z\in\gamma_N} \left|\frac{C(z)}{z(z-\omega)}\right|$$

Från uppg. 28 fanns en konstant M (oberoende av N) så att $|C(z)| \le M$ när $z \in \gamma_N$

$$\frac{|C(z)|}{|z(z-\omega)|} \overset{N\text{ stort}}{\le} \frac{M}{(N+\frac{1}{2})(|z|-|\omega|)} \le \frac{M}{(N+\frac{1}{2})(N+\frac{1}{2}-|\omega|)}$$

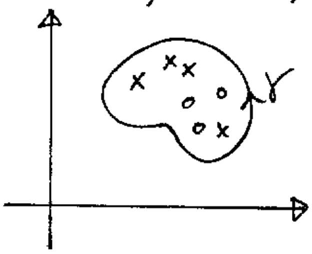
$$\left|\int_{\gamma_N} \frac{C(z)}{z(z-\omega)}\,dz\right| \le \frac{4(2N+1)M}{(N+\frac{1}{2})(N+\frac{1}{2}-|\omega|)} \to 0 \text{ när } N \to \infty$$

Får nu:

$$0 = \pi\cot(\pi\omega) + \sum_{k=1}^{\infty}\left(\frac{\omega}{k(k-\omega)}\right) + \sum_{k=1}^{\infty} \frac{\omega}{k(k+\omega)} - \frac{1}{\omega}$$

ger $\pi\cot(\pi\omega) = \frac{1}{\omega} - \sum_{k=1}^{\infty} \frac{2\cancel{k}\omega}{\cancel{k}(k^2-\omega^2)}$

Argumentprincipen



$h(z)$

o - nollställen

x - poler

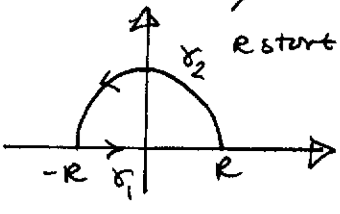
$$\frac{1}{2\pi}\left\{\underset{\text{längs } \gamma}{\text{variationen}} \text{ av } \arg h(z)\right\} =$$

$$= \left\{\begin{matrix}\text{antalet nollst. till} \\ h(z) \text{ innanför } \gamma\end{matrix}\right\} - \left\{\begin{matrix}\text{ant. poler till} \\ h(z) \text{ innanför } \gamma\end{matrix}\right\}$$

sid 180. 8) Bestäm antalet nollställen till

$$h(z) = 2z^4 - 2iz^3 + z^2 + 2iz - 1$$

i övre halvplanet.



inga poler

Max 4 nollst.

Variation längs γ_2:

$$z \in \gamma_2 \quad , \quad z = Re^{it} \quad , \quad 0 \le t \le \pi$$

$$h(z) = \underbrace{R^4 e^{i4t}}_{4\pi}\underbrace{\left(2 - \frac{2i}{Re^{it}} + \frac{1}{R^2 e^{2it}} + \frac{2i}{R^3 e^{3it}} - \frac{1}{R^4 e^{it}}\right)}_{\text{nära noll, när } R \text{ stort.}}$$

Variationen är 4π (ungefär.)

Variation längs γ_1:

$$z = t, \quad -R \le t \le R$$

$$h(t) = u(t) + iv(t) = 2t^4 + t^2 - 1 + i2(t - t^3)$$

$$u = 2t^4 + t^2 - 1$$

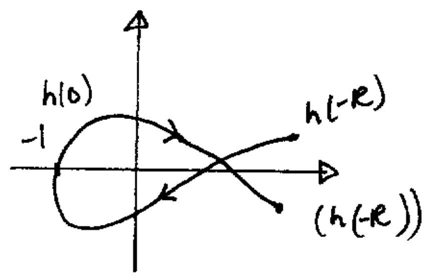
$$v = 2(t - t^3) = 2t(1 - t^2)$$

$$u' = 8t^3 + 2t, \quad 8t(t^2 + \tfrac{1}{4})$$

$$v' = 2 - 6t^2, \quad 2(1 - 3t^2)$$

	$-R$	$-\frac{1}{\sqrt{3}}$		0		$\frac{1}{\sqrt{3}}$		R
u	↘		↘	-1	↗		↗	
v	↘		↗		↗		↘	

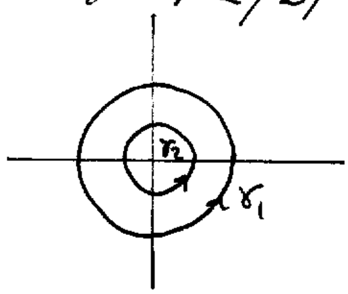
$$\frac{v}{u} \to 0 \quad \text{när } R \to \infty$$



vinkelvariation : -2π

Totalt: $4\pi - 2\pi = 2\pi$ Ger 1 nollställe i övre halvplanet.

