Storgruppsövning 18/9–13

2.2 Potensserier

En (komplex) potensserie är en serie av formen $\sum_{k=0}^{\infty} a_k (z-z_0)^k$, $z_0, a_0, a_1, \ldots \in \mathbb{C}$

Begreppet konvergensradie, samt ~~rot~~-och kvotkriteriet gäller precis som för reella potensserier.

2.2.3 Finn konv.radien för p.s

$$\sum_{j=0}^{\infty} \frac{z^{3j}}{2^j}$$

Lösn.: Sätt $w = z^3 \Rightarrow \sum_{j=0}^{\infty} \frac{z^{3j}}{2^j} = \sum_{j=0}^{\infty} \frac{w^j}{2^j}$

Kvotkriteriet: $\frac{1}{R} = \lim_{j\to\infty} \left| \frac{a_{j+1}}{a_j} \right| = \lim \left| \frac{1/2^{j+1}}{1/2^j} \right| =$

$$= \lim_{j\to\infty} \left| \frac{\cancel{2^j}}{\cancel{2^j}\, 2} \right| = \frac{1}{2} \Rightarrow R = 2$$

$\therefore \sum_{j=0}^{\infty} \frac{w^j}{2^j}$ konv. då $\underbrace{|w|}_{=|z^3| = |z|^3} < 2$

$$\Rightarrow \sum_{j=0}^{\infty} \frac{z^{3j}}{2^j} \text{ konv. då } |z| < 2^{1/3}$$

2.2.17 Finn en sluten form (dvs ett "enkelt" uttryck) för p.s

$$\sum_{n=0}^{\infty} (-1)^n \frac{(z-2\pi i)^n}{n!}$$

Lösn.: Vet att $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Rightarrow e^{-z} = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!}$

forts. →

$$\Rightarrow e^{-z+2\pi i} = \sum_{n=0}^{\infty} \frac{(-z+2\pi i)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-(z-2\pi i))^n}{n!} =$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(z-2\pi i)^n}{n!}$$

Men $e^{-z+2\pi i} = e^{-z} e^{2\pi i} = e^{-z}$, så

$$\sum_{n=0}^{\infty} (-1) \frac{(z-2\pi i)^n}{n!} = e^{-z}$$

2.2.18 Finn en sluten form för p.s

$$\sum_{n=2}^{\infty} n(n-1) z^n$$

Lösn: $\displaystyle\sum_{n=2}^{\infty} n(n-1) z^n \cdot \frac{z^2}{z^2} = z^2 \sum_{n=2}^{\infty} n(n-1) z^{n-2} =$

$$= z^2 \sum_{n=0}^{\infty} n(n-1) z^{n-2} = z^2 \sum_{n=0}^{\infty} \frac{d^2}{dz^2} z^n = \{\text{sats } 2.2.3\} =$$

$$= z^2 \frac{d^2}{dz^2} \sum_{n=0}^{\infty} z^n = z^2 \frac{d^2}{dz^2} \frac{1}{1-z} = \frac{2z^2}{(1-z)^3}$$

2.3 Cauchys sats och integralformel

2.3.7 Använd tekniken i ex. 6 och 7 till att beräkna $I = \displaystyle\int_0^{2\pi} \frac{1}{a + b\cos\theta} d\theta$, $a > b > 0$

Lösn: CS och CIF låter oss beräkna komplexa linjeintegralen på ~~ett~~ smidig ~~sätt~~.

Tekniken i ex. 6 och 7: Gör om "vanliga" 1D-integraler till linjeintegraler.

