Storgruppsövning 27/9-13

2.6 Residykalkyl

Definition

Låt $f \in A(\{0<|z-z_0|<R\})$

$\text{Res}(f; z_0) \overset{\text{def.}}{=} \frac{1}{2\pi i}\int_{|z-z_0|=r} f(z)\,dz$, $r<R$, kallas residyn av f i z_0.

Sats

Om f har Laurentserieutveckling $f(z)= \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$ så är $\text{Res}(f; z_0) = a_{-1}$.

Konsekvens: Om f har en pol av ordn. m i z_0 så $\text{Res}(f; z_0) = \lim_{z\to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)$

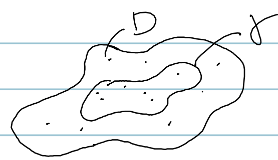
$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \ldots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0)+\ldots$$

$$(z-z_0)^m f(z) = \underbrace{\ldots}_{=0 \text{ vid derivering.}} + (z-z_0)^{m-1}a_{-1} + \ldots$$

Speciellt: Om z_0 enkel pol (dvs $m=1$) så

$$\text{Res}(f; z_0) = \lim_{z\to z_0}(z-z_0)f(z) = \left\{ f(z) = \frac{P(z)}{Q(z)} \right\} =$$

$$= \lim_{z\to z_0} \frac{P(z)}{\frac{Q(z)-Q(z_0)}{z-z_0}} = \frac{P(z_0)}{Q'(z_0)}$$



Sats

$f \in A(D \setminus \{z_1, \ldots, z_n\})$, D och γ som i figur

Då gäller att $\int_\gamma f(z)\,dz = 2\pi i \left(\sum_{z_j \in \text{inre}(\gamma)} \text{Res}(f; z_j) \right)$

2.6.2 Beräkna $I = \int_{-\infty}^{\infty} \frac{x^2}{x^4 - 4x^2 + 5}\,dx$

Lösn: Låt $f(z) = \frac{z^2}{z^4 - 4z^2 + 5}$ (fyra poler)

$z^4 - 4z^2 + 5 = 0 \Leftrightarrow \{t = z^2\}$

$\Leftrightarrow t^2 - 4t + 5 = 0 \Longrightarrow t = 2 \pm \sqrt{4-5} = 2 \pm i = z^2$

$z = a + ib$, $a, b \in \mathbb{R}$ söker $a, b \in \mathbb{R}$ s.a

$(a+ib)^2 = 2 \pm i \Leftrightarrow a^2 - b^2 + 2abi = 2 \pm i$

$$\begin{cases} a^2 - b^2 = 2 \\ 2ab = \pm 1 \Leftrightarrow b = \pm 1/2a \end{cases}$$

$a^2 - 1/4a^2 = 2 \Longrightarrow a^4 - 2a^2 - 1/4 = 0 \Leftrightarrow \{t = a^2\}$

$\Leftrightarrow t^2 - 2t - 1/4 = 0 \Longrightarrow t = 1 \pm \sqrt{1 + \frac{1}{4}} = 1 \underbrace{(\pm) \sqrt{5}/2}_{\text{falsk rot, ty } a \in \mathbb{R}} = a^2$

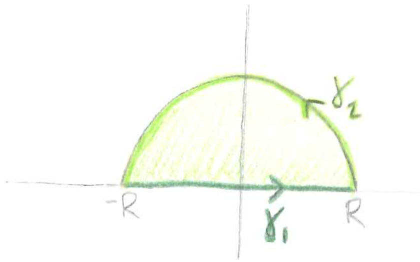
$\Longrightarrow a = \pm \sqrt{1 + \sqrt{5}/2} = \pm \frac{\sqrt{4 + 2\sqrt{5}}}{2}$

Fall 1: $a = \pm \frac{\sqrt{4+2\sqrt{5}}}{2}$, $b = \pm \frac{1}{\sqrt{4+2\sqrt{5}}}$

Fall 2: $a = \pm \frac{\sqrt{4+2\sqrt{5}}}{2}$, $b = \mp \frac{1}{\sqrt{4+2\sqrt{5}}}$

Kommer att studera konturen:

Endast poler i det övre halvplanet ($b > 0$) intressanta



$z_1 = \frac{\sqrt{4+2\sqrt{5}}}{2} + i\frac{1}{\sqrt{4+2\sqrt{5}}}$

$z_2 = -\frac{\sqrt{4+2\sqrt{5}}}{2} + i\frac{1}{\sqrt{4+2\sqrt{5}}}$

$\gamma = \gamma_1 + \gamma_2$

$R >> 0$

Parametriseringar:

$\gamma_1: z = x$, $-R \leq x \leq R$, $dz = dx$

$\gamma_2: z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, $dz = Rie^{i\theta}d\theta$

$$\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} f(x)\,dx \xrightarrow[R\to\infty]{} I$$

$$\left| \int_{\gamma_2} f(z)\,dz \right| = \left| \int_0^{\pi} f(Re^{i\theta})\, iRe^{i\theta}\, d\theta \right| \leq \int_0^{\pi} \frac{O(R^3)}{O(R^4)}\, d\theta \xrightarrow[R\to\infty]{} 0$$

$$I = \underbrace{\int_{\gamma} f(z)\,dz}_{(*)} = 2\pi i \left(\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_2) \right)$$

2.6.4

Beräkna $I = \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{(x^2+1)(x^2+4)}\,dx$, $\alpha \in R$

Lösn: Låt $f(z) = \frac{e^{i\alpha z}}{(z^2+1)(z^2+4)} = \frac{e^{i\alpha z}}{(z-i)(z+i)(z-2i)(z+2i)}$

Vilka poler är intressanta?

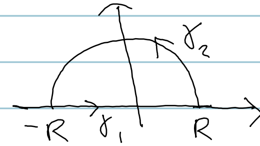
Det beror på tecknet på α.

$e^{i\alpha z} = e^{i\alpha(x+iy)} = e^{-\alpha y} e^{i\alpha x}$

Om $\alpha > 0$ så $e^{i\alpha z}$ begränsad då $y > 0$,

om $\alpha < 0$ så $e^{i\alpha z}$ begränsad då $y < 0$.

Fall 1: $\alpha > 0$, vill att $y > 0$.



$\gamma = \gamma_1 + \gamma_2$

Intressanta poler:

$z_1 = i$, $z_2 = 2i$

$\mathrm{Res}_{z_1} = \lim_{z \to z_1} (z - z_1)(f(z)) =$

$= \frac{e^{i\alpha z}}{2i(-i)3i} = \frac{e^{-\alpha}}{6i}$

$\mathrm{Res}_{z_2} = \frac{e^{-2\alpha}}{i \cdot 3i \cdot 4i} = -\frac{e^{-2\alpha}}{12i}$

Samma param. som tidigare:

$$\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} f(x)\,dx \xrightarrow[R\to\infty]{} \int_{-\infty}^{\infty} f(x)\,dx$$

$$\left| \int_{\gamma_2} f(z)\,dz \right| = \left| \int_0^{\pi} f(Re^{i\theta})\, iRe^{i\theta}\,d\theta \right| \leq$$

$$\leq \int_0^{\pi} \frac{R|e^{i\alpha z}|}{|R^2 e^{2i\theta}+1||R^2 e^{2i\theta}+4|}\,d\theta \leq$$

$$\leq \int_0^{\pi} \frac{R}{(R^2-1)(R^2-4)}\,d\theta \xrightarrow[R\to\infty]{} 0$$

$$\leq \left\{ \begin{array}{l} |e^{i\alpha z}| = e^{-\alpha y} \leq 1 \text{ då } \alpha, y > 0 \\ ||a| - |b|| \leq |a \pm b| \end{array} \right\} \leq \int_0^{\pi} \frac{R}{(R^2-1)(R^2-4)} d\theta \xrightarrow[R \to \infty]{} 0$$

$$\therefore \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{(x^2+1)(x^2+4)} dx = 2\pi i \left(\frac{e^{-\alpha}}{6i} - \frac{e^{-2\alpha}}{12i} \right) = \frac{\pi}{6} e^{-\alpha}(2 - e^{-\alpha})$$

$$\Rightarrow I = \frac{\pi}{6} e^{-\alpha}(2 - e^{-\alpha})$$

Fall 2: $\alpha < 0$, $\alpha = -\beta$, $\beta > 0$

ty cosinus jämn funktion.

$$I = \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{(x^2+1)(x^2+4)} dx = \int_{-\infty}^{\infty} \frac{\cos(\cancel{-}\beta x)}{(x^2+1)(x^2+4)} \overset{\text{fall 1}}{=} \frac{\pi}{6} e^{-\beta}(2 - e^{-\beta}) =$$

$$= \{\alpha = -\beta\} = \frac{\pi}{6} e^{\alpha}(2 - e^{\alpha})$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{(x^2+1)(x^2+4)} dx = \frac{\pi}{6} e^{-|\alpha|}(2 - e^{-|\alpha|}), \quad \alpha \in \mathbb{R}$$

2.6.5 Beräkna $I = \int_{-\infty}^{\infty} \frac{x \sin x}{x^4+1} dx$. Lös denna!