12.) Bestäm antalet nollställen till
$z^3 - 3z + 1$ i $1 < |z| < 2$



1) Nollställen innanför γ_1:
Rouches sats
$|f(z)+g(z)| < |f(z)|$ längs γ
Då har f, g samma antal nollställen innanför γ.
Sätt $f(z) = -z^3$
$g(z) = z^3 - 3z + 1$

$|-3z+1| \leq 3|z| + 1 = 7 \leq 8 = |-z|^3$

$g(z)$ har 3 nollställen innanför γ_2,
(samma som $-z^3$)

2) Nollställen innanför γ_2 ($|z|=1$)
$f(z) = 8z$
$g(z) = z^3 - 3z + 1$
$|f+g| \leq |z|^3 + 1 \leq 2 \leq 3 = |8z|$

g har ett nollställe innanför γ_2

Svar: $g(z)$ har 2 nollställen i området

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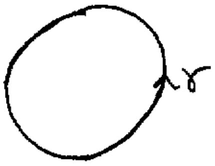
Rouchés sats



f, g analytiska

Om $|f(z) \pm g(z)| < \begin{cases} |f(z)| \\ |g(z)| \end{cases}$

på γ, så har f och g lika många nollställen innanför γ.

sid 181
3.1.23) P ett polynom, $P(z) = a_N z^N + \ldots + a_1 z + a_0$

$a_N \neq 0$, $|P(e^{it})| = 1$ för $0 \leq t \leq 2\pi$

Bilda $Q(z) = z^N \overline{P\left(\frac{1}{\bar{z}}\right)} = z^N \overline{\left(a_N \frac{1}{\bar{z}^N} + \ldots + a_1 \frac{1}{\bar{z}} + a_0\right)}$

$= \overline{a_N} + \ldots \overline{a_1} z^{N-1} + \overline{a_0} z^N$ $\quad Q$ ett polynom

$P(e^{it}) Q(e^{it}) = P(e^{it}) e^{itN} \overline{P(e^{it})} =$

$= \underbrace{|P(e^{it})|}_{=1}{}^2 e^{iNt} - e^{iNt}$

$\therefore P(z) Q(z) = z^N$ då $|z| = 1$ $\quad (z = e^{it})$

Men $P(z) Q(z)$ och z^N är analytiska funktioner som är lika på kurvan $|z| = 1$
De är då <u>lika överallt</u>.

$\therefore P(z) Q(z) = z^N$ överallt

P av grad $N \Rightarrow Q$ av grad 0, $Q = \text{konst} \neq 0$

$\therefore \underline{P(z) = \lambda z^N}$ $\quad |z| = 1$ skall ge $|P(z)| = 1$ varför $|\lambda| = 1$

Möbiusavbildningar $\quad w = \frac{az+b}{cz+d}$

- konform
- tar cirklar/räta linjer på cirkeln/rät linje
- tar inversa punkter på inversa punkter

sid 204.

3.3.4.a) $\begin{cases} z_1 = 1 \\ z_2 = i \\ z_3 = -1 \end{cases}$ $\begin{cases} w_1 = -1 \\ w_2 = i \\ w_3 = -1 \end{cases}$

Ansätt $\frac{z-z_1}{z-z_2} = k \frac{w-w_1}{w-w_2} \quad k \neq 0$

Bestäm k så att z_3 går på w_3

$$\frac{z-1}{z-i} = k \frac{w+1}{w-i}$$

$$\underbrace{\frac{-1-1}{-1-i}}_{\frac{2}{1+i}} = k \underbrace{\frac{1+1}{1-i}}_{\frac{2}{1-i}} \qquad k = \frac{1-i}{1+i} = i$$

$$(z-1)(w-i) = -i(z-i)(w+1)$$

Lös ut $w = -\frac{1}{2}$

3.3.4.b) $\begin{cases} z_1=1\\ z_2=4\\ z_3=\infty\end{cases}$ $\begin{cases} w_1=0\\ w_2=1-i\\ w_3=1+i\end{cases}$