Huvudidé: $\cos\theta = 1/2(e^{i\theta} + e^{-i\theta}) = 1/2(z + 1/z)$
om $z = e^{i\theta}$. Tänk på I som en parametriserad linjeintegral med param. $z = e^{i\theta}$, $0 \le \theta \le 2\pi$
I så fall $dz = ie^{i\theta} d\theta = iz\,d\theta \Leftrightarrow d\theta = \frac{dz}{iz}$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \int_{|z|=1} \frac{1}{a + b\,1/2(z + 1/z)} \frac{dz}{iz} =$$

$$= \frac{2}{i} \int_{|z|=1} \frac{dz}{bz^2 + 2az + b} = \frac{2}{ib} \int_{|z|=1} \frac{dz}{z^2 + \frac{2a}{b} z + 1} = (*)$$

$$z^2 + \frac{2a}{b} z + 1 = 0 \Rightarrow z = -\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} - 1} = \{b > 0\} =$$

$$= \frac{-a \pm \sqrt{a^2 - b^2}}{b} = z_\pm$$

$$\Rightarrow (*) = \frac{2}{ib} \int_{|z|=1} \frac{dz}{(z - z_+)(z - z_-)}$$

Vill använda CIF, måste veta om z_+ och z_- ligger innanför eller utanför $\{|z|=1\}$

$$z_+ = \frac{-a + \sqrt{a^2 - b^2}}{b}, \quad z_- = \frac{-a - \sqrt{a^2 - b^2}}{b}, \quad a > b > 0$$

$$|z_+| \overset{?}{<} 1 \Leftrightarrow |-a + \sqrt{a^2 - b^2}| < b \Leftrightarrow$$
$$\Leftrightarrow \{\sqrt{a^2 - b^2} - a < 0\} \Leftrightarrow a - \sqrt{a^2 - b^2} < b \Leftrightarrow$$
$$\Leftrightarrow a - b < \sqrt{a^2 - b^2} \Leftrightarrow (a - b)^2 < a^2 - b^2 = (\cancel{a-b})(a + b)$$
$$\Leftrightarrow a - b < a + b \quad \text{Ja!!} \quad \therefore |z_+| < 1$$

$$|z_-| \overset{?}{<} 1 \Leftrightarrow |-a - \sqrt{a^2 - b^2}| < b \Leftrightarrow a + \sqrt{a^2 - b^2} < b$$
$$\Leftrightarrow \underbrace{a - b}_{>0} < \underbrace{-\sqrt{a^2 - b^2}}_{<0} \; ⚡ \quad \therefore |z_-| > 1$$

Tillbaka till integralen:

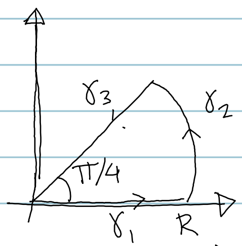
$$I = \frac{2}{ib} \int_{|z|=1} \frac{dz}{(z-z_+)(z-z_-)} = \left\{ |z_-|>1 \Rightarrow f(z) = \frac{1}{z-z_-} \text{ holo på } \{|z|<1\} \right\} = \frac{2}{ib} \int_{|z|=1} \frac{f(z)}{z-z_+}\, dz = \{CIF\} =$$

$$= \frac{2}{\cancel{i}b} \cdot 2\pi \cancel{i} f(z_+) = \frac{4\pi}{b} \cdot \frac{1}{z_+ - z_-} =$$

$$= \frac{4\pi}{b} \cdot \frac{1}{-\frac{\cancel{a}}{\cancel{b}} + \frac{\sqrt{a^2-b^2}}{b} - \left(-\frac{\cancel{a}}{b} - \frac{\sqrt{a^2-b^2}}{b}\right)} =$$

$$= \frac{{}^2\cancel{4}\pi}{\cancel{b}} \cdot \frac{\cancel{b}}{\cancel{2}\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}} \quad \left(\text{fel i facit}\right)$$

2.3.13 Integrera e^{iz^2} kring konturen



för att visa att

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{\sqrt{2\pi}}{4}$$

$$\gamma = \gamma_1 + \gamma_2 + \gamma_3$$

Lösn.: $f(z) = e^{iz^2} \in A(\mathbb{C})$ (ober. av $R>0$)

$$CS \Rightarrow \int_\gamma f(z)\,dz = 0$$

$$\int_\gamma f(z)\,dz = \overbrace{\int_{\gamma_1} f(z)\,dz}^{I} + \overbrace{\int_{\gamma_2} f(z)\,dz}^{II} + \overbrace{\int_{\gamma_3} f(z)\,dz}^{III}$$

