Storgruppsövning 11/10-13

5.1 Fouriertransformer

Definition

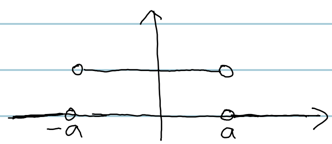
Antag $u : \mathbb{R} \to \mathbb{C}$ s.a $\int_{-\infty}^{\infty} |u(t)|\,dt < \infty \overset{def.}{\Longleftrightarrow} u \in L^1$.

Bilda en ny fkn $\hat{u}(x)$, $\mathcal{F}(u)(x) : \mathbb{R} \to \mathbb{C}$ genom

$$\hat{u}(x) = \mathcal{F}(u)(x) = \int_{-\infty}^{\infty} u(t) e^{-ixt}\,dt \quad (<\infty)$$

$\mathcal{F}(u)$, $\hat{u}$ kallas Fouriertransformen av u.

ex (i) $u(t) = \chi_a(t) = \begin{cases} 1 & \text{om } |t| < a \\ 0 & \text{om } |t| > a \end{cases}$, $a > 0$



$$\mathcal{F}(\chi_a)(x) = \frac{2\sin(ax)}{x}$$

(ii) $u(t) = e^{-t^2} \Rightarrow \hat{u}(x) = \sqrt{\pi}\, e^{-t^2/4}$

(iii) $u(t) = \dfrac{1}{1+t^2} \Rightarrow \hat{u}(x) = \pi e^{-|x|}$

$\Big\}$ Beräknas m.hj.a residykalkyl.

Egenskaper:

(I) $u_1, u_2 \in L^1$, $\lambda_1, \lambda_2 \in \mathbb{C}$

$\Rightarrow \mathcal{F}(\lambda_1 u_1 + \lambda_2 u_2)(x) = \lambda_1 \mathcal{F}(u_1)(x) + \lambda_2 \mathcal{F}(u_2)(x)$

(II) $u \in L^1$, $a \in \mathbb{R}$

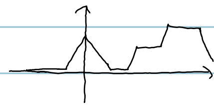
$\Rightarrow \mathcal{F}(u(t+a))(x) = e^{iax}\,\hat{u}(x)$

(III) $u \in L^1$, $b \in \mathbb{R} \setminus \{0\}$

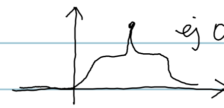
$$\mathcal{F}(u(bt))(x) = \frac{1}{|b|}\, \hat{u}\left(\frac{x}{b}\right)$$

(IV) $u \in L^1$, u kont., $\lim_{h \to 0^\pm} u'(x+h)$ exist. $\forall x$,

$u' \in L^1$ ex.



OK!



ej OK!

Då $\mathcal{F}(u'(t))(x) = ix\, \hat{u}(x)$

(V) Under lämpliga förutsättningar gäller

$$u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(x)\, e^{ixt}\, dx$$

Varför Fouriertransf.?

$$\begin{tikzcd}
\boxed{\text{Diff.ekv}} \arrow[r, "\text{svårt}"] \arrow[d, "\mathcal{F}\ (\mathrm{IV})"] & \boxed{\text{lösn.}} \\
\boxed{\text{Alg. ekv.}} \arrow[r, "\text{"lätt"}"] & \boxed{\text{lösn.}} \arrow[u, "\mathcal{F}^{-1}\ (\mathrm{V})"']
\end{tikzcd}$$

5.1.1

Finn $\hat{u}$ om $u(t) = \begin{cases} t & , |t| < b \\ 0 & , |t| > b \end{cases}$

lösn.1: (Vanlig studentlösning).