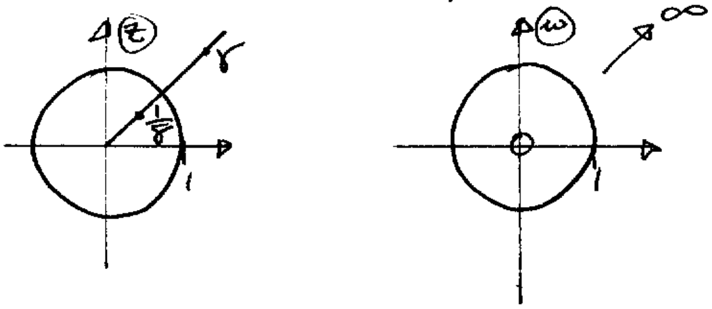
Ansätt $z-1 = k\frac{w}{w-1-i}$

sätt in $z=4,\ w=1-i$: $3 = k\cdot\frac{1-i}{-2i}$ $k=3(1-i)$

lös ut $w = \frac{(1+i)(z-1}{z+3i-4}$

3.3.6) En möbiusavbildning $w=T(z)$

tar $|z|=1$ på $|w|=1$



Någon punkt med $|\gamma|\neq 1$ måste gå på $w=0$

Antag $\gamma\neq\infty$, $\gamma\neq 0$

Den inversa punkten $\frac{1}{\bar\gamma}$ går då på $w=\infty$

T(z) är av formen $w=T(z)=k\frac{z-\gamma}{z-\frac{1}{\bar\gamma}} =$

$$= \underbrace{-k\bar\gamma}_{\lambda}\frac{z-\gamma}{1-\bar\gamma z} = \lambda\frac{z-\gamma}{1-\bar\gamma z}$$

$z=1$ måste ge ett w med $|w|=1$: $1=|\lambda|\frac{|1-\gamma|}{|1-\bar\gamma|}=|\lambda|$

$|\lambda|=1$ Om $\gamma=0$ är $w=T(z)$ av formen $w=\lambda z,\ |\lambda|=1$

Om $\gamma=\infty$ —"— —"— $w=\frac{\lambda}{z},\ |\lambda|=1$

u harmonisk : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$f(z) = u + iv$ analytisk $\Rightarrow$ u och v harmoniska

4.1.1.a)

$$u = x^4 - 6x^2y^2 + y^4$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 12x^2 - 12y^2 - 12x^2 + 12y^2 = 0$$

Det finns harmonisk v så att $f = u + iv$ blir analytisk.

u och v uppfyller Cauchy-Riemanns ekv : $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 4x^3 - 12xy^2$$

$$\Rightarrow v(x,y) = 4x^3y - 4xy^3 + \varphi(x)$$

$$\Rightarrow \frac{\partial v}{\partial x} = 12x^2y - 4y^3 + \varphi'(x)$$

Jämför med $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 12x^2y - 4y^3$

$\therefore$ $\varphi'(x) = 0$ $\quad$ $\varphi(x) = C$

$V(x,y) = 4x^3y + 4xy^3 + C$

4.1.1:d) $u = \sin(x^2-y^2)\cosh(2xy)$

$\xi = z^2 = (x+iy)^2 = x^2-y^2+2ixy = \xi + i\eta$ är analytisk

$u(\xi) = u(\xi,\eta) = \sin\xi\cosh\eta$ är harmonisk

$u(\xi) = \mathrm{Re}\sin\xi$

$\sin(\xi+i\eta) = \sin\xi\cosh + i\cos\xi\sinh\eta$

Då är $u(z^2) = u(x,y)$ harmonisk

$u(x,y) = \mathrm{Re}\sin z^2$

$v(x,y) = \mathrm{Im}\sin z^2$

$= \cos(x^2-y^2)\sinh(2xy)$

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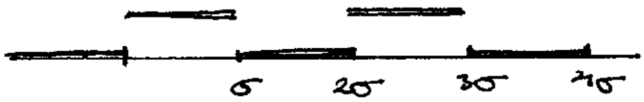
8.) $$u(t) = \begin{cases} 1 & 0 < t < \sigma \\ 0 & \sigma < t < 2\sigma \end{cases}$$