Har parametriseringarna

$\gamma_1 : z = x, \quad 0 \le x \le R, \quad dz = dx$

$\gamma_2 : z = Re^{i\theta}, \quad 0 \le \theta \le \pi/4, \quad dz = iRe^{i\theta}\, d\theta$

$\gamma_3: z = te^{i\pi/4},\ R \ge t \ge 0,\ dz = e^{i\pi/4}dt$

Dessa ger:

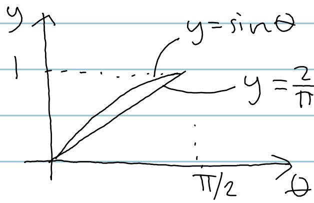
$$\int_{\gamma_1} f(z)\,dz = \int_0^R e^{ix^2}dx \qquad t^2e^{i\pi/2} = it^2$$

$$\int_{\gamma_3} f(z)\,dz = \int_R^0 e^{i(te^{i\pi/4})^2} e^{i\pi/4}\,dt = -e^{i\pi/4}\int_0^R e^{-t^2}dt$$

$$\int_{\gamma_2} f(z)\,dz = \int_0^{\pi/4} e^{iR^2e^{2i\theta}}\, iRe^{i\theta}\,d\theta = \int_0^{\pi/4} e^{iR^2\cos(2\theta)}\cdot e^{-R^2\sin(2\theta)} Rie^{i\theta}\,d\theta$$

Vill nu låta $R\to\infty$, men måste först få en idé om vad som händer II.

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \int_0^{\pi/4} e^{-R^2\sin(2\theta)} R\,d\theta$$



$$\Rightarrow \sin\theta \ge \frac{2\theta}{\pi} \text{ då } 0 \le \theta \le \pi/2$$

$$\Rightarrow -\sin\theta \le -\frac{2\theta}{\pi} \text{ — '' —}$$

$$\Rightarrow -\sin(2\theta) \le -\frac{4\theta}{\pi} \text{ då } 0 \le \theta \le \pi/4$$

$$R\int_0^{\pi/4} e^{-R^2\sin(2\theta)}d\theta \le R\int_0^{\pi/4} e^{-\frac{4R^2}{\pi}\theta}d\theta = \cancel{R}\int_0^{\pi/4}\left(-\frac{\pi}{4R^{\cancel{2}}}\right)\frac{d}{d\theta}e^{-\frac{4R^2}{\pi}\theta}d\theta = -\frac{\pi}{4R}\left[e^{-\frac{4R^2}{\pi}\theta}\right]_0^{\pi/4} =$$

$$= \underbrace{-\frac{\pi}{4R}\left(e^{-R^2}-1\right)}_{g(R)} \xrightarrow[R\to\infty]{} 0$$

forts. →

$\because \quad 0 \leq \left| \int_{\gamma_2} f(z)\,dz \right| \leq g'(R) \xrightarrow[R\to\infty]{} 0$

$\Longrightarrow \int_{\gamma_2} f(z)\,dz \xrightarrow[R\to\infty]{} 0$ (instängnings-regeln)

$$0 = \underbrace{\int_{\gamma_1} f(z)\,dz}_{\downarrow R\to\infty \quad \int_0^\infty e^{ix^2}dx} + \underbrace{\int_{\gamma_2} f(z)\,dz}_{\downarrow R\to\infty \quad 0} + \underbrace{\int_{\gamma_3} f(z)\,dz}_{\downarrow R\to\infty \quad -e^{i\pi/4}\int_0^\infty e^{-t^2}dt}$$

$$\int_0^\infty e^{ix^2}dx = \frac{\sqrt{\pi}}{2}e^{i\pi/4} = \frac{\sqrt{\pi}}{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2\pi}}{4} + i\frac{\sqrt{2\pi}}{4}$$

$$\| $$

$$\int_0^\infty \cos(x^2)\,dx + i\int_0^\infty \sin(x^2)\,dx$$

$$\Longrightarrow \int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{\sqrt{2\pi}}{4} \qquad \blacksquare$$