$u'(t) = \chi_b(t)$

$\Rightarrow \mathcal{F}(u'(t))(x) = 2\sin(bx)/x$

$u'(t) = \{(\mathrm{IV})\} = ix\,\hat{u}(x)$

forts. →

$\Rightarrow \hat{u}(x) = \frac{2\sin(bx)}{ix^2} \cdot \frac{i}{i} = -\frac{2\sin(bx)}{x^2}$ FEL!

u ej kont.!

lösn. 2: (korrekt lösning)

$$\hat{u}(x) = \int_{-\infty}^{\infty} u(t) e^{-ixt}\, dt = \int_{-b}^{b} t e^{-ixt}\, dt =$$

$$= \left[t \frac{e^{-ixt}}{-ix} \right]_{-b}^{b} + \int_{-b}^{b} 1 \cdot \frac{e^{-ixt}}{\cancel{-}ix}\, dt = \frac{ib}{x}\left(e^{-ibx} - e^{ixb}\right) +$$

$$+ \frac{1}{ix}\left[\frac{e^{-ixt}}{-ix}\right]_{-b}^{b} = \frac{2ib\cos(bx)}{x} + \frac{1}{x^2}\left(e^{-xbi} - e^{ixb}\right) =$$

$$= \frac{2ib\cos(bx)}{x} - \frac{2i\sin(bx)}{x^2}, \quad x \neq 0$$

$x=0$: $\hat{u}(0) = \int_{-\infty}^{\infty} u(t) e^{-i0 \cdot t}\, dt = \int_{-b}^{b} t\, dt = 0$

$$\therefore \quad \hat{u}(x) = \begin{cases} \frac{2ib\cos(bx)}{x} - \frac{2i\sin(bx)}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

5.1.4

Finn $\hat{u}(x)$ om $u(t) = 16e^{-4t^2}$

lösn.: Låt $f(t) = e^{-t^2}$ Då är $u(t) = 16f(2t)$

$$\Rightarrow \mathcal{F}(u(t))(x) = \mathcal{F}(16f(2t))(x) = \{(\mathrm{I})\} = 16\mathcal{F}(f(2t))(x)$$

$$= \{(\mathrm{III})\} = 16 \cdot \frac{1}{|2|} \mathcal{F}(f(t))\left(\frac{x}{2}\right) = \{(ii)\} =$$

$$= 8\sqrt{\pi}\, e^{-\frac{1}{4}\left(\frac{x}{2}\right)^2} = 8\sqrt{\pi}\, e^{-x^2/16}$$

5.15

Finn $\hat{u}(x)$ om $u(t) = (20+8t+t^2)^{-1}$.

Lösn: $u(t) = \dfrac{1}{t^2+8t+20} = \dfrac{1}{(t+4)^2+4} =$

$= \dfrac{1}{4} \cdot \dfrac{1}{1+\left(\frac{t+4}{2}\right)^2}$

Låt $f(t) = \dfrac{1}{1+t^2}$. Då är $u(t) = \dfrac{1}{4} f\left(\dfrac{t+4}{2}\right)$

$\Rightarrow \mathcal{F}(u(t))(x) = \{(\mathrm{I})\} = \frac{1}{4}\mathcal{F}\left(f\left(\frac{t+4}{2}\right)\right)(x) =$

$= \{(\mathrm{III}), b = 1/2\} = \frac{1}{4} \cdot \frac{1}{|1/2|} \mathcal{F}(f(t+4))(2x) =$

$= \{(\mathrm{II}), a=4\} = \frac{1}{2} e^{i4x}\mathcal{F}(f(t))(2x) = \{(iii)\} =$

$= \dfrac{1}{2} e^{4ix} \cdot \pi \cdot e^{-|2x|} = \dfrac{\pi}{2} e^{2(2ix-|x|)}$

5.3 Laplacetransform

Definition

Antag att $u : \mathbb{R} \to \mathbb{C}$ s.a

i) $u(t) = 0$ då $t<0$

ii) $|u(t)| \leq Me^{at}$, $M, a > 0$

Låt $(\mathcal{L}u)(s) = \displaystyle\int_0^\infty u(t)e^{-st}\,dt$, $s \in \mathbb{C}$

$\mathcal{L}u$ kallas Laplacetransformen av u

Väldef. om $\mathrm{Re}(s) > a$.

(I alla dagens exempel kommer alltid anta att $u(t) = 0$ då $t<0$.