$$u(t) = u(t+2\sigma) \qquad t > 0$$

Bestäm $\mathcal{L}u$



Om $f(t) = f(t+T)$, $t > 0$

$$\mathcal{L}f(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty f(t+T)e^{-st}\,dt =$$

$$= \{t_1 = t+T\} = \int^\infty f(t_1)e^{-s(t_1-T)}\,dt =$$

$$= e^{sT}\left(\int_0^\infty f(t)e^{-st}\,dt - \int_0^T f(t)e^{-st}\,dt\right)$$

$$\mathcal{L}f(1-e^{sT}) = -e^{sT}\int_0^T f(t)e^{-st}\,dt$$

$$\mathcal{L}f = \frac{-e^{sT}}{\underbrace{1-e^{sT}}_{\frac{+1}{-e^{-sT}+1}}}\int_0^T f(t)e^{-st}\,dt = \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}\,dt$$

I vårat fall:

$$\mathcal{L}u(s) = \frac{1}{1-e^{-2s\sigma}}\int_0^{2\sigma} u(t)e^{-st}\,dt = \frac{1}{1-e^{-2s\sigma}}\int_0^{\sigma} e^{-st}\,dt =$$

$$= \frac{1}{1-e^{-2s\sigma}}\left(\frac{1}{s} - \frac{e^{-s\sigma}}{s}\right) = \frac{1}{s}\,\frac{1}{1+e^{-2s\sigma}}$$

14. $\mathcal{L}u(s) = \frac{1}{(s-1)^2}$ Bestäm u

Kjellfilen:

Om $g(s)$ är analytisk i hela planet utom i polerna $s_1, \dots, s_N$

och $g(s) \to 0$ $|s| \to \infty$

Då är $g(s)$ $\mathcal{L}$tr av

$$f(t) = \sum_{k=1}^{N} \mathrm{Res}(g(s)e^{st}; s_k)$$

Vi har $g(s) = \frac{1}{(s-1)^2}$ dubbelpol i $s=1$

$$f(t) = \mathrm{Res}\left(\frac{e^{st}}{(s-1)^2}; 1\right)$$

Laurentserien: $\frac{e^{st}}{(s-1)^2} = \frac{e^{(s-1)t}e^t}{(s-1)^2} =$

$$= \left[\text{vet att } e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}\right] = \frac{e^t}{(s-1)^2}\left(1 + (s-1)t + \frac{(s-1)^2 t^2}{2!} + \dots\right)$$

Alternativ: $\frac{1}{(s-1)^2} = -D\left(\frac{1}{s-1}\right)$

$$-\frac{d}{ds}\mathcal{L}u(s) = +\mathcal{L}(tu(t))(s)$$

$$-\frac{d}{ds}\int_0^{\infty} u(t)e^{-st}\,dt = \int_0^{\infty} +tu(t)e^{-st}\,dt$$

$g(s) = \frac{1}{s-1}$ är $\mathcal{L}$.tr till

$$f(t) = \mathrm{Res}\left(\frac{e^{st}}{s-1}; 1\right) = e^t, \quad \frac{1}{(s-1)^2} = +\mathcal{L}(te^t)(s)$$

16) $\mathcal{L}u(s) = \dfrac{s-2}{(s-1)(s-3)}e^{-s}$

Bestäm $u(t)$

$\mathcal{L}f \cdot \mathcal{L}f = \mathcal{L}(f * g)$

$\dfrac{s-2}{(s-1)(s-3)} = \dfrac{1}{2}\left(\dfrac{1}{s-1} + \dfrac{1}{s-3}\right)$

$\mathcal{L}u(s) = \dfrac{1}{2}\left(\dfrac{1}{s-1}e^{-s} + \dfrac{1}{s-3}e^{-s}\right)$

$\mathcal{L}(\delta(t-1)) = e^{-s}$ [L.19 i tabellpappret]

$\dfrac{1}{s-1}$ är L.tr till $\operatorname{Res}\left(\dfrac{e^{st}}{s-1}; 1\right) = e^{t}$

$\dfrac{1}{s-3}$ —— " —— $\operatorname{Res}\left(\dfrac{e^{st}}{s-3}; 3\right) = e^{3t}$

$\dfrac{1}{s-1} \cdot e^{-s}$ är $\mathcal{L}$tr till $e^{t} * \delta(t-1)$

$$f * g(t) = \int_0^t f(x)g(t-x)\,dx$$

$$\int_0^t f(x)\delta(x-a)\,dx = \begin{cases} 0 & \text{när } t<a \\ f(a) & \text{när } t>a \end{cases}$$

$$\partial(t-1) * e^{t} = \int_0^t \partial(x-1)e^{t-x}\,dx = \begin{cases} 0 & \text{när } t<1 \\ e^{t-1} & t>1 \end{cases}$$

$\dfrac{1}{s-3}e^{-s}$ är $\mathcal{L}$tr till $\partial(t-1) * e^{3t} = \begin{cases} 0 & \text{när } t<1 \\ e^{3t-3} & \text{när } t>1 \end{cases}$

$$u(t) = \begin{cases} 0 & \text{när } t<1 \\ \frac{1}{2}\left(e^{t-1} + e^{3t-3}\right) & \text{när } t>1 \end{cases}$$

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20./ $u(t) = \begin{cases} 0 & t \le 0 \\ \frac{1}{\sqrt{t}} e^{-c/t} & t > 0 \end{cases}$ svaret givet: $\mathcal{L}u(s) = \sqrt{\frac{\pi}{s}}\, e^{-2\sqrt{cs}}$

$$\mathcal{L}u(s) = \int_0^\infty \frac{1}{\sqrt{t}} e^{-c/t - st}\, dt =$$

$$= \int_0^\infty \frac{1}{\sqrt{t}} e^{-\sqrt{cs}\left(\frac{\sqrt{c}}{\sqrt{s}t} + \frac{\sqrt{s}t}{\sqrt{c}}\right)} dt =$$

$$= \left\{ \begin{array}{l} t_1^2 = \frac{\sqrt{s}}{\sqrt{c}} t \\ dt = \frac{\sqrt{c}}{\sqrt{s}} 2t_1 dt_1 \end{array} \right\} = \frac{2\sqrt{c}}{\sqrt{s}} \left(\frac{s}{c}\right)^{\frac{1}{4}} \int \frac{t_1}{\left(\frac{c}{s}\right)^{\frac{1}{4}} t_1} e^{-\sqrt{cs}\left(t_1^2 + \frac{1}{t_1^2}\right)} dt_1$$

$$\overset{(*)}{=} 2\left(\frac{c}{s}\right)^{\frac{1}{4}} \underbrace{\int_0^\infty e^{-\sqrt{cs}\left(t_1 - \frac{1}{t_1}\right)^2} dt_1}_{I} \cdot e^{-2\sqrt{cs}}$$

$$I = \int_0^\infty e^{-\sqrt{cs}\left(t - \frac{1}{t}\right)^2} dt = \text{trick:} \left\{ \begin{array}{l} t = \frac{1}{t_1} \\ dt = -\frac{1}{t_1^2} dt_1 \end{array} \right\} =$$

$$= \int_0^\infty + \frac{1}{t_1^2} e^{-\sqrt{cs}\left(t_1 - \frac{1}{t_1}\right)^2} dt_1 = \int_0^\infty \left(\frac{1}{t^2} + 1\right) e^{-\sqrt{cs}\left(t - \frac{1}{t}\right)^2} dt - I$$

$$2I = \left\{ \begin{array}{l} t_1 = \left(t - \frac{1}{t}\right) \\ dt_1 = \left(\frac{1}{t^2} + 1\right) dt \end{array} \right\} = \int_{-\infty}^\infty e^{-\sqrt{cs}\, t_1^2} dt_1 = \left\{ \begin{array}{l} t_2 = (cs)^{\frac{1}{4}} t_1 \\ dt_2 = (cs)^{\frac{1}{4}} dt_1 \end{array} \right\} =$$

$$= \frac{1}{(cs)^{\frac{1}{4}}} \int_{-\infty}^\infty e^{-\frac{t^2}{2}} dt_2 = \frac{\sqrt{\pi}}{(cs)^{\frac{1}{4}}}$$

Ger $\mathcal{L}u(s) \overset{(*)}{=}$

$$= 2I \left(\frac{c}{s}\right)^{\frac{1}{4}} e^{-2\sqrt{cs}} =$$

$$= \frac{\sqrt{\pi}}{(cs)^{\frac{1}{4}}} \cdot \left(\frac{c}{s}\right)^{\frac{1}{4}} e^{-2\sqrt{cs}}$$