(ex) (i) $u(t)=1 \Rightarrow (\mathcal{L}u)(s) = 1/s$ om $\mathrm{Re}(s)>0$

(ii) $u(t)=t^k,\ k=1,2,\ldots \Rightarrow (\mathcal{L}u)(s) = k!/s^{k+1},\ \mathrm{Re}(s)>0$

(iii) $u(t)=e^{\alpha t},\ \alpha\in\mathbb{C}$

$$\Rightarrow (\mathcal{L}u)(s) = \frac{1}{s-\alpha} \quad \text{om } \mathrm{Re}(s) > \mathrm{Re}(\alpha)$$

5.3.2

Finn $\mathcal{L}u$ om $u(t)=\sinh(At)$

Lösn: $\sinh(At) = \frac{1}{2}\left(e^{At}-e^{-At}\right)$

$$\mathcal{L}(\sinh(At)) = \tfrac{1}{2}\left(\mathcal{L}(e^{At}) - \mathcal{L}(e^{-At})\right) = \{(iii)\} =$$

$$= \frac{1}{2}\left(\frac{1}{s-A} - \frac{1}{s+A}\right) = \frac{1}{2}\,\frac{\cancel{s}+A-(\cancel{s}-A)}{(s-A)(s+A)} = \frac{A}{s^2-A^2}$$

om $\mathrm{Re}(s) > \mathrm{Re}(A)$ och $\mathrm{Re}(s) > \mathrm{Re}(-A)$

5.3.4

Finn $\mathcal{L}u$ om $u(t) = e^{-Bt}\cos(At)$

Lösn: $u(t) = e^{-Bt}\cdot\frac{1}{2}\left(e^{iAt}+e^{-iAt}\right) =$

$$= \frac{1}{2}\left(e^{(iA-B)t} + e^{-(iA+B)t}\right)$$

$$\mathcal{L}u = \frac{1}{2}\left(\mathcal{L}(e^{(iA-B)t}) + \mathcal{L}(e^{-(iA+B)t})\right) = \{(iii)\} =$$

$$= 1/2\left(\frac{1}{s-iA+B} + \frac{1}{s+iA+B}\right) = 1/2\left(\frac{s+iA+B+s-iA+B}{((s+B)-iA)((s+B)+iA)}\right)$$

$$= \frac{s+B}{(s+B)^2+A^2}$$

om $\mathrm{Re}(s) > \mathrm{Re}(iA-B)$ och $\mathrm{Re}(s) > \mathrm{Re}(-iA-iB)$.

5.3.11

Låt $u(t) = \sin t / t \quad t > 0$, $u(t) = 0 \quad t < 0$

Visa att $(\mathcal{L}u)(s) = \arctan\left(\frac{1}{s}\right)$,

Bevis

$$(\mathcal{L}u)(s) = \int_0^\infty \frac{\sin t}{t} e^{-st}\, dt$$ jobbigt!

Men ser att $\mathcal{L}(tu(t))$ är enkel!

$$\Rightarrow \frac{d}{ds}\mathcal{L}(u(t)) = -\mathcal{L}(tu(t))(s) = -\mathcal{L}(\sin t)(s) =$$

$$= -\frac{1}{2}\left(\mathcal{L}(e^{it}) - \mathcal{L}(e^{-it})\right) = \dots =$$

$$= -\frac{1}{s^2+1} = \frac{1}{1+\left(\frac{1}{s}\right)^2}\left(-\frac{1}{s^2}\right) = \frac{d}{ds}\arctan(1/s)$$

$$\Rightarrow \mathcal{L}(u(t))(s) = \arctan(1/s) + C$$ vill ha $C = 0$

Låt $s \to 0^+$

$\arctan(1/s) \to \pi/2$

$(\mathcal{L}u(t))(s) \to \int_0^\infty \frac{\sin t}{t}\, dt$ Beräknas med residykalkyl.

Låt $f(t) = \chi_1(t) = \begin{cases} 1 & |t| < 1 \\ 0 & |t| > 1 \end{cases}$

Vet att $\hat{f}(x) = 2\sin x / x$ och att

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^\infty \hat{f}(x) e^{ixt}\, dx$$

$t = 0$: $$1 = f(0) = \frac{1}{\cancel{2}\pi}\int_{-\infty}^\infty \frac{\cancel{2}\sin x}{x}\cdot 1\, dx =$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin x}{x}\, dx, \quad \because \int_0^\infty \frac{\sin x}{x}\, dx = \pi/2$$

$$\Rightarrow C = 0 \Rightarrow (\mathcal{L}u)(s) = \arctan(1/s) \quad \blacksquare$$