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2.) $u'' + 9u = f \quad f(t) = \begin{cases} 1, & 0 < t < \pi \\ 0, & \pi < t < \infty \end{cases}$

$u(0) = 1 \quad u'(0) = 0$

Laplacetransf: $\mathcal{L}(u'') + 9\mathcal{L}(u) = \mathcal{L}(f)$

$\mathcal{L}(u') = -u(0) + s\mathcal{L}(u)$

$\mathcal{L}(u'') = -u'(0) - su(0) + s^2\mathcal{L}(u)$

V.L: $-s + (s^2+9)\mathcal{L}u = \mathcal{L}f$

$\mathcal{L}u = \frac{s}{s^2+9} + \frac{1}{s^2+9}\mathcal{L}f$

$u \overset{ws}{=} \cos 3t + \frac{1}{3}\sin(3t) * f(t)$

$f(t) * \sin(3t) = \int_0^t f(x)\cdot\sin(3(t-x))\,dx$

$= \begin{cases} \int_0^t \sin(3(t-x))\,dx & t < \pi \\ \int_0^\pi \sin(3(t-x))\,dx & t > \pi \end{cases}$

3.) (*) $u''(t) + t\,u'(t) + u(t) = 0$

$u(0) = 1, \; u'(0) = 0$

$u'' = -u'(0) - u(0)s + s^2\mathcal{L}u$

$$\int_0^\infty u'(t)e^{-st}\,dt = \left[u(t)e^{-st}\right]_0^\infty + s\int_0^\infty u(t)\,e^{-st}\,dt$$

$$\mathcal{L}u' = -u(0) + s\mathcal{L}u$$

$$\mathcal{L}u'' = -u'(0) + s\mathcal{L}u'$$

$$= -u'(0) - su(0) + s^2\mathcal{L}u$$

$$\frac{d}{ds}\mathcal{L}u = \frac{d}{ds}\int_0^\infty u(t)e^{-st}\,dt =$$

$$= -\int_0^\infty tu(t)e^{-st}\,dt = -\mathcal{L}(tu(t))$$

$$\mathcal{L}(tu'(t)) = -\frac{d}{ds}\mathcal{L}u'$$

$$= -\frac{d}{ds}(-1+s\mathcal{L}u) = -\mathcal{L}u - s(\mathcal{L}u)'$$

(*) blir efter $\mathcal{L}$tr:

$$-s + s^2\mathcal{L}u - \mathcal{L}u - s(\mathcal{L}u)' + \mathcal{L}u = 0$$

$$(\square)\quad s\mathcal{L}u - (\mathcal{L}u)' = 1$$

$$\left(\begin{array}{l} (\mathcal{L}u)' - s\mathcal{L}u = -1 \\ D(\mathcal{L}u\, e^{-\frac{s^2}{2}}) = -e^{-\frac{s^2}{2}} \end{array}\right)$$

nämen! Detta gick ju inte att lösa!

$$(\mathcal{L}u)' = -\mathcal{L}(tu(t))$$

($\square$) är $\mathcal{L}$tr till (kolla tabell)

$$\mathcal{L}\,11: \mathcal{L}(\delta') = s, \quad \mathcal{L}(\delta) = 1$$

(✿) $\delta' * u + tu = \delta$

$$\delta' * u = ?$$

vet att: $\delta * f(t) = \int_0^t \delta(x)\cdot f(t-x)\,dx = f(t)$

$$f' * g = f * g'$$

$$\int_0^t f'(x)\,g(t-x)\,dx = \Big[\underbrace{f(x)\,g(t-x)}_{=0}\Big]_0^t + \int_0^t f(x)\,g'(t-x)\,dx$$

(✿) blir $\delta * u' + tu = \delta$

$$u' + tu = \delta$$

$$D\left(e^{\frac{t^2}{2}}\cdot u\right) = \delta e^{\frac{t^2}{2}}$$

$$e^{\frac{t^2}{2}}\cdot u(t) = \int_0^t \delta(\tau)\,e^{\frac{\tau^2}{2}}\,d\tau = e^{\frac{0^2}{2}} = 1$$

$$u(t) = e^{-\frac{t^2}{2}}$$